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The Properties of the R^n Module over the Matrix Ring $M_{n\times n}(R)$

Triyani Triyani[1,*], Ari Wardayani[1], Alfalfa Amruhasanah[1]

[1] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Jenderal Soedirman, Jl. dr. Suparno, 61, Grendeng, Purwokerto Utara, 53122, Jawa Tengah, Indonesia.*

[*] *Corresponding author. Email: triyani@unsoed.ac.id*

ABSTRACT

This paper discusses the properties of the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$ related to the torsion module, prime module, multiplication module, and faithful module. The study results concluded that the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$ is a torsion module because each element of $\boldsymbol{R^n}$ is a torsion element. The $\boldsymbol{R^n}$ module is also a prime module because the zero element of $\boldsymbol{R^n}$ is a prime submodule. Moreover, the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$ is also a multiplication module because there exists an ideal presentation $I = \boldsymbol{M_{n\times n}(U)}$ where U is ideal of ring R. However, the $\boldsymbol{R^n}$ module is not a faithful module because the annihilator of $\boldsymbol{R^n}$ does not contain only zero element of matrix ring $\boldsymbol{M_{n\times n}(R)}$.

Keywords: *Module, Torsion module, Prime module, Multiplication module, Faithdul module..*

1. INTRODUCTION

A set is a well-defined collection of objects. A non-empty set with one or more binary operations is called an algebraic structure. A group is one of the algebraic structures formed from a non-empty set with one binary operation. Based on [1], a group G is an Abelian group if its binary operation is commutative. The algebraic structure formed from a non-empty set with two binary operations, namely the 'addition' and 'multiplication', is called a ring. A ring R in which the multiplication is commutative is called a commutative ring. Moreover, a ring R with a multiplicative identity element is called a ring with unity.

A non-empty set M is called a module over a ring with unity; if the set M with a binary operation "addition" is Abelian group, there exists a scalar multiplication operation with the ring, and satisfy the module axioms. M is called a torsion module if every element in M is a torsion element. A module M is called a prime module if 0_M is a prime submodule of M. Based on [2], a module M over ring R is called a multiplication module if, for every submodule N in M, there exists an ideal presentation of I in R such that $N = IM$ applies. A module M is called a faithful module if the annihilator of M contains only zero element.

Let R be a commutative ring with unity. The set of matrices of order $n \times n$ whose entries are elements of R, denoted by $\boldsymbol{M_{n\times n}(R)}$, with matrix addition and multiplication is a ring with unity. This concept has been discussed in detail in [3]. Furthermore, [4] also discussed that the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$ is not a free module. Based on these studies, the authors are interested in investigating whether the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$ is a torsion module, a prime module, a multiplication module, or a faithful module. Interest in studying the $\boldsymbol{R^n}$ module is because the results obtained can be used as a basis for studying other algebraic structures, such as quotient module or module homomorphism.

2. METHODOLOGY

The research method used is the study of literature and journals. The steps taken to investigate the properties of the $\boldsymbol{R^n}$module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$ are as follows:

i. Define $\boldsymbol{R^n}$ module over matrix ring $\boldsymbol{M_{n\times n}(R)}$;
ii. Investigate whether the $\boldsymbol{R^n}$module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$ is a torsion module;
iii. Investigate whether the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$is a multiplication module;

iv. Investigate whether the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$ is a prime module;
v. Investigate whether the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$ is a faithful module.

3. RESULTS AND DISCUSSION

This section discusses the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$ whether it satisfies the properties of the torsion module, prime module, multiplication module, or faithful module. Nevertheless, first, let us define the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$.

3.1. The Definition of the R^n Module over the Matrix Ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$

Let R be a commutative ring with unity. The zero element and the unity in R are 0_R and 1_R, respectively. The definition of an n-space whose n-tuples are elements of R is as follows [4].

