Comparative Analysis Between L-Moments and Maximum Product Spacing Method for Extreme PM_{10} Concentration

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Abstract. There are several times where Malaysia suffers from severe air pollution, especially the urban and industrial area. The air quality stations across the country monitor various variables of air pollutants including particulate matter such as PM_{10}. Due to harmful effects of pollution on human health and the environment, especially for extreme cases, air quality is a matter of worldwide concern amongst scientists, policy makers and public alike. In extreme value analysis, the generalized extreme value (GEV) distribution is widely adopted, and its parameters were estimated by various methods. Studies on these estimation methods are of great interest since reliable estimates are needed for modelling and forecasting extreme events. In this study, two methods based on order statistics are compared which are the L-moments (LMOM) and maximum product spacing (MPS) method. The L-moments method is a common method in extreme value analysis while MPS is considered as an alternative for maximum likelihood estimation (MLE) method. Both methods are applied on daily maximums of PM_{10} concentration at eight air quality monitoring stations in Peninsular Malaysia. Both methods provide a relatively close estimates and MPS is shown to be a reasonable alternative for parameter estimation of GEV distribution of extreme PM_{10} concentration in Malaysia.

Keywords: L-Moments (LMOM) · Maximum Product Spacing (MPS) · Generalized Extreme Value (GEV) · air quality

1 Introduction

Air pollutants are among the main factors that contribute to atmospheric pollution and ecosystem degradation. Particulate matter (PM) is an air pollutant which poses more danger to the population's health compared to ground level pollutants such as ozone and carbon monoxide [1]. Haze is an example of particulate pollution and Malaysia is among the worst affected country [2]. High particulate events are frequently associated with high

N. Wahi et al. (Eds.): ICMSS 2022, ACSR 98, pp. 214–227, 2023.
https://doi.org/10.2991/978-94-6463-014-5_21

concentration of PM_{10} that causes poor visibility and bad air quality conditions which effect negatively on human health, environment, and economy [3, 4]. Hence, modelling and forecasting these extreme particulate pollution events are of utmost importance for monitoring and prediction purposes. By proper air quality monitoring and accurate air quality prediction, defensive preparations to minimize threats and loss could be implemented.

Extreme value analysis allows for interpretation of past events and discussion of future occurrences such as by fitting probability distributions and making inferences on the probabilities of extreme events respectively [5]. Extreme value theory (EVT) is a subfield in extreme value analysis which addresses the probability distributions of extreme events [6]. EVT makes use of extreme value distributions that can be generalized into a three-parameter probability distribution that could represent the distribution of block maxima [7]. Block maxima refers to the maximum (or minimum) values of a given variable in a fix time. The GEV distribution had been shown to be suitable in representing the distribution of extreme air quality and pollutants in various countries such as Germany [8], India [9], Indonesia [10] and Malaysia [11].

Many parameter estimation methods for probability distribution functions are being used on extreme value analysis to estimate the parameters of the GEV distribution. Two of the most common methods are the method of moments (MOM) and the maximum likelihood estimation (MLE) method [12, 13]. However, these two classical methods are known to have weaknesses. For example, MOM which sets the moments of distribution function, derived from parameters of probability distribution, equal to the moments of the observed sample, and hence provide biased and inefficient estimators [14]. On the other hand, MLE estimators may not exist or may not be unique [15, 16]. In fact, the parameter estimation of GEV distribution using MLE sometimes results in a very high order of convergence [17]. Hence, methods which is based on order statistics such as the L-moments (LMOM) and the maximum product spacing (MPS) method have been explored to estimates parameter of probability distributions for extreme events [18, 19]. The aim of this study is to explore the suitability of both LMOM and MPS in estimating parameters for probability distribution of extreme PM_{10} concentrations in Peninsular Malaysia. Furthermore, the performance of both LMOM and MPS are compared to determine whether MPS is an appropriate method as an alternative for parameter estimation with respect to the popular LMOM method in extreme value analysis.

