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A Short Note on the Complex Conjugate for Derivatives

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ABSTRACT
The complex conjugate approach could be used easily to solve derivatives analytically for some simple cases in calculus. These cases are common topics in calculus which are functions of trigonometry, hyperbole, exponential and logarithm. In general, the derivative obtained from the component v is rearranged from the result of the Taylor series expansion of the complex conjugate argument function ξ^*. The result of the Taylor series expansion gives the form u-iv where v is the imaginary component. The final result of the derivative using this approach always raises a coefficient of -1/α. Parameter α is the interval from which the function ξ is approximated to the function point or the position of the approximated point. If α tends to 1 or $\alpha\rightarrow 1$, then the derivative result will be the same as the analytical completion. In addition, if it is observed from all cases that have been completed, the component u resulting from the Taylor series expansion is the original function or the function whose derivative is sought.

Keywords: *Complex Conjugate, Taylor Series, Derivatives, Short Note.*

1. INTRODUCTION

In completing derivatives in mathematics many ways have been developed. Solving it becomes interesting when discussing numerical or computational solutions. Currently widely used is the finite-difference method for estimating derivatives both the forward-difference, backward-difference and central-difference method. It's just that this method in its use allows the subtraction of two numbers that are almost the same for very small steps α or when α tends to zero or $\alpha\rightarrow 0$, so that in the computational process can result in the division of zero by zero or NaN. Further complex variable approach has been developed for estimating derivatives at this time. Generally, this approach is developed from the results of the Taylor series expansion for functions that have real, continuous and analytic values. Some researchers who have examined this approach can be found in [1-5].

In this paper, analytical derivative completion will be presented by utilizing the complex conjugate approach. Complex conjugate is basically negative form of complex variable in its imaginary component. In this approach, the final result is an equation which, besides showing the estimation results of the derivatives, also shows an estimate of the original functions. The approach for derivative estimation can be found in papers published by [6] for first-order and in [7] for second-order. The derivatives solved in this paper only focus on the first-order for a few simple cases, and are common topics in calculus such as trigonometric, hyperbolic, exponential and logarithmic functions.

Regarding this approach, in the two papers mentioned, the authors have shown the accuracy of this approach by comparing it with manual completion. They tested it using the total magnetic field anomaly function, and obtained Relative Errors ranging from $3.159607284251312x10^{-17}$ to $7.079603533496899x10^{-12}$ for first-order and 0.088271927039904 to $6.679993448987438x10^{-07}$ for second-order. In the first-order, the uniqueness is that the choice of step α, similar to the ordinary finite-difference method, it can be carried out arbitrarily, does not require complicated combinations of parameters involved or requires special treatment, especially for very small α.

2. COMPLEX CONJUGATE APPROACH

Basically, a complex conjugate is a complex variable that has a negative value in its imaginary component. The commonly known form of complex conjugate is expressed as $z^*=x\text{-}i\alpha$ with the notation i has a value of $\sqrt{-1}$ and represents an imaginary component. If the function ξ is expanded in a Taylor series and has a complex conjugate argument, then the equation for estimating the derivatives is based on [6].

$$\xi'(x) = -\frac{1}{\alpha}\,\mathrm{Im}\,\xi^* + S_T \quad (1)$$

Whereas the approximation for the original function is

$$\xi(x) = \mathrm{Re}\,\xi^* + S_T \quad (2)$$

where ξ^* is the complex conjugate argument function or $\xi(z^*)$. The notation α in the above equation expresses the step or interval between the x-position and the position of the approximated point, and S_T for the cumulative magnitude of the terms removed when truncating the Taylor series. Equations (1) and (2) are analogous to those described by [3] but are derived from complex variables.

3. SOLVING DERIVATIVES

In general, the function ξ^* resulting from the Taylor series expansion in the complex conjugate argument is stated by

$$\xi^* = u - iv \quad (3)$$

where u denotes the real component and v is the imaginary one. In this approach, h^2-term and higher terms are ignored for first-order derivative. From [6] it is known that the derivative is described from the imaginary component v as obtained as in equation (1), and the real one represents an approximation to the original function as in equation (2).