Definition 3.1 *Let R be a commutative ring with unity. The set of matrices of order $n\times 1$ is called the n-space over ring R if the entries in the matrix are elements of R and are denoted by $\boldsymbol{R^n}$.*

The form of $\boldsymbol{R^n}$ is

$$\boldsymbol{R^n}=\left\{\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}\middle| a_1,a_2,\dots,a_n\in R\right\}.$$

The addition operation in $\boldsymbol{R^n}$ is defined as

$$\boldsymbol{x}\oplus\mathbf{y}=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\oplus\begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}=\begin{pmatrix}x_1+y_1\\x_2+y_2\\\vdots\\x_n+y_n\end{pmatrix},$$

for each $\boldsymbol{x}=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix},\mathbf{y}=\begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}\in\boldsymbol{R^n}$. According to [4], the algebraic structure of $(\boldsymbol{R^n},\oplus)$ is an Abelian group.

Based on [3], the set of matrices of order $n\times n$ over ring R with matrix addition and multiplication operations, denoted by $(\boldsymbol{M_{n\times n}}(\boldsymbol{R}),+,\times)$, is a ring with unity with the definition of

$$\boldsymbol{M_{n\times n}}(\boldsymbol{R})=\{[a_{ij}]|a_{ij}\in R,\forall i,j=1,2,\dots,n\}.$$

Furthermore, [4] states that the $\boldsymbol{R^n}$ is a module over the matrix ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$ with the definition of the scalar multiplication operation is as follows.

For each $\boldsymbol{A}=[a_{ij}]\in\boldsymbol{M_{n\times n}}(\boldsymbol{R})$ and $\boldsymbol{x}=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\in\boldsymbol{R^n}$,

$$\boldsymbol{A}\bullet\boldsymbol{x}=\boldsymbol{Ax}$$

$$=[a_{ij}]\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$$

$$=\begin{pmatrix}\sum_{i=1}^n a_{1i}x_i\\\sum_{i=1}^n a_{2i}x_i\\\vdots\\\sum_{i=1}^n a_{ni}x_i\end{pmatrix}\in\boldsymbol{R^n}.$$

3.2. The Properties of the R^n Module over the Matrix Ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$

This section will discuss the related properties of the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$ namely torsion module, prime module, multiplication module, and faithful module. These properties will be stated in the theorem as follows.

Theorem 3.1 *The $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$ is a torsion module.*

Proof. Suppose the $\boldsymbol{R^n}$ is a module over the matrix ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$. In order to prove that the $\boldsymbol{R^n}$ is a torsion module over the matrix ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$, it must be shown that every element of the $\boldsymbol{R^n}$ is a torsion element. In this case, there are two cases to show that the $\boldsymbol{R^n}$ module is a torsion module.

Case 1. For $\boldsymbol{t}=\begin{pmatrix}0_R\\0_R\\\vdots\\0_R\end{pmatrix}\in\boldsymbol{R^n}$, we can always find any non-zero element $\boldsymbol{A}=[a_{ij}]\in\boldsymbol{M_{n\times n}}(\boldsymbol{R})$ so that

$$\boldsymbol{A}\bullet\boldsymbol{t}=[a_{ij}]\begin{pmatrix}0_R\\0_R\\\vdots\\0_R\end{pmatrix}=\begin{pmatrix}0_R\\0_R\\\vdots\\0_R\end{pmatrix}.$$

Thus, $\boldsymbol{t}=\begin{pmatrix}0_R\\0_R\\\vdots\\0_R\end{pmatrix}\in\boldsymbol{R^n}$ is the torsion element of the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}}(\boldsymbol{R})$.

