Black-Scholes Process and Monte Carlo Simulation-Based Options Pricing

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Abstract. Monte Carlo simulation is one of the most important algorithms in finance and numerical computational science. It plays an important role in option pricing and risk management. The Monte Carlo method can easily deal with high-dimensional problems. The upper complexity and computational requirements usually increase linearly. This paper mainly introduces a special Monte Carlo method – the application of the quasi-Monte Carlo method in American option pricing problems.

Keywords: Black Scholes Process · Monte Carlo Methods · Options Pricing

1 Introduction

With the rapid development of option pricing theory in recent decades, the classic Black-Scholes option pricing theory can no longer describe the market well under its too harsh assumptions, and the option pricing model based on the Black-Scholes process is relative to the BS model which can better characterize the market characteristics [4]. In addition to overcoming the harsh conditions that the underlying asset must obey the geometric Brownian motion, the BS model can also describe the process of continuous jumping. In addition, it can also model with the characteristics of market spikes and thick tail layouts, which can better reflect the superiority of describing the market process.

This option pricing model treats the time interval of asset price changes as independent variables and assumes that changes in price or asset returns over time follow a normal distribution. In other words, transactions are evenly distributed in various periods. Since the daily, weekly, or monthly transaction volume is huge, according to the Central Limit Theorem, these prices will conform to a normal or Gaussian distribution [11]. When the asset's return distribution is normal, the probability of different situations is known. Understanding these probabilities can provide investors with an idea to better quantify the risks that may arise when holding these assets.

In the past few decades, the Monte Carlo method has been widely used in American option pricing problems. However, the convergence rate of the Monte Carlo method is very slow. To overcome this shortcoming, a quasi-Monte Carlo method based on low deviation sequences is proposed, and the convergence rate is increased from $O(\frac{1}{N^{\frac{1}{2}}})$ to $O(\frac{1}{N})$, where N is the number of simulation paths used. This paper mainly combines the

N. Radojević et al. (Eds.): ICAID 2022, AHIS 7, pp. 733–741, 2023.
https://doi.org/10.2991/978-94-6463-010-7_75

quasi-Monte Carlo method and the dual method to the American option pricing problem. The dual method is based on the American option pricing problem and the optimal stopping time problem [11]. By seeking the minimum value of an appropriate martingale process, the upper bound of the price of American options can be estimated. This method can overcome the limitation of the dimensionality of some previous algorithms in solving American option pricing problems, and can quickly solve the high Victoria American option pricing problem [8]. In addition, we introduce several martingale process construction methods and then give the specific steps of the dual method to solve the American option pricing problem [9]. The results of numerical experiments show that the combination of the quasi-Monte Carlo method and the dual method is obtained, and the calculated results are accurate and require a short time, which can greatly improve the computational efficiency of Monte Carlo simulation, so this method is very effective in practical applications.

Since the 1990s, due to the continuous acceleration of financial innovation, financial stabilization, and financial globalization, financial derivative securities have developed rapidly [6]. Because financial derivative securities are the basic tools of important financial activities such as financial risk management and prevention, speculative arbitrage, investment financing, consumer credit, etc., the pricing of financial derivative securities has become an extremely important research field in the modern financial industry [1]. In today's financial market, on the one hand, as the degree of market uncertainty continues to increase, the change process of major underlying assets such as stocks has become more and more complex. Therefore, to accurately reflect these complex characteristics, it must be based on these processes. It is a complex assumption, which greatly increases the number of underlying variables; on the other hand, various types of customers have more and more personalized requirements for financial instruments, which promotes the production and rapid development of new exotic derivative securities with complex profit and loss characteristics [12]. These two reasons make high-dimensional derivative securities with complex nature and structure increasingly occupy an important position in the financial derivatives market [13]. According to the exiting financial asset pricing theory, most option prices must be determined by numerical analysis methods. The commonly used numerical analysis techniques of financial derivative securities pricing can be divided into three basic types: grid analysis technique, finite difference technique, and Monte Carlo simulation technique [7].

