Solution Formula of the Half-Space Model Problem for Incompressible Fluid Flow

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ABSTRACT
In this paper we determine a slightly detailed the solution formula of the incompressible fluid flows by using Fourier transform in N-dimensional Euclidean space (N≥ 2) for the linearized equations. For further research, from this result we can estimate the boundedness of the operator families. This research is based on Shibata and Shimizu article.

Keywords: *Incompressible, Fourier transform, Fluid flows, Linearized equations.*

1. INTRODUCTION

The Let $\mathbf{u}$ is the velocity and θ is the density. We consider the solution of the Stokes equation system which describes the motion of incompressible fluid in bounded domain in half-space. We define $\mathbb{R}_+^n$ and $\mathbb{R}_0^n$ be the half-space and its boundary, respectively by

$$\mathbb{R}_+^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\},$$

and

$$\mathbb{R}_0^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}. \quad (1)$$

The resolvent problem of Stokes equations are being described by the set of equations,

$$\begin{cases} \lambda\mathbf{u} - \mathrm{Div}\, \mathrm{S}(\mathbf{u}, \theta) = \mathbf{f} & \text{in } \mathbb{R}_+^n \\ \mathrm{div}\, \mathbf{u} = 0 & \text{in } \mathbb{R}_+^n \\ \mathrm{S}(\mathbf{u}, \theta)\mathbf{n} = \mathbf{h} & \text{on } \mathbb{R}_0^n \end{cases} \quad (2)$$

where $\mathbf{f} = (f_1, \dots, f_n)$, $\mathbf{h} = (h_1, \dots, h_n)$ and $\mathrm{S}(\mathbf{u}, \theta)$ is the stress tensor which defined by

$$\mathbf{S}(\mathbf{u}, \theta) = -\theta\mathbf{I} + \mu\mathbf{D}(\mathbf{u}), \quad (3)$$

and $\mathbf{n} = (0,0, \dots, -1)$ stands for the unit outer normal. The doubled deformation $\mathbf{D}(\mathbf{u})$ tensor whose (i,j) components are $\mathbf{D}_{ij}(\mathbf{u}) = \mathrm{D}_i u_j + \mathrm{D}_j u_i$ $(\mathrm{D}_i = \partial/\partial x_i)$, $\mathbf{I}$ the $n \times n$ identity matrix, and also $\mathrm{div}\, \mathbf{u} = \sum_{j=1}^n \partial_j u_j$.

Before stating our main results precisely, in this part we shall explain the notation which used in whole of the paper. For row vector valued functions are denoted by bold-face letter which is corresponding to the velocity filed. $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of all natural numbers, real numbers and complex numbers, respectively, and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For the differentiations of scale functions f and N- vector functions $\mathbf{g} = (g_1, \dots, g_n)$, we use the following symbols:

$$\nabla f = (\partial_1 f, \dots, \partial_N f), \quad \nabla^2 f = (\partial_i \partial_j f \mid i,j = 1, \dots, N),$$
$$\nabla\mathbf{g} = (\partial_i g_j \mid i,j = 1, \dots, N), \quad \nabla^2\mathbf{g} = (\partial_i \partial_j g_k \mid i,j,k = 1, \dots, N).$$

Let $\mathcal{F}_x = \mathcal{F}$ and $\mathcal{F}_\xi^{-1} = \mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively, which are defined by [1], [2]

$$\hat{f}\,(\xi) = \mathcal{F}_x[f]\,(\xi) = \hat{u}\,(x) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)dx, \quad \mathcal{F}_\xi^{-1}[g](x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}\, g(\xi)d\xi.$$

Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ the denote the Laplace transform and the Laplace inverse transform, which defined by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\, dt = \mathcal{F}_t[e^{-\gamma t} f(t)](\tau),$$

$$\mathcal{L}^{-1}[g](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t}\, g(\tau)\, d\tau = e^{\gamma t}\mathcal{F}_t^{-1}[g(\tau)](t),$$

with $\lambda = \gamma + i\,\tau \in \mathbb{C}$, respectively. Set

$$\widehat{W}_q^m(\Omega) := \{\theta \in L_{q,\mathrm{loc}}(\Omega) \mid \nabla\theta \in L_q(\Omega)^n\},$$

and

$$\Sigma_{\epsilon,\gamma_0} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \leq \pi - \epsilon, |\lambda| \geq \gamma_0\}.$$

Before we state the main result, first of all we introduce the definition of Sobolev space $W_q^m(\Omega)$.