Case 2. For every $\boldsymbol{x}=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\in\boldsymbol{R^n}$ where $\boldsymbol{x}\neq\boldsymbol{t}$, we can always find a non-zero element $\boldsymbol{B}\in\boldsymbol{M_{n\times n}}(\boldsymbol{R})$ i.e. $\boldsymbol{B}=[b_{ij}]$, where $b_{11}=x_2$, $b_{12}=-x_1$, and else is 0_R, so that

$$\boldsymbol{B}\bullet\boldsymbol{x}=[b_{ij}]\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$$

$$=\begin{pmatrix}x_2x_1+(-x_1x_2)+0_R+\cdots+0_R\\0_R+0_R+0_R+\cdots+0_R\\\vdots\\0_R+0_R+0_R+\cdots+0_R\end{pmatrix}$$

$$= \begin{pmatrix} x_2x_1 + (-x_1x_2) \\ 0_R \\ \vdots \\ 0_R \end{pmatrix}$$

$$= \begin{pmatrix} 0_R \\ 0_R \\ \vdots \\ 0_R \end{pmatrix}$$

Thus, for any $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \boldsymbol{R^n}$, with $\boldsymbol{x} \neq \boldsymbol{t}$, is the torsion element of the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$. Because every element of the $\boldsymbol{R^n}$ is a torsion element, then the $\boldsymbol{R^n}$ is a torsion module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$. ■

Theorem 3.2 *The module $\boldsymbol{R^n}$ over matrix ring $\boldsymbol{M_{n\times n}(R)}$ is a prime module.*

Proof. To prove that the $\boldsymbol{R^n}$ module is a prime module, based on [5], it is sufficient to prove that $\{\boldsymbol{0_{R^n}}\}$ is a prime submodule. The set $\{\boldsymbol{0_{R^n}}\}$ is a trivial submodule of $\boldsymbol{R^n}$. The next step is to prove that $\{\boldsymbol{0_{R^n}}\}$ is a prime submodule. Note that since the $\boldsymbol{R^n}$ is a module over matrix ring $\boldsymbol{M_{n\times n}(R)}$, there exists a zero element$\boldsymbol{0_{R^n}}$, so that for each$\boldsymbol{A} \in \boldsymbol{M_{n\times n}(R)}$ satisfies $\boldsymbol{A} \bullet \boldsymbol{0_{R^n}} = \boldsymbol{0_{R^n}}$. Based on the prime submodule definition on [5], this indicates that $\{\boldsymbol{0_{R^n}}\}$ is a prime submodule ofthe$\boldsymbol{R^n}$. Thus, it is proved that the $\boldsymbol{R^n}$ is a prime module over the matrix ring$\boldsymbol{M_{n\times n}(R)}$. ■

Before we present a theorem about the multiplication property of the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$, we present a lemma about the ideal of matrix over R [6].

Lemma 3.1 *If U is the ideal of the ring R, then matrix over U is the ideal of matrix over R.*

Proof. Let R is a commutative ring with unity 1_R. If U is ideal of ring R, then according to [6], for every $x, y \in U$, and $r \in R$, it satisfies:

i. $x - y \in U$;

ii. $xr \in U$ and $rx \in U$.

Next, matrix over U, denoted by $\boldsymbol{M_{n\times n}(U)}$, is a set of matrices of order $n \times n$ whose entries are elements in U. Note that for each

$\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{M_{n\times n}(U)}$, and$\boldsymbol{A} \in \boldsymbol{M_{n\times n}(R)}$, where $\boldsymbol{X} = [x_{ij}], \boldsymbol{Y} = [y_{ij}]$, and $\boldsymbol{A} = [a_{ij}]$, satisfy:

i. $\boldsymbol{X} - \boldsymbol{Y} = [x_{ij}] - [y_{ij}] = [x_{ij} - y_{ij}]$
Because $x_{ij}, y_{ij} \in U$ and U is ideal of ring R, then $x_{ij} - y_{ij} \in U$. Therefore,
$$\boldsymbol{X} - \boldsymbol{Y} \in M_{n\times n}(U);$$

ii. $\boldsymbol{XA} = [x_{ij}][a_{ij}] = [\sum_{q=1}^{n} x_{iq}a_{qj}] \in \boldsymbol{M_{n\times n}(U)}$ and
$\boldsymbol{AX} = [a_{ij}][x_{ij}] = [\sum_{q=1}^{n} a_{iq}x_{qj}] \in \boldsymbol{M_{n\times n}(U)}$.
Thus, $\boldsymbol{M_{n\times n}(U)}$ is the ideal of $\boldsymbol{M_{n\times n}(R)}$. ■