As a kind of derivative tool, the option has the function of diversifying risk, which helps to strengthen the overall anti-risk ability of the financial market. Therefore, studying the options market is of great significance for improving the capital market business. On February 9, 2015, the launch of the SSE 50ETF option contract, the first stock options product in the Chinese securities market, marked the formal entry of the options era in my country's financial market. However, there are currently fewer types of on-market options, and over-the-counter options have developed rapidly since they were officially launched in 2013. Because there is no third-party supervision in the over-the-counter options market, there is no special clearing agency to force option sellers to perform their obligations, so option holders will face credit risk. With the advancement of RMB internationalization and exchange rate market reforms, the massive infrastructure investment in the Belt and Road initiative, and the ever-expanding cross-border capital transactions have increased the need for companies to manage risks in foreign currency assets. Therefore, under the current market environment, it has important theoretical and practical significance to study the SSE 50ETF option pricing, option pricing with credit risk, and option pricing with exchange rate risk.

2 Literature Review

Since the birth of options, the research on options has mainly revolved around the issue of option pricing is the core and foundation of option theory research [2] In 1973, Black and Scholes proposed the BS option pricing model, which formally opened the door of option pricing research. The theoretical starting point of the BS model is that the process of stock price changes is subject to the geometric Wiener process [5]. Under a series of constraints, a partial differential equation (BS partial differential equation) is derived using the risk-neutral pricing principle and the principle of no-arbitrage pricing, and then the analytical solution of this partial differential equation is obtained [3]. Then through the option parity formula, the pricing formula of the European put option is obtained. However, because a series of assumptions of the BS model is too ideal and deviates from the real situation of the market, the pricing results of the BS model often deviate from the true price of the option [10]. To improve and solve the shortcomings of the BS model, follow-up scholars began to study the issue of option pricing under the relaxation of the assumptions of the BS model. An important assumption in the BS model is that the volatility of the underlying asset is always constant, and in practice, the volatility of financial asset returns usually has heteroscedasticity, that is, the volatility is random, and it is also accompanied by volatility at a certain time. The phenomenon of agglomeration appears in the segment, which is manifested by the characteristics of spikes and thick tails in the distribution of return on assets. Follow-up scholars researched option pricing under stochastic volatility. By assuming that volatility obeys a stochastic process, a series of stochastic volatility models were derived, such as the CEV model, jump-diffusion model, Heston model, SABR model, etc. However, the form of the model is extremely complex, there are many parameters to be estimated, and there are a large number of local extreme points, which makes the estimation of model parameters very difficult. Traditional nonlinear least-squares estimation cannot estimate the parameters of this type of model, while optimization algorithms such as differential evolution and simulated annealing are prone to fall into a local minimum or lose the optimal solution in the optimization process.

As the core content of modern financial engineering and financial mathematics, option pricing has always been the research focus of the industry and academia. Based on the introduction of several classic option pricing methods, this paper introduces the Monte Carlo method with relative advantages, and discusses and realizes its application in option pricing. This paper first introduces the risk-neutral theory of modern option pricing, and then gives an analytical solution form represented by the Black-Scholes formula, and then further introduces the binary tree options pricing model and the finite difference option pricing model. In the application of the Monte Carlo method, the variance reduction technique can effectively improve the calculation accuracy. Based on the Monte Carlo method of sampling a large number of samples and finally averaging, the variance can be reduced by proper sampling. The focus of this paper is to introduce the variance reduction technique of importance sampling and give empirical results. Variance reduction techniques introduce common random numbers, dual variables, control variables, importance sampling, and stratified sampling. The variance reduction technique of importance sampling is giving important weights to some samples to reduce the variance. This paper directly applies the technique to the pricing practice of Asian options and compares the calculation results with the calculation results that have not been processed by the variance reduction technology to illustrate the effectiveness of the variance reduction technology.

Financial innovation, financial liberalization, and financial globalization in the financial market have promoted the variety of major financial derivatives such as options to become more and more diverse. At the same time, various types of customers have more and more personalized needs for financial instruments. Strange derivative securities have developed rapidly, and real options methods based on option pricing theory have also received more and more attention. These development trends make the financial market urgently need a powerful mathematical tool to solve the pricing problem of financial derivatives. The Monte Carlo method has the characteristics of simple program structure, probability of convergence, and convergence speed independent of the dimension of the problem and strong adaptability. It can make up for the deficiencies of grid analysis technology and finite difference technology in the face of high-dimensional problems. It is increasingly applied to option pricing.

