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The Existence of the Moore-Penrose Inverse in Symmetrized Max-Plus Algebraic Matrix

Suroto Suroto[1,*] Najmah Istikaanah[1], Renny Renny[1]

[1]*Departement of Mathematics, Universitas Jenderal Soedirman, Purwokerto, Central Java, Indonesia*
[*]*Corresponding author. Email: suroto@unsoed.ac.id*

ABSTRACT
In this paper we discusses Moore-Penrose inverse in symmetrized max-plus algebraic matrix. The existence of Moore-Penrose inverse is shown using a link among symmetrized max-plus algebra and conventional algebra. The result is a Moore-Penrose inverse in symmetrized max-plus algebraic matrix exists. Furthermore, the balanced inverse and the max-plus inverse are also the Moore-Penrose inverse in symmetrized max-plus algebraic matrix..

Keywords: *Existence, Moore-Penrose inverse, Symmetrized max-plus algebra, Conventional algebra.*

1. INTRODUCTION

Max-plus algebra is the set of $\mathbb{R} \cup \{-\infty\}$ equipped with maximum (simply written "max") as addition and usual addition (simply written "plus") as multiplication, where $\mathbb{R}$ is the set of all real numbers. Henceforth, the max-plus algebra is denoted by $\mathbb{R}_{\max}$. It is different from conventional algebra, since there is no inverse element under addition for every element in max-plus algebra, except for zero element [1][2][3]. The symmetrization process can be done to solve the additive inverse problem. This process is carried out using a balance relation (denoted by ∇) in order to obtain the minus and balance of all elements in $\mathbb{R}_{\max}$. The result of this symmetrization is called symmmetrized max-plus algebra and denoted by $\mathbb{S}$ [4][5].

In conventional algebra, it is known that A^{-1} denotes the inverse of an invertible square matrix $A_{n\times n}$ [6]. It is known that there is an inverse for an $m \times n$ matrix called the Moore-Penrose inverse and usually denoted by A^{+}[7]. The concept of the inverse of an $n \times n$ matrix can be used as an alternative way to find a solution to a system of linear equations in the form $Ax = b$. If A is an invertible matrix, then the solution of the system of linear equations can be solved using the formula $x = A^{-1}b$. If the matrix A of the system has a size of $m \times n$, the solution of the system cannot be found using these rules. The discussion about application of the Moore-Penrose inverse in linear equation systems was discussed in [8].

The discussion about the Moore-Penrose inverse on arbitrary ring and integral domain were discussed in [9] and [10], respectively. In this paper, we discuss the Moore-Penrose inverse of matrix over $\mathbb{S}$. We use a link between $\mathbb{S}$ and conventional algebra in [11] to show the existence of Moore-Penrose inverse in $\mathbb{S}$. We adopt the Moore-Penrose inverse in conventional algebra [12] to define the Moore-Penrose inverse in $\mathbb{S}$, by changing equal relation in the conventional Moore-Penrose inverse into balance relation in symmetrized max-plus algebra. The results in this paper can potentially be used as an alternative tool to solve the solution of the systems of linear balance in $\mathbb{S}$.

2. BASIC TERMINOLOGY

This section discusses basic terminologies in symmetrized max-plus algebra. Let $\mathbb{R}$ be the set of all real numbers, $\mathcal{E} \overset{\text{def}}{=} -\infty$ and $\mathbb{R}_{\mathcal{E}} \overset{\text{def}}{=} \mathbb{R} \cup \{-\infty\}$. The basic operations in $\mathbb{R}_{\mathcal{E}}$ are defined by

$$a \oplus b = \max\{a, b\} \quad (1)$$
$$a \otimes b = a + b \quad (2)$$

where $\max\{a, -\infty\} = a$ and $a + (-\infty) = -\infty$, for all $a, b \in \mathbb{R}_{\mathcal{E}}$. The mathematical system $\mathbb{R}_{\max} = (\mathbb{R}_{\mathcal{E}},\oplus,\otimes)$ is called the max-plus algebra, with the zero element is $\mathcal{E}$, the unity element is e and the zero element $\mathcal{E}$ is absorbsing for $\otimes$. Furthermore, $\mathbb{R}_{\max}$ is an idempotent commutative semiring. There is no inverse element under addition for all a in $\mathbb{R}_{\max}$ except for $a = \mathcal{E}$.