Lemma 3.2 *If U is the ideal of ring R, then $\boldsymbol{U^n}$ is the submodule of the $\boldsymbol{R^n}$.*

Proof. Let$R$ is a commutative ring with unity 1_R and U is the ideal of ring R. The n-space U, denoted by $\boldsymbol{U^n}$, is a set of $n \times 1$ matrices whose entries are elements of U.

Note that for every $x, y \in \boldsymbol{U^n}$, where $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$,and $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M_{n\times n}(R)}$, where $\boldsymbol{A} = [a_{ij}]$ and $\boldsymbol{B} = [b_{ij}]$ satisfy :

$$\boldsymbol{A} \bullet \boldsymbol{x} + \boldsymbol{B} \bullet \boldsymbol{y} = \boldsymbol{Ax} + \boldsymbol{By}$$

$$= [a_{ij}]\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + [b_{ij}]\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} a_{1i}x_i \\ \sum_{i=1}^{n} a_{2i}x_i \\ \vdots \\ \sum_{i=1}^{n} a_{ni}x_i \end{pmatrix} + \begin{pmatrix} \sum_{i=1}^{n} b_{1i}y_i \\ \sum_{i=1}^{n} b_{2i}y_i \\ \vdots \\ \sum_{i=1}^{n} b_{ni}y_i \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} (a_{1i}x_i + b_{1i}y_i) \\ \sum_{i=1}^{n} (a_{2i}x_i + b_{2i}y_i) \\ \vdots \\ \sum_{i=1}^{n} (a_{ni}x_i + b_{ni}y_i) \end{pmatrix}$$

since$x_i, y_i \in U, a_{ij}, b_{ij} \in R$ and $U$is ideal of ring R, then $a_{ij}x_i + b_{ij}y_i \in U$. Thus, $\boldsymbol{A} \bullet \boldsymbol{x} + \boldsymbol{B} \bullet \boldsymbol{y} \in \boldsymbol{U^n}$.

Therefore, based on [7], $\boldsymbol{U^n}$ is a submodule of the $\boldsymbol{R^n}$. ■

The following is a theorem regarding whether the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$ is a multiplication module.

Theorem 3.3 *The $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$ is a multiplication module.*

Proof. In order to prove that the $\boldsymbol{R^n}$ module over the matrix ring $\boldsymbol{M_{n\times n}(R)}$ is a multiplication module, it must be shown that for each submodule $\boldsymbol{S}$ of $\boldsymbol{R^n}$, there exists an ideal presentation of I in $\boldsymbol{M_{n\times n}(R)}$ such that $\boldsymbol{S} = \boldsymbol{IR^n}$. Based on Lemma 3.2, if U is the ideal of ring R, then $\boldsymbol{U^n}$, the n-space in whose entries are the elements of U, is a submodule of $\boldsymbol{R^n}$. Based on Lemma 3.1, if U is the ideal of ring R, then $\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{U})$, which is a set of $n\times n$ - ordered matrices whose entries are the elements of U, is the ideal of $\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})$.