The option is an important part of financial derivatives, and its pricing theory is one of the core issues of financial mathematics. How to provide option pricing theoretically has become a hot issue that many mathematicians, financial scientists, and other scholars pay attention to. This work passes the Esscher transformation to obtain a martingale measure equivalent to the original measure and then uses Cossanov's theorem, the transformation of pricing units, and other theories to study the pricing problems of several types of options. The main research work of this paper is 1. In risk neutrality under the measurement, first, we assume that the price of the underlying risk asset obeys the differential equation of the continuous diffusion process, defines digital power options, and extends the application range of power options. Then, with the help of the Esscher transformation theory, the Esscher transformation is taken as a derivative, determine another equivalent martingale measure, using Cossanov's theorem and other theories, we respectively get the pricing of two-way European options, geometric average Asian options, and digital power options. Under the true probability measurement, it is assumed that the price of the underlying asset obeys the differential equation of the jump-diffusion process. Through the Esscher transformation, the appropriate Esscher parameters are selected to obtain a martingale measurement equivalent to the original measure, which is the risk-neutral measure.

3 Models and Methods

There are a lot of objective or subjective uncertainties in the financial market, such as randomness and ambiguity. With the deepening of empirical research, people find that this uncertainty affects the decision-makers behavioral choices and then affects the changes in asset prices. People are paying more and more attention to how uncertainty can be better reflected in the model, to provide effective ideas and methods for decision-making. This work studies the option pricing problem under an uncertain environment puts forward some more realistic option pricing models and designs the corresponding solution algorithm. We discussed the existing option pricing models, made a detailed analysis and comparison of the research status, and pointed out the advantages and disadvantages, and improvements. The volatility of fuzzy variables is studied, discrete and continuous fuzzy volatility option pricing models are established respectively, and the symmetric pricing formulas of fuzzy European call and put options are given. According to fuzzy simulation technology, an algorithm for estimating the value of options is designed, and the application of the method is illustrated with examples. The uncertainty of the future value of securities is studied, the credibility theory is used to express investors' subjective inferences and trade-offs of the price of the underlying securities, and the uncertainty of the future value of the securities is expressed by the fuzzy process, and discrete-time and continuous-time American options are established.

3.1 Historical Volatility

From the perspective of calculation time, historical volatility calculates the volatility of the price of the underlying asset in the past period, while future volatility and predicted volatility calculates the volatility of the price of the underlying asset for some time in the future.

From the point of view of the calculation method: the implied volatility is calculated by the B-S formula, while the realized volatility is the standard deviation obtained by the variance calculation formula.

3.2 Implied Volatility

Implied volatility is the rate of change that helps investors to estimate the future price change of a particular asset. The difference with historical volatility is Implied volatility estimates the future price, while Historical volatility analyzes prices in the past. Implied volatility is calculated by substituting the corresponding option price into the Black-Scholes model, which reflects investors' expectations of future underlying securities volatility.

3.3 American Basket Options

A basket of options is a portfolio of multiple underlying assets. It is usually cheaper than the total value of a single asset option. Therefore a basket of options is more cost-efficient than a basket of individual options. With the increasing requirements of investors for the diversification of their portfolios, the demand for such portfolio options is also increasing. When the dimensionality continues to increase, using Monte Carlo simulation to price high-dimensional options is relatively less computationally expensive, because the computational complexity does not increase exponentially like other pricing methods. Monte Carlo simulation is only easy to solve the pricing problem of European options.