Let $P_{\mathcal{E}} \overset{\text{def}}{=} \mathbb{R}_{\mathcal{E}} \times \mathbb{R}_{\mathcal{E}}$. The basic operations in $P_{\mathcal{E}}$ are defined by

$$(a, b) \oplus (c, d) = (a \oplus c, b \oplus d) \quad (3)$$
$$(a, b) \otimes (c, d) = (a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c) \quad (4)$$

for all $(a, b), (c, d) \in P_{\mathcal{E}}$. The zero element is $(\mathcal{E}, \mathcal{E})$, the unity element is $(0, \mathcal{E})$ and $(\mathcal{E}, \mathcal{E})$ is absorbing for multiplication. The mathematical system $P_{\max} = (P_{\mathcal{E}},\oplus$

,⊗) is a commutative idempotent semiring and called the algebra of pairs. Some terminologies in algebra of pairs refers to [4]. If $u = (a, b) \in P_{\max}$, then the absolute value of u is defined as $|u|_{\oplus} = a \oplus b$, the minus of u is $\ominus u = (b, a)$ and the balance of u is $u^{\bullet} = u \oplus (\ominus u) = (|u|_{\oplus}, |u|_{\oplus})$. Furthermore, for all $u, v \in P_{\max}$, the following statements are satisfied: $u^{\bullet} = (\ominus u)^{\bullet} = (u^{\bullet})^{\bullet}$, $u \otimes v^{\bullet} = u^{\bullet} \otimes v = (u \otimes v)^{\bullet}$, $\ominus (\ominus u) = u$, $\ominus (u \oplus v) = (\ominus u) \oplus (\ominus v)$ and $\ominus (u \otimes v) = (\ominus u) \otimes v$.

In the conventional algebra, for all $x \in \mathbb{R}$, $x - x = 0$, but for all $u \in P_{\max}$, $u \ominus u = u^{\bullet} \neq (\mathcal{E}, \mathcal{E})$, except $u = (\mathcal{E}, \mathcal{E})$. It is important to introduce "balance" relation for substituting "equal" relation in conventional algebra. If $u = (a, b), v = (c, d) \in P_{\max}$, balance relation (denoted by ∇) in $P_{\max}$ is defined as follows :

$$u \nabla v \text{ if } a \oplus d = b \oplus c. \quad (5)$$

The balance relation reflexive, symmetric but it is not transitive, so that it is impossible to define the quotient set of $P_{\mathcal{E}}$ by ∇. For example, $(5,4)\nabla(5,5)$ and $(5,5)\nabla(4,5)$ but $(5,4)\not\nabla(4,5)$. The new relation will be introduced in order to solve "transitive problem" in balance relation. Let $u = (a, b), v = (c, d) \in P_{\max}$, relation $\mathcal{B}$ in $P_{\max}$ is defined as follows :

$$u\,\mathcal{B}\,v \ \ if \ \begin{cases} (a,b)\nabla(c,d), \ if \ a \neq b \ and \ c \neq d \\ (a,b) = (c,d), \ if \ a = b \ or \ c = d \end{cases}. \quad (6)$$

For all $u \in P_{\max}$, $u \ominus u\,\mathcal{B}\,(\mathcal{E}, \mathcal{E})$ except for $u = (\mathcal{E}, \mathcal{E})$ and $\mathcal{B}$ is an equivalence relation. So, it is possible to obtain a quotient set of $P_{\max}$ by $\mathcal{B}$. The equivalence classes generated by $\mathcal{B}$ are

1. $\overline{(w, -\infty)} = \{(w, x) \in P_{\max} | x < w\}$ is called max-positive,
2. $\overline{(-\infty, w)} = \{(x, w) \in P_{\max} | x < w\}$ is called max-negative, and
3. $\overline{(w, w)} = \{(w, w) \in P_{\max}\}$ is called balanced.

The quotient set of $P_{\max}$ by $\mathcal{B}$ is denoted ${P_{\max}}/_{\mathcal{B}} \stackrel{\text{def}}{=} \mathbb{S}$. Note that (5,4) balance with (5,5) and (5,5) also balance with (4,5), but (5,4) is not $\mathcal{B}$ relation to (5,5), neither are (5,5) and (4,5).

The mathematical system $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$ is called the symmetrized max-plus algebra. The zero element is the class $\bar{\mathcal{E}} = \overline{(\mathcal{E}, \mathcal{E})}$ class, the unity element is the class $\bar{e} = \overline{(0, \mathcal{E})}$ and the zero element $\overline{(\mathcal{E}, \mathcal{E})}$ is absorbing for ⊗. Furthermore, $\overline{(w, -\infty)}$, $\overline{(-\infty, w)}$ and $\overline{(w, w)}$ are sufficiently written by w, $\ominus w$ and $w^{\bullet}$, respectively. The set of all max-positive class or zero class, max-negative class or zero class and balanced class are denoted by $\mathbb{S}^{\oplus}$, $\mathbb{S}^{\ominus}$ and $\mathbb{S}^{\bullet}$, respectively. The set $\mathbb{S}^{\vee} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}$ is called the set of all signed element. Note that $\mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus} \cup \mathbb{S}^{\bullet} = \mathbb{S}$ and $\mathbb{S}^{\oplus} \cap \mathbb{S}^{\ominus} \cap \mathbb{S}^{\bullet} = \{\overline{(\mathcal{E}, \mathcal{E})}\}$.