3.1. Trigonometric Function

The analytical derivatives of trigonometry for the functions of cosine, sine, tangent, secant, cosecant and cotangent are well known. The completions can be found in many sources or literatures, especially in calculus books (e.g. in [8], [9] and [10]). For the complex conjugate approach, the completions of the trigonometric derivatives ξ' are presented in Table 1. The functions $\xi(x)$ completed include cos x, sin x, tan x, sec x, csc x and cot x.

Table 1. Analytical completion of derivative for trigonometric functions

Function $\xi(x)$	Complex Conjugate		
	ξ^*	ξ	ξ'
$\cos x$	$\cos(x - i\alpha) = \cos x + i\alpha \sin x$	$\mathrm{Re}\,\xi^* = \cos x$	$-\frac{1}{\alpha}\mathrm{Im}\,\xi^*$
$\sin x$	$\sin(x - i\alpha) = \sin x - i\alpha \cos x$	$\mathrm{Re}\,\xi^* = \sin x$	$-\frac{1}{\alpha}\mathrm{Im}\,\xi^*$
$\tan x$	$\tan(x - i\alpha) = \frac{\sin(x - i\alpha)}{\cos(x - i\alpha)}$	$\mathrm{Re}\,\xi^* = \tan x$	$-\frac{1}{\alpha}\mathrm{Im}\,\xi^*$
	$\tan(x - i\alpha) = \frac{(1-\alpha^2)\sin x \cos x - i\alpha}{\cos^2 x + \alpha^2 \sin^2 x}$		
	$\tan(x - i\alpha) = \frac{(1-\alpha^2)\tan x - i\alpha \sec^2 x}{1 + \alpha^2 \tan^2 x}$		
	for $\alpha \to 0$		
	$\tan(x - i\alpha) = \tan x - i\alpha \sec^2 x$		
$\sec x$	$\sec(x - i\alpha) = \frac{1}{\cos(x - i\alpha)}$	$\mathrm{Re}\,\xi^* = \sec x$	$-\frac{1}{\alpha}\mathrm{Im}\,\xi^*$
	$\sec(x - i\alpha) = \frac{\cos x + i\alpha \sin x}{\cos^2 x + \alpha^2 \sin^2 x}$		
	$\sec(x - i\alpha) = \frac{\sec x - i\alpha \tan x \sec x}{1 + \alpha^2 \tan^2 x}$		
	for $\alpha \to 0$		
	$\sec(x - i\alpha) = \sec x - i\alpha \tan x \sec x$		

$\csc x$	$\csc(x-i\alpha)=\dfrac{1}{\sin(x-i\alpha)}$	$\operatorname{Re}\xi^*=\csc x$	$-\dfrac{1}{\alpha}\operatorname{Im}\xi^*$
	$\csc(x-i\alpha)=\dfrac{\sin x+i\alpha\cos x}{\sin^2 x+\alpha^2\cos^2 x}$		
	$\csc(x-i\alpha)=\dfrac{\csc x+i\alpha\cot x\csc x}{1+\alpha^2\cot^2 x}$		
	for $\alpha\to 0$		
	$\csc(x-i\alpha)=\csc x+i\alpha\cot x\csc x$		
$\cot x$	$\cot(x-i\alpha)=\dfrac{\cos(x-i\alpha)}{\sin(x-i\alpha)}$	$\operatorname{Re}\xi^*=\cot x$	$-\dfrac{1}{\alpha}\operatorname{Im}\xi^*$
	$\cot(x-i\alpha)=\dfrac{(1-\alpha^2)\sin x\cos x+i\alpha}{\sin^2 x+\alpha^2\cos^2 x}$		
	$\cot(x-i\alpha)=\dfrac{(1-\alpha^2)\cot x+i\alpha\csc^2 x}{1+\alpha^2\cot^2 x}$		
	for $\alpha\to 0$		
	$\cot(x-i\alpha)=\cot x+i\alpha\csc^2 x$		

3.2. Hyperbolic Function

Completing derivatives of hyperbolic functions is identical to the trigonometric completions in Section 3.1. The functions $\xi(x)$ that are completed include cosh *x*, sinh *x*, tanh *x*, sech *x*, csch *x* and coth *x*, and the results of their decomposition ξ' can be seen in Table 2.