However, as far as we know, the most difficult problem to solve in the pricing of a basket of options for multiple underlying assets is its high dimensionality. To solve high-dimensional problems, we usually use the Monte Carlo simulation method and finite difference method which can also be used to price high-dimensional options, their storage and calculations increase exponentially with the increase of option dimension. For solving the problem that can be executed in advance, we often use the binary tree method and the finite difference method. This paper attempts to apply the feature of the finite difference method that can solve the pre-executable high-dimensional derivative securities (American style basket option) pricing problem. The basic steps are as follows: First, we use the finite difference method to analyze the optimal executable boundary of the American basket of options. According to the results of the analysis, a boundary with only one free parameter is proposed. By changing the value of the free parameter, the maximum expected discounted income is found and the optimal executable boundary in the form of parameters is found. This expected discounted return is based on the optimal executable boundary in the forms of parameters if the price of the American-style basket of options we seek.

4 Numerical Experiments

We can change the time steps M as well as the number of simulated paths. Meanwhile, we obtain the option price using simulation and FFT respectively. Viewing the price from FFT as a benchmark, we can calculate the relative price error to compare the price difference for different M and N. The option price obtained by FFT is $ 17.53. After calculating the relative error with N varying from 10 to 40000, we find out when N is not large enough, the average of our simulation is still fluctuating. At the same time, when N is too large, the result is fluctuating around the coverage value as well.

Choosing N from 10 to 10000, we calculated the moving average of the calculated relative error, which is indeed decreasing as N becomes larger, which makes me think that maybe we should do it several times with the same N and calculated the mean of the results. As the figure below indicates, the call option price converges when N approaches 10000.

As a result, we presented the results with N varying from 1250 to 40000. When N is above 10000, the relative error is small enough for M = (63, 126, 252).

From the Table 1, we can use N = 10000, M = 252 to get an accurate result since it can achieve a low relative error.

We use 2.91, the value obtained by setting N = 200000 as the benchmark value to compute the relative price error. And we need N = 40000 to get an accurate price. With control variate, we get c = −0.0994. As shown in the Table 2, when we price up-and-out calls using the European call as a control variate, the result is not so different from results obtained without a control variate. When N is not large enough, the average of our simulation is still fluctuating. At the same time, when N is too large, the result is fluctuating around the coverage value as well. It seems the result starts to converge when N is larger than 10000.

Here we compared the Black-Litterman model with Markowitz mean-variance portfolio based on model stability, net value performance, Sharpe ratio, maximum drawdown,

Table 1. Prices and Relative Errors under Different Sets of Hyperparameters

Price, Relative Error	M = 63	M = 126	M = 252
N = 1250	18.29; 4.31%	17.54; 0.08%	17.63; 0.58%
N = 2500	17.92; 2.26%	16.95; 3.33%	17.61; 4.79%
N = 5000	18.04; 2.94%	17.84; 1.77%	17.55; 0.13%
N = 10000	17.78; 1.48%	17.67; 0.82%	17.51; 0.11%
N = 20000	17.91; 2.21%	17.61; 0.47%	17.79; 1.47%
N = 40000	18.13; 3.40%	17.67; 0.82%	17.65; 0.67%

Table 2. Prices and Relative Errors without and with Control Variates

Price, Relative Error	Without Control Variate	With Control Variate
N = 1250	3.08; 6.05%	3.16; 8.58%
N = 2500	3.14; 8.00%	2.96; 1.72%
N = 5000	2.81; 3.17%	2.82; 2.92%
N = 10000	2.84; 2.33%	2.90; 0.30%
N = 20000	2.85; 2.00%	2.78; 1.03%
N = 40000	2.90; 0.97%	3.00; 3.19%
N = 80000	2.94; 1.17%	2.96; 1.69%

and the value. We can conclude that Markowitz portfolio is highly leveraged and portfolio allocations change drastically with small changes in the forecasts. The weights of Black-Litterman model are scaler-stable. The sum of absolute value of weight change in BL model remains within a small range. And under general good views, the Black-Litterman performance based on net value can always over perform Markowitz portfolio are generally higher and more volatile than that of CAPM portfolio.

5 Conclusion

As the implied volatility is determined by the options market price volatility, it is the exact reflection of the market price and the effective market price is the product of the relationship between demand and supply balance, so the implied volatility reflects the change of the market on the underlying product, which is applied extensively in option transactions.