2 Methods

This study focuses on parameter estimation of the generalized extreme value distribution (GEV) using both the L-moments (LMOM) and maximum product spacing (MPS) method. Both LMOM and MPS methods are based on order statistics. The estimates obtained from both methods are then compared by looking at the differences between estimated parameters of GEV distribution and the root mean squared error (RMSE) between observed values of daily maximum PM10 concentrations and estimated values of daily maximum PM10 concentrations found from the GEV distribution with parameters estimated through both LMOM and MPS method.

Table 1. The individual family and support of the extreme value distributions with respect to GEV distribution.

Distribution	Range for κ	Support
Fréchet	$\kappa < 0$	$\mu + \alpha/\kappa \le x < \infty$
Gumbel	$\kappa = 0$	$x \in \mathbb{R}$
Weibull	$\kappa > 0$	$-\infty < x \le \mu + \alpha/\kappa$

2.1 Generalized Extreme Value (GEV) Distribution

Jenkinson in 1995 [20] combined the three families of extreme value distributions introduced by Fisher and Tippett in 1928 [21] into a generalized version of the probability distributions which is called the generalized extreme value (GEV) distribution. The distribution function of GEV is written as [22, 23]

$$F(x) = \begin{cases} \exp\left[-\left(1 - \frac{\kappa}{\alpha}(x-\xi)\right)^{1/\kappa}\right], & \kappa \neq 0 \\ \exp\left[-\exp\left(-\left(\frac{x-\xi}{\alpha}\right)\right)\right], & \kappa = 0 \end{cases} \tag{1}$$

with $\xi \in \mathbb{R}$, $\alpha > 0$ and $\kappa \in \mathbb{R}$ are the location, scale, and shape parameter, respectively. The individual family of extreme distributions is obtained based on the different range of values for the shape parameter, κ, as in Table 1.

In this study, the GEV distribution is fitted to the daily maximum of PM_{10} concentration. Hence, the relevant distribution is only the Fréchet and Gumbel distribution since the Weibull distribution has an upper limit on the values that it supports.

2.2 L-Moments (LMOM) Method

L-moments, λ, are constructed using linear combinations of expected values of ordered statistics and can be written as [23, 24]

$$\lambda_h = \frac{1}{h}\sum_{k=0}^{h-1}(-1)^k\binom{h-1}{k}E\left[X_{h-k:h}\right], h = 1, 2, \ldots \tag{2}$$

With

$$E[X_{(j:h)}] = \frac{h!}{(j-1)!(h-j)!}\int x(F(x))^{(j-1)}(1-F(x))^{(h-j)}dF(x) \tag{3}$$

L-skewness, τ_3, can then be found by taking the ratio of λ_3 to λ_2; i.e.

$$\tau_3 = \frac{\lambda_3}{\lambda_2} \tag{4}$$

The approximation for L-moments in Eq. (2) and L-skewness in Eq. (4) which are known as sample L-moments and sample L-skewness respectively, can be found from a finite sample of size n, arranged in ascending order, $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{n:n}$. The sample L-moments, l, are written as

$$l_{h+1} = \sum\nolimits_{k=0}^{h} p_{h,k} b_k,\ h = 0, 1, \ldots, n-1 \tag{5}$$

with

$$p_{h,k} = (-1)^{h-k} \binom{h}{k} \binom{h+1}{k} \tag{6}$$

and

$$b_k = \frac{1}{n} \binom{n-1}{k}^{-1} \sum\nolimits_{j=k+1}^{n} \binom{j-1}{k} X_{j:n} \tag{7}$$

Thus, the sample L-skewness, t_3, can then be found by taking the ratio of l_3 to l_2 such that

$$t_3 = \frac{l_3}{l_2} \tag{8}$$

Then, by substituting Eq. (1) in Eq. (3) and equating the population L-moments in Eq. (2) to their corresponding sample L-moments in Eq. (5), the parameter estimates for the GEV distribution can be found as follows [25]:

$$\begin{gathered} \hat{\kappa} = 7.8590c + 2.9554c^2,\ \text{with}\ c = \tfrac{2}{(3+t_3)} - \tfrac{log\,2}{log\,3} \\ \hat{\alpha} = \tfrac{(l_2\hat{\kappa})}{(1-2^{-\hat{\kappa}})\Gamma(1+\hat{\kappa})}\ \hat{\xi} = l_1 - \tfrac{\hat{\alpha}}{\hat{\kappa}}\left[1 - \Gamma(1+\hat{\kappa})\right] \end{gathered} \tag{9}$$