Definition 1.1 (Adams and Fournier, [4])
Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty)$ then the Sobolev Space $W_q^m(\Omega)$ is defined by

$$W_q^m(\Omega) := \{\mathbf{u} \in L_q(\Omega) \mid D^\alpha \mathbf{u} \in L_q(\Omega), \forall\, \alpha \text{ with } |\alpha| \le m\}$$

Next, we state the main theorem of this paper.

Theorem 1.2 *Let $N < q < \infty$, $2 < p < \infty$ and $\lambda \in \Sigma_{\epsilon,\gamma_0}$*, $\gamma_0 \geq 1$ depending on ϵ such that problem (2) admits a unique solution $(\boldsymbol{u}, \theta) \in W_q^2(\mathbb{R}_+^n) \times \widehat{W}_q^1(\mathbb{R}_+^n)$, *with*

$$u_j(x) = -\mathcal{F}_{\xi'}^{-1}\left[\frac{2i\xi_j A}{\mu D(A.B)} M(A,B,x_n)\big(i\xi' \cdot \hat{h}' - B\hat{h}_n\big)\right](x') - \mathcal{F}_{\xi'}^{-1}\left[\frac{\xi_j(B-A)e^{-Bx_n}}{\mu B D(A.B)}\xi' \cdot \hat{h}'\right](x') + \mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_j e^{-Ax_n}}{\mu D(A.B)}\big(2i\xi' \cdot \hat{h}' - (B-A)\hat{h}_n\big)\right](x') + \mathcal{F}_{\xi'}^{-1}\left[\frac{e^{-Bx_n}}{\mu B}\hat{h}_j\right](x'),$$

$$u_n(x) = \mathcal{F}_{\xi'}^{-1}\left[\frac{A}{\mu D(A.B)} M(A,B,x_n)\big(2Bi\xi' \cdot \hat{h}' - (A^2+B^2)\hat{h}_n\big)\right](x') + \mathcal{F}_{\xi'}^{-1}\left[\frac{e^{-Bx_n}}{\mu D(A.B)}\big((B-A)i\xi' \cdot \hat{h}' + A(A+B)\hat{h}_n\big)\right](x'),$$

$$\theta(x) = -\mathcal{F}_{\xi'}^{-1}\left[\frac{(A+B)e^{-Ax_n}}{D(A.B)}\left[2Bi\xi' \cdot \hat{h}' - (A^2+B^2)\hat{h}_n\right]\right](x').$$

for $j = 1, \dots, n-1$. With $A^2 = |\xi'|^2$, $B^2 = \frac{\lambda}{\mu} + |\xi'|^2$ and $D(A.B) = B^3 + AB^2 + 3A^2B - A^3$.

There are many researcher who concern studying fluid dynamic not only for incompressible case but also compressible case. As we know that fluid dynamic motion described by Navier Stokes equations. And Stokes equations known as linearized of the Navier Stokes equations. In 2016, Maryani [5] studied the model of the fluid flows for polymer which known as Oldroyd-B with free boundary problems. She investigated local well-posedness of the model problem for bounded and unbounded problems. For PDE model problems, investigating local well-posedness and also globall well-posedness are the important point. Since, we can know whether the solution exist, unique and have some behaviour in small data.

The physical geometry of the model problem that in the bounded domain which occupied by a compressible fluid such as water. In this paper, we consider the model problem of (2) in half-space case which is defined in (1).

