

Coefficient Estimates in the Class of Bazilevic Functions $\mathcal{B}_1(\alpha)$ Related to the Lemniscate Bernoulli

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ABSTRACT

In this research, we estimate the coefficient in the class of Bazilevic functions $\mathcal{B}_1(\alpha)$ related to the Lemniscate Bernoulli on the unit disk $\mathbb{D} = \{z: |z| < 1\}$, satisfying subordination condition $\left[f'(z) \left(\frac{f(z)}{z} \right)^{\alpha-1} \right] < \sqrt{1+z}$, for $z \in \mathbb{D}$. The upper bound of the modulus of a_2 and a_3 are determined.

Keywords: Coefficient, Bazilevic functions $\mathcal{B}_1(\alpha)$, Lemniscate Bernoulli, Subordination.

1. INTRODUCTION

Let S denote the class of analytic normalized univalent functions f defined in unit disk $\mathbb{D} = \{z: |z| < 1\}$, normalized by $f(z) = z$ and $f'(z) = 1$, and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

Then for $\alpha \geq 0$, $f \in \mathcal{B}(\alpha) \subset S$ if and only if for $z \in \mathbb{D}$

$$Re \left[f'(z) \left(\frac{f(z)}{g(z)} \right)^{\alpha-1} \right] > 0 \tag{2}$$

From equation (2), if $g(z) \equiv z$ then it gives the class $\mathcal{B}_1(\alpha)$ of Bazilevic functions with logarithmic growth. $\mathcal{B}_1(\alpha)$ is well known as class \mathbb{R} of functions whose derivative has positive real part in \mathbb{D} . The properties of $\mathcal{B}_1(\alpha)$ have been studied extensively by previous researchers [2], [3], and [5].

Therefore, for $\alpha \geq 0$, $f \in \mathcal{B}_1(\alpha) \subset S$ if and only if for $z \in \mathbb{D}$,

$$Re \left[f'(z) \left(\frac{f(z)}{z} \right)^{\alpha-1} \right] > 0 \tag{3}$$

We say that an analytic function f is subordinate to an analytic function g and write $f(z) < g(z)$, if and only if

there exists a function ω analytic in \mathbb{D} , such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$ and $f(z) < g(\omega(z))$.

First, we give definition of the class Bazilevic functions $\mathcal{B}_1(\alpha)$ related to the Lemniscate Bernoulli on the unit disk $\mathbb{D} = \{z: |z| < 1\}$.

Definition 1. If $f(z)$ and $g(z)$ analytic in unit disk \mathbb{D} and $f(z) = g(z)$. The assumed f univalent and $R_g \subseteq R_f$ gives $\omega(z) = f^{-1}(g(z))$, and satisfying the condition,

$$\left[f'(z) \left(\frac{f(z)}{z} \right)^{\alpha-1} \right] < \sqrt{1+z} =: q(z), \tag{4}$$

where the branch of the square root is chosen to be $q(z) = 1$. Bernoulli Lemniscate function $\zeta = (x^2 + y^2)^2 - 2(x^2 - y^2) = 0$ and $q(\mathbb{D})$, lies in the region bounded by the right part of the Bernoulli Lemniscate. [8].

Next, we need to determine the modulus of initial coefficient. We use some Lemmas in [1], [2], [5], [8], and [9] to obtain the upper bound of modulus of a_2 and a_3 . Modulus a_2 and a_3 can be written as $|a_2|$ and $|a_3|$. The main advantage of this research is to estimate the value of functions by using its coefficients.

2. PRELIMINARY LEMMAS

Definition 2. [1]. If $p \in \mathcal{P}$ the class of function satisfying $Re(p(z)) > 0$ for $z \in \mathbb{D} = \{z: |z| < 1\}$.

Let,

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{5}$$

The following lemmas used in [2], [5], [8], and [9] as given as follows

Lemma 2.1. If $p \in \mathcal{P}$ analytic in \mathbb{D} with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ for $n \geq 1$ then

$$|p_n| \leq 2.$$

For the $p(z) = \frac{1+z}{1-z}$ this lemma known as inequality Caratheodory Toeplitz. [2], [8].