The basic operation of matrix over $\mathbb{S}$ can be done in the usual way as that in conventional algebra. The zero matrix in $\mathbb{S}^{m \times n}$ is $[\mathcal{E}]$ with $\mathcal{E}_{ij} = \mathcal{E}$ for all $i = 1,2, \ldots, m$ and $j = 1,2, \ldots, n$. The identity matrix is $I = [a] \in \mathbb{S}^{n \times n}$ with $a_{ij} = e$ if $i = j$ and $a_{ij} = \mathcal{E}$ if $i \neq j$, for $i, j = 1,2, \ldots, n$. For all $A, B \in \mathbb{S}^{m \times n}$, $A \nabla B$ *if* $a_{ij} \nabla b_{ij}$ for $i = 1,2, \ldots, m$ and $j = 1,2, \ldots, n$. If $A = \begin{bmatrix} 1 & \ominus 2 \\ \mathcal{E} & 3^{\bullet} \end{bmatrix}$ and $B = \begin{bmatrix} 2^{\bullet} & \ominus 2 \\ 4^{\bullet} & 3 \end{bmatrix}$, then $A \nabla B$ since $a_{ij} \nabla b_{ij}$ for $i = 1,2$ and $j = 1,2$. Note that the corresponding entries of A and B are not always equal.

3. A LINK BETWEEN CONVENTIONAL ALGEBRA AND SYMMETRIZED MAX-PLUS ALGEBRA

This section discusses a link between $\mathbb{S}$ and conventional algebra. It is used to solve the Moore-Penrose inverse in $\mathbb{S}$ via conventional algebra approach. In this paper, this link is used to show the existence of axioms of the Moore-Penrose inverse in $\mathbb{S}$ sense. The link is refered from [11].

Definition 1

A mapping $\mathcal{F}$ with domain of definition $\mathbb{S} \times \mathbb{R}_0 \times \mathbb{R}_0^+$ is defined as

$$\mathcal{F}(a, \mu, s) = \begin{cases} |\mu| e^{as}, \ if \ a \in \mathbb{S}^{\oplus} \\ -|\mu| e^{|a|_{\oplus} s}, if \ a \in \mathbb{S}^{\ominus} \\ \mu e^{|a|_{\oplus} s}, if \ a \in \mathbb{S}^{\bullet} \end{cases} \quad (7)$$

where $a \in \mathbb{S}, \mu \in \mathbb{R}_0, s \in \mathbb{R}_0^+$.

Definition 2

Let $f(s) \sim ve^{|a|_{\oplus} s}$ be in the neighbourhood of ∞, here is the function $\mathcal{R}$ is defined as

$$\mathcal{R}(f) = \begin{cases} |a|_{\oplus}, if \ v \ positive \\ \ominus |a|_{\oplus}, if \ v \ negative \end{cases} \quad (8)$$

The function in (7) and (8) are used to correspond elements in symmetrized max-plus algebra into conventional algebra and otherwise, respectively. If $\mu = 1$ then $\mathcal{F}(5,1,s) = e^{5s}$, $\mathcal{F}(\ominus 5,1,s) = e^{-5s}$ and $\mathcal{F}(5^{\bullet}, 1, s) = e^{5s}$. Note that $\mathcal{R}(\mathcal{F}(5,1,s)) = \mathcal{R}(\mathcal{F}(5,1,s)) = \mathcal{R}(e^{5s}) = 5$ and $\mathcal{R}(\mathcal{F}(\ominus 5,1,s)) = \mathcal{R}(e^{-5s}) = \ominus 5$. Range of $\mathcal{R}$ is $\mathbb{S}^{\vee} = \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}$ i.e the set of all signed element.

The folowing theorem explain the correspondence between addition and multiplication in symmetrized max-plus algebra into conventional algebra.

Theorem 3

Let $a, b, c \in \mathbb{S}$.

1. *If $a \oplus b = c$ then there are $\mu_a, \mu_b, \mu_c \in \mathbb{R}_0$ such that $\mathcal{F}(a, \mu_a, s) + \mathcal{F}(b, \mu_b, s) \sim \mathcal{F}(c, \mu_c, s)$, $s \to \infty$.*
2. *If there are $\mu_a, \mu_b, \mu_c \in \mathbb{R}_0$ such that $\mathcal{F}(a, \mu_a, s) + \mathcal{F}(b, \mu_b, s) \sim \mathcal{F}(c, \mu_c, s)$, for $s \to \infty$ then $a \oplus b \nabla c$.*
3. *If $a \otimes b = c$ then there are $\mu_a, \mu_b, \mu_c \in \mathbb{R}_0$ such that $\mathcal{F}(a, \mu_a, s) \times \mathcal{F}(b, \mu_b, s) = \mathcal{F}(c, \mu_c, s)$, $s \in \mathbb{R}_0^+$.*
4. *If there are $\mu_a, \mu_b, \mu_c \in \mathbb{R}_0$ such that*

$\mathcal{F}(a,\mu_a,s) \times \mathcal{F}(b,\mu_b,s) = \mathcal{F}(c,\mu_c,s)$, for $s \in \mathbb{R}_0^+$ then $a \otimes b \nabla c$.