The research methodology of this article is literature review of the article [3]. We give more detail how to find the solution formula of the model problems by using partial Fourier transform in half-space without surface tension. Using this transformation, we get new equation system then the solution formula is furnished. As we known that the fluid motion was formed by conservation of mass and conservation of momentum. These conservations guarantee the boundary condition of the model problem. First of all, we elaborate the model problem (2) by definition (3) in half-space, then we have the following section which we called reduce system.

2. REDUCE SYSTEM

In this section, we formulated the model problem (2) and state our main result. Let $\mathbb{R}_+^n$ and $\mathbb{R}_0^n$ as defined in (1). We consider for $\lambda \neq 0$, then we have

$$\begin{cases} \lambda\mathbf{u} - \mu\Delta\mathbf{u} + \nabla\theta = \mathbf{0} & \text{in } \mathbb{R}_+^n \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \mathbb{R}_+^n \\ \mu(D_n u_j + D_j u_n) = -h_j & \text{on } \mathbb{R}_0^n \\ 2\mu D_n u_n - \theta = -h_n & \text{on } \mathbb{R}_0^n \end{cases} \tag{4}$$

for $j = 1, \dots, n-1$.

Moreover, we derive a solution formula of (4). For this purpose, applying the partial Fourier transform to (4) i.e

$$\hat{u} = \hat{u}(x_n) = \hat{u}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} u(x', x_n)\, dx'$$

$$\mathcal{F}_{\xi}^{-1}[\hat{u}(\xi', x_n)](x') = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \hat{u}(\xi', x_n) d\xi',$$

where $\xi' = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, we have

$$\begin{cases} \lambda\hat{u}_j + \mu\hat{u}_j|\xi'|^2 - \mu D_n^2\hat{u}_j + i\xi_j\hat{\theta} = 0 & (x_n > 0) \\ \lambda\hat{u}_n + \mu\hat{u}_n|\xi'|^2 - \mu D_n^2\hat{u}_n + D_n\hat{\theta} = 0 & (x_n > 0) \\ \sum_{j=1}^{n-1} i\xi_j\,\hat{u}_j + D_n\hat{u}_n = 0 & (x_n > 0)\ , \\ \mu\big(D_n\hat{u}_j + i\xi_j\hat{u}_n\big) = -\widehat{h_j}(\xi', 0) & \\ 2\mu D_n\hat{u}_n - \hat{\theta} = -\hat{h}_n(\xi', 0) & \end{cases} \tag{5}$$

for $j = 1, \dots, n-1$.

Let $\hat{\mathbf{u}} = (\hat{u}_1, \dots, \hat{u}_n)$ and $\hat{\theta}$ have general formula in the following

$$\hat{u}_\ell = \alpha_\ell e^{-Ax_n} + \beta_\ell e^{-Bx_n}, \tag{6}$$
$$\hat{\theta} = \gamma e^{-Ax_n}, \tag{7}$$

then we get

$$D_n\hat{u}_n = -A\alpha_n e^{-Ax_n} - B\beta_n e^{-Bx_n}, \tag{8}$$
$$D_n^2\hat{u}_n = A^2\alpha_n e^{-Ax_n} + B^2\beta_n e^{-Bx_n}, \tag{9}$$

$$D_n\hat{\theta} = -A\gamma e^{-Ax_n}. \quad (10)$$

Applying div to first equation of (4) we have

$$(\lambda - \mu\Delta)\mathrm{div}\,\mathbf{u} + \Delta\theta = 0. \quad (11)$$

Since div $\mathbf{u} = 0$, we have

$$\Delta\theta = 0. \quad (12)$$

Multiplying first equation of (4) by Δ then using (12), we have

$$\Delta(\lambda - \mu\Delta)\mathbf{u} = \mathbf{0}. \quad (13)$$

Multiplying (13) by $\frac{1}{\mu}$, we have

$$\Delta\left(\frac{\lambda}{\mu} - \Delta\right)\mathbf{u} = \mathbf{0}. \quad (14)$$