Lemma 2.2 If $p \in \mathcal{P}$ analytic in \mathbb{D} with $0 \leq B \leq 1$ and $B(2B - 1) \leq D \leq B$, than

$$|p_1 - 2Bp_1p_2 + Dp_1^3| \leq 2.$$

Lemma 2.3. If $p \in \mathcal{P}$ analytic in \mathbb{D} with coefficient p_n , than $|p_n| \leq 2$ for $n \geq 1$ than

$$\begin{aligned} \left| p - \frac{\mu}{2} p_1^2 \right| &\leq \max \{2, 2|\mu - 1|\} \\ &= \begin{cases} 2, & 0 \leq \mu \leq 2 \\ 2|\mu - 1|, & \text{elsewhere} \end{cases} \end{aligned}$$

In this reseach we use three lemmas such as the lemmas 2.1, 2.2 and 2.3.

3. RESULT

This reseach are working on the Bazilevic functions $\mathcal{B}_1(\alpha)$ related to Bernoulli Lemniscate based on the result of Sokol and Thomas [9] for Starlike function related to Bernoulli Lemniscateas. This idea is also supported by the result of Singh [10] for coefficient of Bazilevic functions $\mathcal{B}_1(\alpha)$. We first define functions $f \in \mathcal{S}$, from lemma 2.1 and lemma 2.3 gives,

$$\left[f'(z) \left(\frac{f(z)}{z} \right)^{\alpha-1} \right] < \sqrt{1 + \omega(z)} \tag{6}$$

where $\omega(0) = 0, |\omega(z)| < 1$ for $z \in \mathbb{D}$. On the other hand it is known that

$$\omega(z) = \frac{\delta p(z) - 1}{\delta p(z) + 1} \tag{7}$$

For some $p \in \mathcal{P}$ which gives,

$$\left[\frac{f'(z) f(z)^{\alpha-1}}{z^{\alpha-1}} \right] = \sqrt{1 + \frac{\delta p(z) - 1}{\delta p(z) + 1}} \tag{8}$$

Theorem 1. If $f \in \mathcal{B}_1(\alpha)$ and is given by (1), then

$$|a_2| \leq \frac{2\sqrt{\alpha}}{\sqrt{2}(1 + \delta)^{\frac{3}{2}}}, \quad \text{for } \alpha \geq 0 \text{ and } \delta \geq 0.$$

We also have,

$$|a_3| \leq \frac{\sqrt{2}\sqrt{\alpha}}{(2 + \alpha)(1 + \delta)^{\frac{3}{2}}}, \quad \text{for } 0 \leq \alpha \leq 1,596$$

and $\delta \geq 0$,

$$\leq \frac{\sqrt{2}\sqrt{\alpha}}{(2 + \alpha)(1 + \delta)^{\frac{3}{2}}}, \quad \text{for } \alpha > 1,596$$

$$\text{and } 0 \leq \delta \leq \frac{9}{-1 - 8\alpha - 6\alpha^2 + 4\alpha^3 + 2\alpha^4}$$

$$\leq \frac{4\sqrt{2} 2\sqrt{\delta} (1 + \delta)^2}{8(2 + \alpha)(1 + \delta)^{\frac{7}{2}}}, \quad \text{for } \alpha > 1,596$$

$$\text{and } \delta \geq \frac{9}{-1 - 8\alpha - 6\alpha^2 + 4\alpha^3 + 2\alpha^4}.$$

For boundaries $|a_2|, |a_3|$ and the inequalities are sharp.

Proof.

We need the following steps to proof the Theorem.

1. To express the related function into function with positive real part,
2. Equating coefficients,
3. Obtain some equations,
4. Solve the above equations by considering some cases and using known lemmas,

From equation (8), we equating the initial coefficient which gives,

$$a_2 = \frac{p_1 \sqrt{\alpha}}{\sqrt{2}(1 + \delta)^{\frac{3}{2}}}$$

and

$$\begin{aligned} a_3 = \frac{1}{8(2 + \alpha)(1 + \delta)^{\frac{7}{2}}} &(\sqrt{\delta} (4p_2 \sqrt{2}(1 + \delta)^2 \\ - p_1^2 (\sqrt{2} + 5\sqrt{2}\delta + 4\sqrt{2}\delta^2 \\ + 2(-2 + \alpha + \alpha^2)\sqrt{\delta}\sqrt{1 + \delta})) \end{aligned}$$

Next, by using lemma 2.1 gives,

$$|a_2| = \frac{\sqrt{\alpha}}{\sqrt{2}(1 + \delta)^{\frac{3}{2}}} |p_1|$$

$$\leq \frac{2\sqrt{\alpha}}{\sqrt{2}(1+\delta)^{\frac{3}{2}}}$$

The inequality for $|a_2|$ is sharp when $p_1 = 2$.