We can expand Theorem 3 into matrix over $\mathbb{S}$. If $A = [a_{ij}] \in \mathbb{S}^{m\times n}$ and $M_A = [(m_A)_{ij}] \in \mathbb{R}_0^{m\times n}$ then $\tilde{A} = [\tilde{a}_{ij}] = \mathcal{F}(A, M_A, \cdot)$ is the $m \times n$ real matrix-valued function of A with $\tilde{a}_{ij}(s) = \mathcal{F}(a_{ij}, (m_A)_{ij}, s)$.

Theorem 4
Let A, B, C are matrices whose entries are in $\mathbb{S}$.
1. *If $A \oplus B = C$ then there are matrices M_A, M_B, M_C whose entries are in $\mathbb{R}_0$ such that $\mathcal{F}(A,M_A,s) + \mathcal{F}(B,M_B,s) \sim \mathcal{F}(C,M_C,s)$, $s \to \infty$.*
2. *If there are matrices M_A, M_B, M_C whose entries are in $\mathbb{R}_0$ such that $\mathcal{F}(A,M_A,s) + \mathcal{F}(B,M_B,s) \sim \mathcal{F}(C,M_C,s), s \to \infty$ then $A \oplus B \nabla C$.*
3. *If $A \otimes B = C$ then there are matrices M_A, M_B, M_C whose entries are in $\mathbb{R}_0$ such that $\mathcal{F}(A,M_A,s) \times \mathcal{F}(B,M_B,s) \sim \mathcal{F}(C,M_C,s)$, $s \to \infty$.*
4. *If there are matrices M_A, M_B, M_C whose entries are in $\mathbb{R}_0$ such that $\mathcal{F}(A,M_A,s) \times \mathcal{F}(B,M_B,s) \sim \mathcal{F}(C,M_C,s), s \to \infty$ then $A \otimes B \nabla C$.*

The entries of the matrix in $\mathbb{S}$ will be considered as sums or series of exponentials. So we give the definition such functions and their properties.

Definition 5
Let S_e be the set of real functions which are analytic and can be written as a sum of exponentials in a neighborhood of ∞, i.e
$S_e = \{f: A \to \mathbb{R} | A \subseteq \mathbb{R}, \exists K \in \mathbb{R}_0^+$ such that $[K,\infty) \subseteq A$ and f is analytic in $[K,\infty)$ and either $\forall x \geq K, f(x) = \sum_{i=0}^{n} \alpha_i e^{b_i x}$, $n \in \mathbb{N}$, $\alpha_i \in \mathbb{R}_0$, $b_i \in \mathbb{R}_\varepsilon$ for all i and $b_0 > b_1 > \cdots > b_n$ or $\forall x \geq K, f(x) = \sum_{i=0}^{\infty} \alpha_i e^{b_i x}$, $\alpha_i \in \mathbb{R}_0$, $b_i \in \mathbb{R}$, $b_i > b_{i+1}$, $\lim_{i\to\infty} b_i = \varepsilon$ and the series are absolutely convergent for each $x \geq K\}$.

Theorem 6
If $f \in S_e$ then $f(x) \sim \alpha_0 e^{b_0 x}$ for $x \to \infty$.

Theorem 7
1. *If f and g belong to S_e then $\rho f, f+g, fg, f^l$ and $|f|$ are also in S_e for all $\rho \in \mathbb{R}$ and $l \in \mathbb{N}$.*
2. *If there exists $K \in \mathbb{R}$ such that $f(x) \neq 0$ for each $x \geq K$ then $\frac{1}{f}$ and $\frac{g}{f}$ restricted to $[K,\infty)$ are also in S_e.*
3. *If there exist $Q \in \mathbb{R}$ such that $f(x) > 0$ for every $x \geq Q$ then $\sqrt{f}$ restricted to $[Q,\infty)$ are also in S_e.*

4. THE MOORE-PENROSE INVERSE IN SYMMETRIZED MAX-PLUS ALGEBRAIC MATRIX

This section discusses the existence of the Moore-Penrose inverse in $\mathbb{S}$ sense. The link between $\mathbb{S}$ and conventional algebra is used to derive several properties in determining the existence of the Moore-Penrose inverse on matrix over $\mathbb{S}$.

The following theorems explain the existence of the balance of matrix over $\mathbb{S}$, which is similiar to the first and second axioms of the Moore-Penrose in conventional algebra.

Theorem 8
Let A and X be matrices over $\mathbb{S}$. If there are matrices N_A, N_X whose entries are in $\mathbb{R}_0$ such that $\mathcal{F}(A,N_A,s).\mathcal{F}(X,N_X,s).\mathcal{F}(A,N_A,s) \sim \mathcal{F}(A,N_A,s)$, for $s \to \infty$, then $A \otimes X \otimes A \nabla A$.