Applying Fourier transform to (14), we have formula

$$A^2 = |\xi'|^2 \text{ and } B^2 = \frac{\lambda}{\mu} + |\xi'|^2. \quad (15)$$

Substituting (6), (7), (8), (9), (10) and (15) to equation system of (5) and equating the coefficients of e^{-Ax_n} and e^{-Bx_n}, we have new equation system

$$\begin{cases} \lambda\alpha_j(B^2 - A^2) + i\xi_j\gamma = 0 \\ \lambda\alpha_n(B^2 - A^2) - A\gamma = 0 \\ \sum_{k=1}^{n-1} i\xi_k\,\alpha_k - A\alpha_n = 0 \\ \sum_{k=1}^{n-1} i\xi_k\,\beta_k - B\beta_n = 0 \\ \mu\left(A\alpha_j + B\beta_j - i\xi_j(\alpha_n + \beta_n)\right) = \hat{h}_j(\xi', 0) \\ 2\mu(A\alpha_n + B\beta_n) + \gamma = \hat{h}_n(\xi', 0)\ , \end{cases} \quad (16)$$

and then from first, second, third, and fourth equation of (16), we get the formula

$$\alpha_j = \frac{-i\xi_j\gamma}{\mu(B^2 - A^2)}, \quad (17)$$

$$\alpha_n = \frac{A\gamma}{\mu(B^2 - A^2)}, \quad (18)$$

$$\gamma = \frac{\alpha_n\mu(B^2 - A^2)}{A}, \quad (19)$$

$$A\alpha_n = \sum_{k=1}^{n-1} i\xi_k\,\alpha_k\ , \quad (20)$$

and

$$B\beta_n = \sum_{k=1}^{n-1} i\xi_k\,\beta_k\ . \quad (21)$$

Substituting (19) to equation (17), we have

$$\alpha_j = -\frac{\alpha_n i\xi_j}{A}, \quad (22)$$

multiplying fifth equation of (16) by $i\xi_j$ then summing from $j = 1$ to $j = n - 1$ and inserting (20) and (21), we have

$$\sum_{k=1}^{n-1} i\xi_k\,\hat{h}_k = \mu(2A^2\alpha_n + (A^2 + B^2)\beta_n). \quad (23)$$

Eliminating (23) and sixth equation of (16) then inserting (18), we have formula

$$\gamma = -\frac{A+B}{D(A.B)}\left[2B\sum_{k=1}^{n-1} i\xi_k\,\hat{h}_k - (A^2 + B^2)\hat{h}_n\right], \quad (24)$$

where $D(A.B) = B^3 + AB^2 + 3A^2B - A^3$.

Substituting (24) to (17) and (18), we have formula

$$\alpha_j = \frac{-i\xi_j}{\mu(B-A)D(A.B)}\left[2B\sum_{k=1}^{n-1} i\xi_k\,\hat{h}_k - (A^2 + B^2)\hat{h}_n\right], \quad (25)$$

$$\alpha_n = -\frac{A}{\mu(B-A)D(A.B)}\left[2B\sum_{k=1}^{n-1} i\xi_k\,\hat{h}_k - (A^2 + B^2)\hat{h}_n\right], \quad (26)$$

respectively.

Substituting (18) to sixth equation of (16), we have

$$\beta_n = \frac{(B^2 - A^2)\hat{h}_n + (-A^2 - B^2)\gamma}{2\mu B(B^2 - A^2)}, \quad (27)$$

and then substituting (24) to (27), we have formula

$$\beta_n = \frac{1}{\mu(B-A)D(A.B)}\left[(A^2 + B^2)\sum_{k=1}^{n-1} i\xi_k\,\hat{h}_k - 2A^3\hat{h}_n\right]. \quad (28)$$

Substituting (18), (22), (27) to fifth equation of (16), we have

$$\beta_j = i\xi_j\left(\frac{4AB - (A^2 + B^2)}{2\mu B^2(B^2 - A^2)}\gamma + \frac{1}{2\mu B^2}\hat{h}_n\right) + \frac{\hat{h}_j}{\mu B}, \quad (29)$$

and then substituting (24) to (29), we have formula

$$\beta_j = \frac{i\xi_j}{\mu B(B-A)D(A.B)}\left[(A^2 + B^2 - 4AB)\sum_{k=1}^{n-1} i\xi_k\,\hat{h}_k - 2AB^2\hat{h}_n\right] + \frac{\hat{h}_j}{\mu B}. \quad (30)$$