Next, we estimate $|a_3|$ by applying lemma 2.3 and write

$$|a_3| = \frac{\sqrt{\delta}(4\sqrt{2}(1+\delta))^{\frac{2}{7}}}{8(2+\alpha)(1+\delta)^{\frac{2}{7}}} \left| p_2 - \frac{\mu}{2} p_1^2 \right|$$

with,

$$\mu = \frac{1}{2\sqrt{2}(1+\delta)^2} (\sqrt{2} + 5\sqrt{2}\delta + 4\sqrt{2}\delta^2 + 2(-2 + \alpha + \alpha^2)\sqrt{\delta}\sqrt{1+\delta}).$$

Since $0 \leq \mu \leq 2$ provided $0 \leq \alpha \leq 1,596$ and $\delta \geq 0$, and when $\alpha > 1,596$ and $\delta \geq \frac{9}{-1-8\alpha-6\alpha^2+4\alpha^3+2\alpha^4}$, then the first and second inequality for $|a_3|$ are proven. For elsewhere condition in lemma 2.3, we get the third inequality for $|a_3|$.

The first and second inequality for $|a_3|$ are sharp when $p_1 = 0$ and $p_2 = 2$. The third inequality for $|a_3|$ is sharp when $p_1 = 2$ and $p_2 = 2$. \square

Next, determine the coefficient of a_4 . From (8) equating initial coefficient gives,

$$a_4 = \left(\frac{1}{48(2+\alpha)(1+\delta)^{\frac{9}{2}}} \right) (\sqrt{\alpha}(24\sqrt{2} p_3 (2+\alpha) (1+\delta)^3 - 12p_1 p_2 (1+\delta)(2\alpha^2\sqrt{\delta}\sqrt{1+\delta} + 2(\sqrt{2} + 5\sqrt{2} + 4\sqrt{2}\delta^2 - 3\sqrt{\delta}\sqrt{1+\delta}) + \alpha(2(\sqrt{2} + 5\sqrt{2} + 4\sqrt{2}\delta^2 - 4\sqrt{\delta}\sqrt{1+\delta})) + p_1^3(14\sqrt{2}\alpha^3\delta + 4\sqrt{2}\alpha^4\delta + 6(2(\sqrt{2} + 5\sqrt{2} + 4\sqrt{2}\delta^2 + 8\sqrt{2}\delta^2 - 3\sqrt{\delta}\sqrt{1+\delta}) - 12\delta^{\frac{3}{2}}\sqrt{1+\delta}) + \delta^2(-4\sqrt{2}\delta + 6\sqrt{\delta}\sqrt{1+\delta} + 24\delta^{\frac{3}{2}}\sqrt{1+\delta}) + \alpha(3\sqrt{2} - 11\sqrt{2}\delta + 36\sqrt{2}\delta^2 + 24\sqrt{2}\delta^3 + 12\delta\sqrt{1+\delta} + 48\delta^{\frac{3}{2}}\sqrt{1+\delta}))$$

To complete the theorem we use lemma 2.1, lemma 2.2 and lemma 2.3, then allows for additional entries for sharp boundaries, [1], [2], [5], [8], and [9].

4. CONCLUSION

This research is extended from Sokol and Thomas research [9]. We use the Bazilevic function to determine the boundary for the coefficient estimates in the class of Bazilevic functions $\mathcal{B}_1(\alpha)$ related to the Lemniscate

Bernoulli for the modulus of a_2 and a_3 . The results are sharp.

AUTHORS' CONTRIBUTIONS

All authors, NMA, MM, AF, and RBEW, have contributions about CONCEPT, METHOD, EDITING, and ANALYSIS. NMA provided feedback, discussed result and contributed to the final manuscript.

ACKNOWLEDGMENTS

Thanks and appreciation for my promotor Marjono and my co promotor Sa'adatul Fitri and Ratno Bagus E.W from the Department of Mathematics and Natural Sciences Brawijaya University, for their support and guidance in completing this research and this paper. Thanks to the committee of the Soedirman's International Conference on Mathematics and Applied Sciences (SICoMAS 2021) Soedirman University for organizing this event and finishing proceeding.

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