Proof. Let a and x be entries of A and X, respectively. By using Definition 1, the exponential form of all entries in A and X as in conventional algebra are obtained. All entries in exponential form are in S_e. Therefore, according to Theorem 6 and Theorem 7, all of algebraic operation of those exponential are in S_e. Let $\mathcal{F}(A,N_A,s)$ and $\mathcal{F}(X,N_X,s)$ be real matrix-valued function for A and X, with N_A and N_X are matrices whose entries are in $\mathbb{R}_0$. Suppose there are matrices N_A, N_X whose entries are in $\mathbb{R}_0$ such that $\mathcal{F}(A,N_A,s) \,.\, \mathcal{F}(X,N_X,s) \,.\, \mathcal{F}(A,N_A,s) \sim \mathcal{F}(A,N_A,s)$, for $s \to \infty$. If the asymtotic equivalent form in part 4 of Theorem 4 is replaced by $\mathcal{F}(A,N_A,s) \,.\mathcal{F}(X,N_X,s) \,.\mathcal{F}(A,N_A,s) \sim \mathcal{F}(A,N_A,s)$, for $s \to \infty$ then it is obtained that $A \otimes X \otimes A \nabla A$. ■

Theorem 9
Let A and X be matrices over $\mathbb{S}$. If there are matrices N_A, N_X whose entries are in $\mathbb{R}_0$ such that $(X,N_X,s) \,.\, \mathcal{F}(A,N_A,s) \,.\, \mathcal{F}(X,N_X,s) \sim \mathcal{F}(X,N_X,s)$, for $s \to \infty$ then $X \otimes A \otimes X \nabla X$.

Proof. Let $\mathcal{F}(A,N_A,s)$ and $\mathcal{F}(X,N_X,s)$ be real matrix-valued function for A and X, with N_A and N_X are matrices whose entries are in $\mathbb{R}_0$. Suppose there are matrices N_X, N_A whose entries are in $\mathbb{R}_0$ such that $F(X,N_X,s) \,.\mathcal{F}(A,N_A,s) \,.\mathcal{F}(X,N_X,s) \sim \mathcal{F}(X,N_X,s)$, for $s \to \infty$. If the asymtotic equivalent form in part 4 of Theorem 4 is replaced by $F(X,N_X,s) \,.\mathcal{F}(A,N_A,s) \,.\mathcal{F}(X,N_X,s) \sim \mathcal{F}(X,N_X,s)$ for $s \to \infty$ then it is obtained that $X \otimes A \otimes X \nabla X$. ■

The following theorems explain the existence of the balance of matrix over $\mathbb{S}$, which is similiar to the third and fourth axioms of the Moore-Penrose in conventional algebra.

Theorem 10
Let A and X be matrices over $\mathbb{S}$. If there are N_A, N_X whose entries are in $\mathbb{R}_0$ such that $\left(\mathcal{F}(A,N_A,s) \,.\, \mathcal{F}(X,N_X,s)\right)^T \sim \mathcal{F}(A,N_A,s) \,.\, \mathcal{F}(X,N_X,s)$ for $s \to \infty$ then $(A \otimes X)^T \nabla A \otimes X$

Proof. Let $\mathcal{F}(A,N_A,s)$ and $\mathcal{F}(X,N_X,s)$ be real matrix-valued function for A and X, with N_A and N_X are matrices whose entries are in $\mathbb{R}_0$. Suppose there are matrices N_A, N_X whose entries are in $\mathbb{R}_0$ such that

$\left(\mathcal{F}(A,N_A,s)\,.\,\mathcal{F}(X,N_X,s)\right)^T \sim \mathcal{F}(A,N_A,s)\,.\,\mathcal{F}(X,N_X,s)$, for $s\to\infty$. If the asymtotic equivalent form in part 4 of Theorem 4 is replaced by $\left(\mathcal{F}(A,N_A,s)\,.\,\mathcal{F}(X,N_X,s)\right)^T \sim \mathcal{F}(A,N_A,s)\,.\,\mathcal{F}(X,N_X,s)$ for $s\to\infty$ then we have $(A\otimes X)^T\ \nabla\ A\otimes X$. ■

Theorem 11

Let A and X be matrices over $\mathbb{S}$. If there are N_A, N_X whose entries are in $\mathbb{R}_0$ such that

$\left(\mathcal{F}(X,N_X,s)\,.\,\mathcal{F}(A,N_A,s)\right)^T \sim \mathcal{F}(X,N_X,s)\,.\,\mathcal{F}(A,N_A,s)$

for $s\to\infty$, then $(X\otimes A)^T\ \nabla\ X\otimes A$

Proof. Let $\mathcal{F}(X,N_X,s)$ and $\mathcal{F}(A,N_A,s)$ be real matrix-valued function for X and A, with N_X and N_A are matrices whose entries are in $\mathbb{R}_0$. Suppose there are matrices N_X, N_A whose entries are in $\mathbb{R}_0$ such that $\left(\mathcal{F}(X,N_X,s)\,.\,\mathcal{F}(A,N_A,s)\right)^T \sim \mathcal{F}(X,N_X,s)\,.\,\mathcal{F}(A,N_A,s)$, for $s\to\infty$. If the asymtotic equivalent form in part 4 of Theorem 4 is replaced by $\left(\mathcal{F}(X,N_X,s)\,.\,\mathcal{F}(A,N_A,s)\right)^T \sim \mathcal{F}(X,N_X,s)\,.\,\mathcal{F}(A,N_A,s)$ for $s\to\infty$ then we have $(X\otimes A)^T\ \nabla\ X\otimes A$. ■

The following example illustrates the existence of the matrix balance form in Theorem 8 until Theorem 11.