Table 2. Derivative completion for hyperbolic functions

Function $\xi(x)$	**Complex Conjugate**		
	ξ^*	ξ	ξ'
$\cosh x$	$\cosh(x-i\alpha)=\cosh x-i\alpha\sinh x$	$\operatorname{Re}\xi^*=\cosh x$	$-\dfrac{1}{\alpha}\operatorname{Im}\xi^*$
$\sinh x$	$\sinh(x-i\alpha)=\sinh x-i\alpha\cosh x$	$\operatorname{Re}\xi^*=\sinh x$	$-\dfrac{1}{\alpha}\operatorname{Im}\xi^*$
$\tanh x$	$\tanh(x-i\alpha)=\dfrac{\sinh(x-i\alpha)}{\cosh(x-i\alpha)}$	$\operatorname{Re}\xi^*=\tanh x$	$-\dfrac{1}{\alpha}\operatorname{Im}\xi^*$
	$\tanh(x-i\alpha)=\dfrac{(1+\alpha^2)\sinh x\cosh x-i\alpha}{\cosh^2 x+\alpha^2\sinh^2 x}$		
	$\tanh(x-i\alpha)=\dfrac{(1+\alpha^2)\tanh x-i\alpha\operatorname{sech}^2 x}{1+\alpha^2\tanh^2 x}$		
	for $\alpha\to 0$		
	$\tanh(x-i\alpha)=\tanh x-i\alpha\operatorname{sech}^2 x$		
$\operatorname{sech} x$	$\operatorname{sech}(x-i\alpha)=\dfrac{1}{\cosh(x-i\alpha)}$		
	$\operatorname{sech}(x-i\alpha)=\dfrac{\cosh x+i\alpha\sinh x}{\cosh^2 x+\alpha^2\sinh^2 x}$		

	$\mathrm{sech}(x-i\alpha)=\dfrac{\mathrm{sech}x+i\alpha\tanh x\,\mathrm{sech}x}{1+\alpha^2\tanh^2 x}$	$\mathrm{Re}\,\xi^*=\mathrm{sech}x$	$-\dfrac{1}{\alpha}\mathrm{Im}\,\xi^*$
	for $\alpha\to 0$		
	$\mathrm{sech}(x-i\alpha)=\mathrm{sech}x+i\alpha\tanh x\,\mathrm{sech}x$		
csch x	$\mathrm{csch}(x-i\alpha)=\dfrac{1}{\sinh(x-i\alpha)}$	$\mathrm{Re}\,\xi^*=\mathrm{csch}x$	$-\dfrac{1}{\alpha}\mathrm{Im}\,\xi^*$
	$\mathrm{csch}(x-i\alpha)=\dfrac{\sinh x+i\alpha\cosh x}{\sinh^2 x+\alpha^2\cosh^2 x}$		
	$\mathrm{csch}(x-i\alpha)=\dfrac{\mathrm{csch}x+i\alpha\coth x\,\mathrm{csch}x}{1+\alpha^2\coth^2 x}$		
	for $\alpha\to 0$		
	$\mathrm{csch}(x-i\alpha)=\mathrm{csch}x+i\alpha\coth x\,\mathrm{csch}x$		
coth x	$\coth(x-i\alpha)=\dfrac{\cosh(x-i\alpha)}{\sinh(x-i\alpha)}$	$\mathrm{Re}\,\xi^*=\coth x$	$-\dfrac{1}{\alpha}\mathrm{Im}\,\xi^*$
	$\coth(x-i\alpha)=\dfrac{(1+\alpha^2)\sinh x\cosh x+i\alpha}{\sinh^2 x+\alpha^2\cosh^2 x}$		
	$\coth(x-i\alpha)=\dfrac{(1+\alpha^2)\coth x+i\alpha\mathrm{csch}^2 x}{1+\alpha^2\coth^2 x}$		
	for $\alpha\to 0$		
	$\coth(x-i\alpha)=\coth x+i\alpha\mathrm{csch}^2 x$		

3.3. Exponential Function

The exponential function completed here is $\xi(x)=e^{kx}$. Analytically, the derivative of the function $\xi(x)$ with respect to x is $\xi'=ke^{kx}$. How to estimate the derivative analytically using the complex conjugate approach?