2.3 Maximum Product Spacing (MPS) Method

Parameter estimation based on order statistics known as maximum product spacing (MPS) method was introduced by Cheng and Amin in 1983 [26]. For an ordered sequence, $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{n:n}$, , for any continuous random variable with the cumulative distribution function (CDF)$F(x; \boldsymbol{\theta})$, the space between two CDF of the consecutive points can be denoted as $Z_i(\boldsymbol{\theta})$ where $\boldsymbol{\theta}$ refers to the parameters in the CDF [19]. Hence, for the CDF of GEV distribution shown in Eq. (1), the spacing between two consecutive points can written as [17].

$$\begin{gathered} Z_i, (\xi, \alpha, \kappa) = F(x_{i:n}; \xi, \alpha, \kappa) - F(x_{i-1:n}; \xi, \alpha, \kappa), \\ i = 1, 2, \ldots, n+1 \end{gathered} \tag{10}$$

where $F(x_0; \xi, \alpha, \kappa) = 0$ and $F(x_{n+1}; \xi, \alpha, \kappa) = 1$.

The MPS method finds the optimum estimates for the parameters, ξ, α and κ, by maximizing the geometric mean of the spacings in Eq. (10) with respect to each of the parameter. In other words, the estimates are the values of the parameters that maximize the following

$$G(\xi, \alpha, \kappa) = \left\{ \prod_{i=1}^{n+1} Z_i(\xi, \alpha, \kappa) \right\}^{1/(n+1)} \tag{11}$$

or equivalently

$$H(\xi, \alpha, \kappa) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log Z_i(\xi, \alpha, \kappa) \tag{12}$$

with respect to ξ, α and κ.

Hence, the estimators of ξ, α and κ are found by solving the following nonlinear equations:

$$\frac{\delta H(\xi, \alpha, \kappa) =}{\delta \xi} 0, \frac{\delta H(\xi, \alpha, \kappa) =}{\delta \alpha} 0, \frac{\delta H(\xi, \alpha, \kappa) =}{\delta \kappa} 0 \tag{13}$$

2.4 Root Mean Squared Error (RMSE)

The root mean squared error (RMSE) id a goodness-of-fit index which can be written as

$$\mathrm{RMSE} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(x_{i:n} - \hat{x}_{i:n} \right)^2} \tag{14}$$

with n is the sample size or the length of the observed daily maximum of PM_{10} concentration at each station, $x_{i:n}$ is the observed values of daily maximum of PM_{10} concentration and $\hat{x}_{i:n}$ is the estimated values of daily maximum of PM_{10} concentration obtained from the GEV distribution with the estimated parameters from LMOM and MPS methods. The smaller value of RMSE shows that the GEV distribution produced by the estimated parameters is better in representing the distribution of daily maximum PM_{10} at the station.

3 Data

This study used the daily maximum of PM_{10} concentration from eight air quality monitoring station in Peninsular Malaysia. The data is obtained from the Department of Environment Malaysia. The duration of daily maximum of PM_{10} concentration used is from 5th of July 2017 to 31st January 2019. The location as well as the minimum, median, mean and maximum values of PM_{10} concentration at all eight stations are given in Table 2. Meanwhile, the locations of the eight air quality monitoring stations are also shown in Fig. 1.