Example 12

Let A and X be matrices over $\mathbb{S}$, respectively where $A = \begin{bmatrix} 1 & 2 & \ominus 1 \\ \ominus 2 & \ominus 3 & 2 \end{bmatrix}$ and $X = \begin{bmatrix} -5 & \ominus(-4) \\ -4 & \ominus(-3) \\ \ominus(-5) & -4 \end{bmatrix}$.

If $N_A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $N_X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $\mathcal{F}(A,N_A,s) = \begin{bmatrix} e^{s} & e^{2s} & -e^{s} \\ -e^{2s} & -e^{3s} & e^{2s} \end{bmatrix}$ and $\mathcal{F}(X,N_X,s) = \begin{bmatrix} e^{-5s} & -e^{-4s} \\ e^{-4s} & -e^{-3s} \\ -e^{-5s} & e^{-4s} \end{bmatrix}$. Therefore, there are $N_A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $N_X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ such that

a. $\mathcal{F}(A,N_A,s)\,.\,\mathcal{F}(X,N_X,s)\,.\,\mathcal{F}(A,N_A,s)$
$\sim \begin{bmatrix} e^{s} & e^{2s} & -e^{s} \\ -e^{2s} & -e^{3s} & e^{2s} \end{bmatrix} = \mathcal{F}(A,N_A,s),\ s\to\infty.$

b $\mathcal{F}(X,N_X,s)\,.\,\mathcal{F}(A,N_A,s)\,.\,\mathcal{F}(X,N_X,s)$
$\sim \begin{bmatrix} e^{-5s} & -e^{-4s} \\ e^{-4s} & -e^{-3s} \\ -e^{-5s} & e^{-4s} \end{bmatrix} = \mathcal{F}(X,N_X,s),\ s\to\infty.$

c. $\left(\mathcal{F}(A,N_A,s)\,.\,\mathcal{F}(X,N_X,s)\right)^T$
$\sim \begin{bmatrix} 2e^{-4s}+e^{-2s} & -e^{1s}-2e^{-3s} \\ -e^{-s}-2e^{-3s} & 1+2e^{-2s} \end{bmatrix}$
$= \left(\mathcal{F}(A,N_A,s)\,.\,\mathcal{F}(X,N_X,s)\right),\ s\to\infty.$

d. $\left(\mathcal{F}(X,N_X,s)\,.\,\mathcal{F}(A,N_A,s)\right)^T$
$\sim \begin{bmatrix} e^{-4s}+e^{-2s} & e^{-3s}+e^{-s} & -e^{-4s}-e^{-2s} \\ e^{-3s}+e^{-s} & e^{-2s}+1 & -e^{-3s}-e^{-s} \\ -e^{-4s}-e^{-2s} & -e^{-3s}-e^{-s} & e^{-4s}+e^{-2s} \end{bmatrix}$
$= \mathcal{F}(X,N_X,s)\,.\,\mathcal{F}(A,N_A,s),\ s\to\infty.$

Since A and X satisfy a, b, c and d, then it is obtained that

1. $A\otimes X\otimes A\ \nabla\ A$
2. $X\otimes A\otimes X\ \nabla\ X$
3. $(A\otimes X)^T\ \nabla\ A\otimes X$
4. $(X\otimes A)^T\ \nabla\ X\otimes A$. ■

The discussion in Theorem 8 until Theorem 11 were used to show the existence of the matrix balances in order to define the Moore-Penrose inverse in $\mathbb{S}$.

The following definition explains the Moore-Penrose inverse in the matrix over $\mathbb{S}$. This definition is defined by modifying the definition of the Moore-Penrose inverse in a conventional matrix, i.e by changing "equals" relation in conventional algebra to "balance" relation in $\mathbb{S}$.

Definition 13

Let $M\in\mathbb{S}^{m\times n}$. The Moore-Penrose inverse of M is an $n\times m$ matrix M^+ which satisfies

1. $M\otimes M^+\otimes M\nabla M$
2. $M^+\otimes M\otimes M^+\nabla M^+$
3. $(M\otimes M^+)^T\nabla M\otimes M^+$
4. $(M^+\otimes M)^T\nabla M^+\otimes M$

Since $X = \begin{bmatrix} -5 & \ominus(-4) \\ -4 & \ominus(-3) \\ \ominus(-5) & -4 \end{bmatrix}$ in Example 12 fulfills the axioms in Definiton 13, then X is the Moore-Penrose inverse of $A = \begin{bmatrix} 1 & 2 & \ominus 1 \\ \ominus 2 & \ominus 3 & 2 \end{bmatrix}$.