Note that submodules of the $\boldsymbol{R}^{\boldsymbol{n}}$ module over the matrix ring $\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})$are $\boldsymbol{S}=\boldsymbol{U}^{\boldsymbol{n}}$, and for any submodule $\boldsymbol{S}$ there exists an ideal presentation

$$I = \{\boldsymbol{A}\in\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R}) | \boldsymbol{A}\bullet\boldsymbol{R}^{\boldsymbol{n}}\subseteq\boldsymbol{U}^{\boldsymbol{n}}\}$$

$$=\left\{[a_{ij}]\middle|\boldsymbol{A}\bullet\begin{pmatrix}r_1\\r_2\\\vdots\\r_n\end{pmatrix}\subseteq\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix},\begin{pmatrix}r_1\\r_2\\\vdots\\r_n\end{pmatrix}\in\boldsymbol{R}^{\boldsymbol{n}},\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}\in U^n\right\}$$

$$=\{[u_{ij}]|u_{ij}\in\boldsymbol{U}^{\boldsymbol{n}}\}$$

$$=\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{U}).$$

Because for each submodule $\boldsymbol{S}=\boldsymbol{U}^{\boldsymbol{n}}$ in $\boldsymbol{R}^{\boldsymbol{n}}$, there exists $\boldsymbol{I}=\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{U})$, such that

$\boldsymbol{S}=\boldsymbol{U}^{\boldsymbol{n}}=\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{U})\boldsymbol{R}^{\boldsymbol{n}}=\boldsymbol{I}\boldsymbol{R}^{\boldsymbol{n}}$, then the $\boldsymbol{R}^{\boldsymbol{n}}$ is multiplication module over matrix ring $\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})$. ■

The following is a theorem showing that the $\boldsymbol{R}^{\boldsymbol{n}}$ module over matrix ring $\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})$ is not a faithful module.

Theorem 3.4 *The $\boldsymbol{R}^{\boldsymbol{n}}$ module over matrix ring $\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})$ is not a faithful module.*

*Proof.*Suppose the $\boldsymbol{R}^{\boldsymbol{n}}$ is a module over the matrix ring $\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})$. Based on [2], a module is faithful module if and only if

$$Ann_{\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})}(\boldsymbol{R}^{\boldsymbol{n}})=\{0_{\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})}\}.$$

In Theorem 3.1, it has been shown that $\boldsymbol{R}^{\boldsymbol{n}}$ module is a torsion module. It has already been shown that for every $\boldsymbol{x}=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\in\boldsymbol{R}^{\boldsymbol{n}}$,we can always find $\boldsymbol{A}=[a_{ij}]$, where$a_{11}=x_2$,$a_{12}=-x_1$, and else is 0_R, that satisfies $\boldsymbol{A}\bullet\boldsymbol{x}=\boldsymbol{0}_{\boldsymbol{R}^{\boldsymbol{n}}}$. It implies that

$$Ann(\boldsymbol{R}^{\boldsymbol{n}}):=\{A\in\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})|A\bullet\boldsymbol{R}^{\boldsymbol{n}}=\{\boldsymbol{0}_{\boldsymbol{R}^{\boldsymbol{n}}}\}\}$$

$$=\left\{\begin{pmatrix}x_2&-x_1&\cdots&0_R\\0_R&0_R&&0_R\\\vdots&&\ddots&\vdots\\0_R&0_R&\cdots&0_R\end{pmatrix}\middle|x_1,x_2\in R\right\}$$

$$\neq\{0_{\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})}\}$$

Therefore, the $\boldsymbol{R}^{\boldsymbol{n}}$ is not a faithful module over the matrix ring $\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})$. ■

4. CONCLUSION

Let R is a commutative ring with unity. The $\boldsymbol{R}^{\boldsymbol{n}}$ is a module over the matrix ring $\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})$, denoted by $\boldsymbol{R}^{\boldsymbol{n}}=\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})-$ module. From Section 3, it has been discussed that the $\boldsymbol{R}^{\boldsymbol{n}}$ module over the matrix ring $\boldsymbol{M}_{\boldsymbol{n}\times\boldsymbol{n}}(\boldsymbol{R})$ is a torsion module, a prime module, a multiplication module, but not a faithful module.

AUTHORS' CONTRIBUTIONS

AA conceived of the presented idea. TT and AW developed the theory and verified the analytical methods. TT encouraged AA to investigate the material and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

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