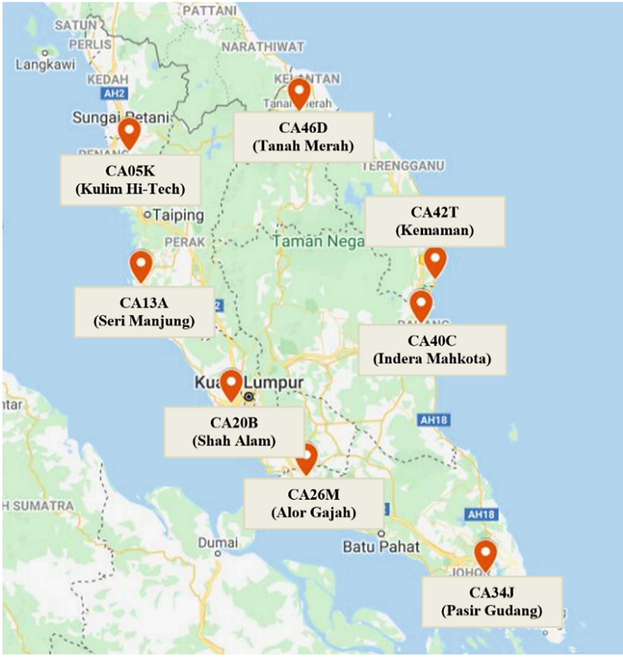

Fig. 1. Locations of the eight air quality monitoring stations used in this study.

Table 2. The minimum, median, mean and maximum values of PM_{10} concentration at each station

Station ID	Location	Daily Maximum PM_{10} Concentrations ($\mu g/m^3$)			
		Minimum	Median	Mean	Maximum
CA05K	Kulim Hi-Tech, Kedah	10.00	29.67	34.83	188.36
CA13A	Seri Manjong, Perak	11.00	33.00	38.36	197.18
CA20B	Shah Alam, Selangor	14.00	42.62	45.58	168.49
CA26M	Alor Gajah, Melaka	10.99	30.00	33.91	132.53
CA34J	Pasir Gudang, Johor	11.08	35.85	39.18	199.95
CA40C	Indera Mahkota, Pahang	7.46	24.46	28.28	273.87
CA42T	Kemaman, Terengganu	9.00	29.04	34.68	488.44
CA46D	Tanah Merah, Kelantan	8.95	33.26	38.30	176.82

4 Results and Discussion

The GEV distribution is fitted to the daily maximum of PM_{10} concentrations at each of the air quality monitoring station in Peninsular Malaysia. Both the LMOM and MPS methods are used to estimate parameters of the GEV distribution at each of these stations.

Then, the PM_{10} values obtained from the GEV distribution with the estimated parameters are compared to the observed values of daily maximum of PM_{10} concentrations by calculating the values of RMSE as in Eq. (14). Table 3 shows the estimated values for all three parameters of the GEV distribution and the RMSE values obtained for both LMOM and MPS methods at all eight stations. The cumulative distribution plots of PM_{10} concentrations for the observed values as well as estimated values found from the GEV distribution with parameters estimated from both LMOM and MPS methods are shown in Fig. 2.

Based on Table 3, all three estimated parameters of the GEV distribution (location, scale, and shape parameters) obtained from both the LMOM and MPS methods are almost similar in values. Both LMOM and MPS methods was able to provide good estimates for the parameters of GEV distribution that represents the distribution of daily maximum PM_{10} concentration in Peninsular Malaysia. This can be seen from the very small values of RMSE which are all less than 15 which indicates a very small error with respect to the average values of daily maximum PM_{10} concentrations at all eight air quality monitoring stations.

There are six stations with RMSE value from LMOM method smaller than RMSE value from MPS method. Meanwhile, there are two stations with RMSE value from the MPS method smaller than RMSE value from the LMOM method. However, the differences between the RMSE values from both methods at all eight stations are very small with a range between 0.010875 and 3.837654. This is because the values for the estimated parameters are also very close to each other. Thus, this implies that the MPS method is able to produce estimated parameters that are comparable to the estimated parameters obtained through LMOM method. This further indicates that the MPS method is a suitable alternative in parameter estimation of GEV distribution since LMOM method is a common and popular method used for fitting probability distributions of extreme events.

Based on Fig. 2, the GEV distribution with estimated parameters from both the LMOM and MPS method are shown to be able to fit the distribution of daily maximum PM_{10} concentrations well at all eight stations. Both methods are able to capture the general pattern of the distribution of daily maximum PM_{10} concentration. Aside from the closeness shown from the distribution obtained and the observed distribution in Fig. 2, the percentage of errors, which indicates the percentage of difference between the observed values of PM_{10} and the corresponding values found from the GEV distribution with parameters estimated from both LMOM and MPS method, are also given in Fig. 3. Based on Fig. 3, the percentage of errors are very small, which is less than 10%, in the middle and for most values of probabilities in the distribution at all eight stations. However, the percentage of errors are higher at the tails of the distribution. Nevertheless, almost all the percentage of errors are less than 50% for all stations under study.