The discussion of symmetrization of $\mathbb{R}_{\max}$ shows that $\mathbb{R}_{\max}$ is the positive part or zero of $\mathbb{S}$. Therefore, $\mathbb{R}_{\max}$ is a special case of the symmetrized max-plus algebra, i.e $\mathbb{R}_{\max}$ is a special case of $\mathbb{S}$. According to Theorem 8 until Theorem 11, the existence of the Moore-Penrose inverse in max-plus algebraic matrix is guaranteed.

Based on the properties of the weak balances, the weak transitive and the reduction of balance in symmetrized max-plus algebra, Definition 13 can be changed into the following definition.

Definition 14

Let $M\in(\mathbb{S}^{\vee})^{m\times n}$. The Moore-Penrose inverse of M is an $n\times m$ matrix $M^+\in(\mathbb{S}^{\vee})^{n\times m}$ which satisfies

1. $M\otimes M^+\otimes M = M$
2. $M^+\otimes M\otimes M^+ = M^+$
3. $(M\otimes M^+)^T = M\otimes M^+$
4. $(M^+\otimes M)^T = M^+\otimes M$

According to Definition 14, we can define the Moore-Penrose in max-plus algebra sense.

Definition 15

Let $A\in(\mathbb{R}_{\max})^{m\times n}$. The Moore-Penrose inverse of M is an $n\times m$ matrix $A^+\in(\mathbb{R}_{\max})^{n\times m}$ which satisfies

1. $A\otimes A^+\otimes A = A$
2. $A^+\otimes A\otimes A^+ = A^+$
3. $(A\otimes A^+)^T = A\otimes A^+$
4. $(A^+\otimes A)^T = A^+\otimes A$

The balanced inverse of a square symmetrized max-plus algebraic matrix plays a similiar role as an inverse in coventional matrix.

Definition 16 (Balanced Inverse)
Let $A \in \mathbb{S}^{n\times n}$. If there is $B \in \mathbb{S}^{n\times n}$ such that $A \otimes B \nabla I_n$ and $B \otimes A \nabla I_n$ then A is said to be balanced invertible and B is a balanced inverse of A. Furthermore, the balanced inverse of A is denoted by A_{∇}^{-1}.

The balanced inverse of a square matrix in $\mathbb{S}$ can be solved using Definition 1, Definition 2 and the properties of the link between $\mathbb{S}$ with conventional algebra. The following example explains the balanced inverse of square matrix over $\mathbb{S}$.

Example 17

Let $A = \begin{bmatrix} 1 & 1^\bullet \\ 2^\bullet & 4 \end{bmatrix}$ where $\det(A) = 5 \nabla \mathcal{E}$. The real matrix-valued function which corresponds to A by the function in (7) is $\tilde{A}(s) = \begin{bmatrix} e^s & e^s \\ e^{2s} & e^{4s} \end{bmatrix}$. Therefore $\det(\tilde{A}(s)) = e^{5s} - e^{3s}$ and $\operatorname{cof}(\tilde{A}(s))^T = \begin{bmatrix} e^{4s} & -e^s \\ -e^{2s} & e^s \end{bmatrix}$ for $s \in \mathbb{R}_0^+$. Since $\frac{\operatorname{cof}(\tilde{A}(s))^T}{\det(\tilde{A}(s))} = \begin{bmatrix} \frac{e^{4s}}{e^{5s}-e^{3s}} & \frac{-e^s}{e^{5s}-e^{3s}} \\ \frac{-e^{2s}}{e^{5s}-e^{3s}} & \frac{e^s}{e^{5s}-e^{3s}} \end{bmatrix}$ then

$$\tilde{A}(s)\,\frac{\operatorname{cof}(\tilde{A}(s))^T}{\det(\tilde{A}(s))} = \begin{bmatrix} \frac{e^{5s}-e^{3s}}{e^{5s}-e^{3s}} & \frac{-e^{2s}+e^{2s}}{e^{5s}-e^{3s}} \\ \frac{e^{6s}-e^{6s}}{e^{5s}-e^{3s}} & \frac{-e^{3s}+e^{5s}}{e^{5s}-e^{3s}} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \tilde{I}_2 \text{ and}$$

$$\frac{\operatorname{cof}(\tilde{A}(s))^T}{\det(\tilde{A}(s))}\,\tilde{A}(s) = \begin{bmatrix} \frac{e^{5s}-e^{3s}}{e^{5s}-e^{3s}} & \frac{e^{5s}-e^{5s}}{e^{5s}-e^{3s}} \\ \frac{-e^{3s}+e^{3s}}{e^{5s}-e^{3s}} & \frac{-e^{3s}+e^{5s}}{e^{5s}-e^{3s}} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \tilde{I}_2 \text{ for}$$

$s \to \infty$.