Substituting (24), (25), (26), (28), (30) to (6) and (7), we have

$$\hat{u}_j = -\frac{2i\xi_j A}{\mu D(A.B)}M(A, B, x_n)\left(i\xi' \cdot \hat{h}' - B\hat{h}_n\right) - \frac{\xi_j(B - A)e^{-Bx_n}}{\mu BD(A.B)}\xi' \cdot \hat{h}' + \frac{i\xi_j e^{-Ax_n}}{\mu D(A.B)}\left(2i\xi' \cdot \hat{h}' - (B - A)\hat{h}_n\right) + \frac{e^{-Bx_n}}{\mu B}\hat{h}_j, \quad (31)$$

$$\hat{u}_n = \frac{A}{\mu D(A.B)}M(A, B, x_n)\left(2Bi\xi' \cdot \hat{h}' - (A^2 + B^2)\hat{h}_n\right) + \frac{e^{-Bx_n}}{\mu D(A.B)}\left((B - A)i\xi' \cdot \hat{h}' + A(A + B)\hat{h}_n\right), \quad (32)$$

$$\hat{\theta} = -\frac{(A+B)e^{-Ax_n}}{D(A.B)}\left[2Bi\xi' \cdot \hat{h}' - (A^2 + B^2)\hat{h}_n\right], \quad (33)$$

where $\xi' = (\xi_1, \xi_2, \dots, \xi_{n-1})$, $\hat{h}' = (\hat{h}_1, \hat{h}_2, \dots, \hat{h}_{n-1})$, and $M(A, B, x_n) = \frac{e^{-Bx_n} - e^{-Ax_n}}{B-A}$.

Therefore, we have the solution formula of $\mathbf{u}$ and θ of the model problem (2)

$$u_j(x) = -\mathcal{F}_{\xi'}^{-1}\left[\frac{2i\xi_j A}{\mu D(A.B)}M(A, B, x_n)\left(i\xi' \cdot \hat{h}' - B\hat{h}_n\right)\right](x') - \mathcal{F}_{\xi'}^{-1}\left[\frac{\xi_j(B - A)e^{-Bx_n}}{\mu BD(A.B)}\xi' \cdot \hat{h}'\right](x') + \mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_j e^{-Ax_n}}{\mu D(A.B)}\left(2i\xi' \cdot \hat{h}' - (B - A)\hat{h}_n\right)\right](x') + \mathcal{F}_{\xi'}^{-1}\left[\frac{e^{-Bx_n}}{\mu B}\hat{h}_j\right](x'), \quad (34)$$

$$u_n(x) = \mathcal{F}_{\xi'}^{-1}\left[\frac{A}{\mu D(A.B)}M(A, B, x_n)\left(2Bi\xi' \cdot \hat{h}' - (A^2 + B^2)\hat{h}_n\right)\right](x') + \mathcal{F}_{\xi'}^{-1}\left[\frac{e^{-Bx_n}}{\mu D(A.B)}\left((B - A)i\xi' \cdot \hat{h}' + A(A + B)\hat{h}_n\right)\right](x'), \quad (35)$$

$$\theta(x) = -\mathcal{F}_{\xi'}^{-1}\left[\frac{(A+B)e^{-Ax_n}}{D(A.B)}\left[2Bi\xi'\cdot\hat{h}' - (A^2 + B^2)\hat{h}_n\right]\right](x'), \quad (36)$$

For $j = 1, \dots, N-1$ which completes the proof of **Theorem 1.2.**

AUTHORS' CONTRIBUTIONS

All authors (MLBCD, SM, AW, and BHG) contributed equally to the WRITING of this article. All authors READ and APPROVED the final manuscript.

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