Volatility can help investors to make their movements. Investors can profit by simply anticipating a rise (or fall) in volatility without having to worry too much about whether the underlying asset will rise or fall. For example, if you believe that the price of the underlying asset will fluctuate significantly over some time, the direction is unclear, but volatility will increase, then you can buy a straddle portfolio.

Using Monte Carlo simulation and binary tree method to price the American-style basket of options containing underlying assets, we draw the following conclusions: (1) The option prices obtained by these two numerical analysis methods are very high and close, which shows that we can also use Monte Carlo simulation to effectively price American options that can be executed in advance. (2) To solve the pricing problem of high-dimensional derivative securities of multi-standard assets, Monte Carlo simulation is a very effective numerical analysis method, and the estimated standard error and convergence speed are independent of the dimensionality of the problem so that it can be better used in the pricing of multi-variable high-dimensional derivative securities. (3) When the Monte Carlo simulation is used to price American options if other methods of thinking are incorporated, it will show greater advantages over the binary tree method and the finite difference method.

In addition, there is a class of arbitrage based on volatility also known as the volatility trade. Its trading strategy is to find the outliers of implied volatility based on the historical performance and current value of implied volatility. Making decisions like selling a portfolio with high implied volatility and buying a portfolio with low implied volatility in this outlier point and therefore they can benefit.

The price of options is affected by many factors, including both objective and subjective factors. In an environment where the market contains uncertain factors, the factors that affect option prices are not only random, but also vague. On the one hand, due to the interference of objective factors, some variables cannot be estimated from accurate data, and some data inevitably cause errors in observations and statistics, which has an uncertain effect; on the other hand, people are making market investments. It is inevitable to bring one's own subjective judgements, thereby affecting the market, which makes the pricing problem more complicated. This work uses traditional pricing ideas, based on the B-S model and the equivalent martingale measurement model. First, assuming that the stock price obeys a fuzzy random process, using fuzzy mathematics knowledge, the pricing formula of single-factor European options is obtained, the fuzzy price parity formula of European options is proved, and the binary tree discrete fuzzy model is also given for single-factor is also a fuzzy stochastic process. According to the fuzzy expansion operation, the fuzzy term structure of the interest rate is obtained.

References

1. Bouchard B, Touzi N (2004) Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. Stoch Process Appl 111(2):175–206
2. Broadie M, Glasserman P (1996) Estimating security price derivatives using simulation. Manag Sci 42(2):269–285
3. Brandimarte P (2014) Handbook in Monte Carlo Simulation: Applications in Financial Engineering, Risk Management, and Economics. John Wiley & Sons
4. Cheng S, Sanming L (2012) Option pricing analysis of Monte Carlo simulation based on black-scholes model. J Shanghai Dianji Univ 3
5. Cvetanoska V, Toni S (2012) Using high performance computing and Monte Carlo simulation for pricing American options. arXiv preprint arXiv:1205.0106
6. Gong H, Thavaneswaran A, Singh J (2010) A black-scholes model with garch volatility. Math Sci 35(1)

7. Giles MB (2008) Multilevel Monte Carlo path simulation. Oper Res 56(3):607–617
8. Jia Q (2009) Pricing American options using Monte Carlo methods
9. Khazraeian S, Hadi M (2018) Monte Carlo simulation-based benefit-cost analysis combined with analytical hierarchy process to support ITS investment with consideration of connected vehicle technology. Transp Res Rec 2672(19):1–12
10. Liu RH, Qian Z, Gang Y (2006) Option pricing in a regime-switching model using the fast Fourier transform. Int J Stoch Anal
11. Mehrdoust F, Babaei, and Somayeh Fallah. (2017) Efficient Monte Carlo option pricing under CEV model. Commun Stat Simul Comput 46(3):2254–2266
12. Shahbandarzadeh H, Khodakaram S, Reza M (2013) *Application of monte Carlo simulation in the assessment of european call options.* 7–25
13. Xie F, He Z, Wang X (2019) An importance sampling-based smoothing approach for quasi-Monte Carlo simulation of discrete barrier options. Eur J Oper Res 274(2):759–772