In complex conjugate argument, the function $\xi(x)$ can be written

$$\xi^*=e^{kz^*}=e^{k(x-i\alpha)} \tag{4}$$

If the above equation is parsed, then it will become

$$\xi^*=e^{kx}e^{-i\alpha} \tag{5}$$

$$\xi^*=e^{kx}\left[\cos(k\alpha)-i\sin(k\alpha)\right] \tag{6}$$

For α tends to zero or $\alpha\to 0$, then equation (6) changes to

$$\xi^*=e^{kx}\left[1-ik\alpha\right] \tag{7}$$

$$\xi^*=e^{kx}-i\alpha ke^{kx} \tag{8}$$

If seen, the real component of ξ^* or the component u contains the original function whose derivative is sought. While the derivative of $\xi(x)$ with respect to x can be completed from its imaginary component or component v. So that it can be written down

$$\xi=\mathrm{Re}\,\xi^* \tag{9}$$

and

$$\xi'=-\frac{1}{\alpha}\mathrm{Im}\,\xi^* \tag{10}$$

3.4. Logarithmic Function

An example of the logarithmic function completed here is $\xi(x)=\log_a x$ where its derivative with respect to x is $(1/x)\ln a$. In complex conjugate argument, the function $\xi(x)$ can be expressed as

$$\xi^*=\log_a z^*=\frac{\ln z^*}{\ln a}=\frac{\ln x+\ln\left(1-\dfrac{i\alpha}{x}\right)}{\ln a} \tag{11}$$

If the $\ln\left(1-\dfrac{i\alpha}{x}\right)$-term is expanded in a series, equation (11) will change to

$$\xi^*=\frac{\ln x+\left\{-\dfrac{i\alpha}{x}-\dfrac{\alpha^2}{2x^2}+\ldots\right\}}{\ln a} \tag{12}$$

For α tends to zero or $\alpha \to 0$, then

$$\xi^* = \frac{\ln x}{\ln a} - i\frac{\alpha}{x \ln a} \qquad (13)$$

or

$$\xi^* = \log_a x - i\frac{\alpha}{x \ln a} \qquad (14)$$

It appears that from the above equation that

$$\xi = \operatorname{Re} \xi^* \qquad (15)$$

and

$$\xi' = -\frac{1}{\alpha} \operatorname{Im} \xi^* \qquad (16)$$

An interesting thing that can be observed from the examples above is that the completion of the derivative ξ' satisfies equation (1) for all functions expanded in the complex conjugate argument ξ^* which is generally expressed by equation (3). So, the result of expanding the function ξ in the complex conjugate argument always results in a pair of the original function ξ and its derivative ξ'.

4. CONCLUSION

For some simple cases in calculus, the derivatives can be easily completed analytically using the complex conjugate approach. In this paper, the simple cases referred to are part of the trigonometric, hyperbolic, exponential and logarithmic functions which are basically general topics in calculus. The function ξ in the complex conjugate arguments ξ^* when expanded in a Taylor series will give the general form *u*-i*v*. The derivative is obtained by rearranging the imaginary component or component *v*. The final results of completing these cases always give rise to a coefficient of $1/\alpha$. If α is arranged in such a way that it is close to 1 or $\alpha \to 1$, then the complex conjugate completion will be the same as the analytical completion. This parameter α is similar to parameter α in the well-known finite-difference method, namely the step or interval between the point used to approximate the function and its function point. In addition, it is also known that the real component or component *u* resulting from the expansion of the series is the original function of the sought derivative. So the result of a Taylor series expansion rearranged is basically a combination of the original function and the result of its derivative.

The final form of the equation of the complex conjugate approach is very simple and easy to implement in programming languages by just making simple codes. This can be seen in the previous studies using this approach. As can be seen, this paper only tries to present a way to solve derivatives analytically using the approach. Examples of completing derivatives analytically both for the approach and for the complex variable approach in general are still very rarely given. Readers can see several examples of completing derivatives analytically using this approach.

AUTHORS' CONTRIBUTION

This manuscript was performed as the single authors, AM. Contribution covers the whole the whole CONCEPT, METHOD, and ANALYSIS or the whole CONTENTS of the manuscript.

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