Consequently, $\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \otimes \begin{bmatrix} -1 & \ominus(-4) \\ \ominus(-3) & -4 \end{bmatrix} \nabla \begin{bmatrix} 0 & \mathcal{E} \\ \mathcal{E} & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & \ominus(-4) \\ \ominus(-3) & -4 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \nabla \begin{bmatrix} 0 & \mathcal{E} \\ \mathcal{E} & 0 \end{bmatrix}$. are obtained. Since $\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \nabla \begin{bmatrix} 1 & 1^\bullet \\ 2^\bullet & 4 \end{bmatrix}$ where $\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$ is a signed matrix in symmetrized max-plus algebra, then $\begin{bmatrix} 1 & 1^\bullet \\ 2^\bullet & 4 \end{bmatrix} \otimes \begin{bmatrix} -1 & \ominus(-4) \\ \ominus(-3) & -4 \end{bmatrix} \nabla \begin{bmatrix} 0 & \mathcal{E} \\ \mathcal{E} & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & \ominus(-4) \\ \ominus(-3) & -4 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1^\bullet \\ 2^\bullet & 4 \end{bmatrix} \nabla \begin{bmatrix} 0 & \mathcal{E} \\ \mathcal{E} & 0 \end{bmatrix}$. Therefore, a balanced inverse $A_{\nabla}^{-1} = \begin{bmatrix} -1 & \ominus(-4) \\ \ominus(-3) & -4 \end{bmatrix}$ such that $A \otimes A_{\nabla}^{-1} \nabla I_2$ and $A_{\nabla}^{-1} \otimes A \nabla I_2$ is obtained ■

The following theorem explains that the balanced inverse of A is also the Moore-Penrose inverse of A.

Theorem 18
A balanced inverse of $A \in \mathbb{S}^{n\times n}$ is the Moore-Penrose of A.
Proof. Let A_{∇}^{-1} be a balanced inverse of A. According to Definition 16, it satisfies

$$A \otimes A_{\nabla}^{-1} \nabla I_n \text{ and } A_{\nabla}^{-1} \otimes A \nabla I_n.$$

Consequently, it also satisfies
1. $A \otimes A_{\nabla}^{-1} \otimes A \nabla I_n \otimes A = A$
2. $A_{\nabla}^{-1} \otimes A \otimes A_{\nabla}^{-1} \nabla I_n \otimes A_{\nabla}^{-1} = A_{\nabla}^{-1}$
3. $(A \otimes A_{\nabla}^{-1})^T \nabla (I_n)^T = I_n \nabla A \otimes A_{\nabla}^{-1}$
4. $(A_{\nabla}^{-1} \otimes A)^T \nabla (I_n)^T = I_n \nabla A_{\nabla}^{-1} \otimes A$.

Therefore, A_{∇}^{-1} is the Moore-Penrose inverse of A. ■

According to Example 17, the balanced inverse matrix $A_{\nabla}^{-1} = \begin{bmatrix} -1 & \ominus(-4) \\ \ominus(-3) & -4 \end{bmatrix}$ is the Moore-Penrose inverse of $A = \begin{bmatrix} 1 & 1^\bullet \\ 2^\bullet & 4 \end{bmatrix}$.

Corollary 19
An inverse matrix of $A \in (\mathbb{R}_{\max})^{n\times n}$ is the Moore-Penrose of A.
Proof. Let $A^{\otimes -1}$ be an inverse of A in max-plus algebra sense. It satisfies

$$A \otimes A^{\otimes -1} = I_n \text{ and } A^{\otimes -1} \otimes A = I_n.$$

Consequenly, it also satisfies
1. $A \otimes A^{\otimes -1} \otimes A = I_n \otimes A = A$
2. $A^{\otimes -1} \otimes A \otimes A^{\otimes -1} \nabla I_n \otimes A^{\otimes -1} = A^{\otimes -1}$
3. $\left(A \otimes A^{\otimes -1}\right)^T \nabla (I_n)^T = I_n \nabla A \otimes A^{\otimes -1}$
4. $\left(A^{\otimes -1} \otimes A\right)^T \nabla (I_n)^T = I_n \nabla A^{\otimes -1} \otimes A$.

Therefore, $A^{\otimes -1}$ is the Moore-Penrose inverse of A. ■

5. CONCLUSION

The existence of the Moore-Penrose inverse in symmetrized max-plus algebra can be determined using a link between symmetrized max-plus algebra and conventional algebra. The Moore-Penrose inverse in symmetrized max-plus algebra can be defined as that in conventional algebra by replacing "equal" relation by "balance" relation. The balanced inverse is the Moore-Penrose inverse in symmetrized max-plus algebra. The future research potentially can be done in construction Moore-Penrose inverse using matrix decomposition in symmetrized max-plus algebra.

AUTHORS' CONTRIBUTIONS

S is a researcher whose research object is symmetrized max-plus algebra and the main researcher in this study. NI and R contributed to drafting and editing the manuscript.

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