Hence, the GEV distribution is a suitable probability distribution to represent the extreme event of high PM_{10} concentration in Peninsular Malaysia and both the LMOM and MPS methods are adequate to estimate parameters for the GEV distribution.

The plots for return level against return period of daily maximum PM_{10} based on GEV distribution with parameters estimated from LMOM and MPS method are shown in Fig. 4.

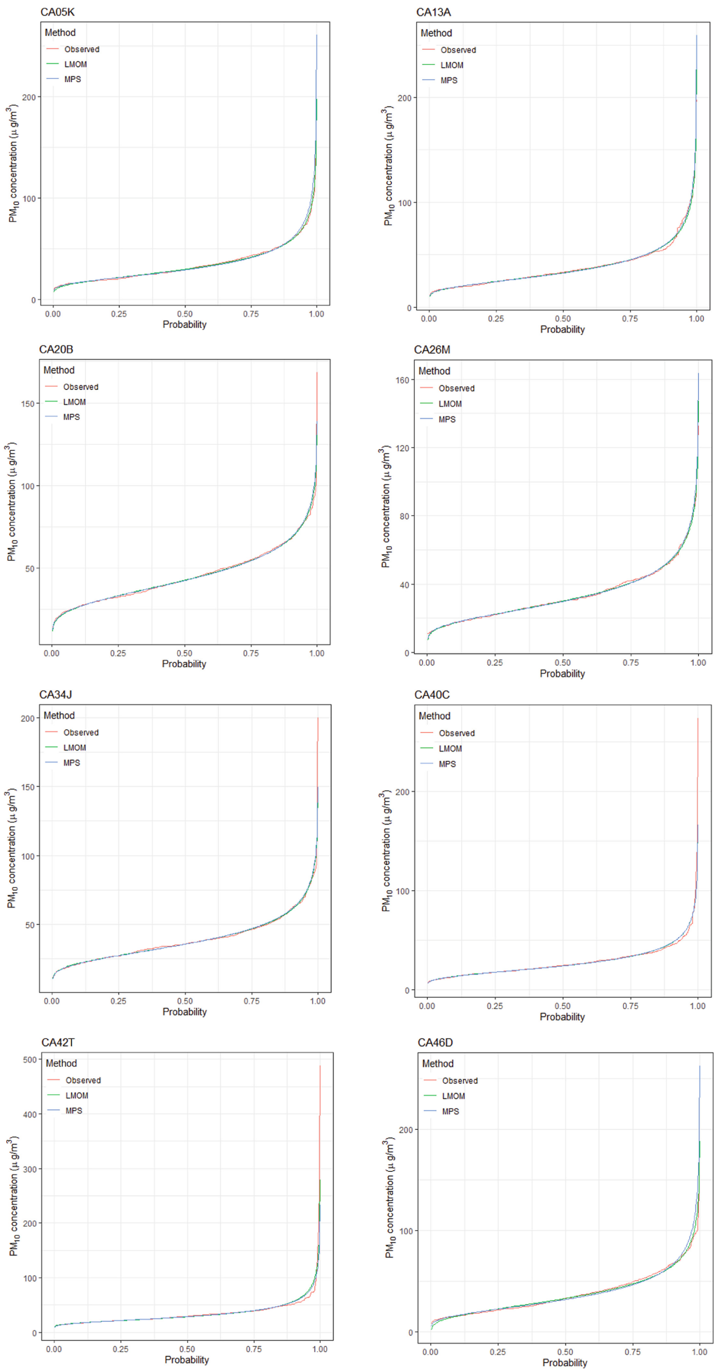

Fig. 2. Cumulative distribution plots for the values of daily maximum PM_{10} concentration and estimated values of PM_{10} concentration found from GEV distribution with estimated parameters from LMOM and MPS method, respectively.

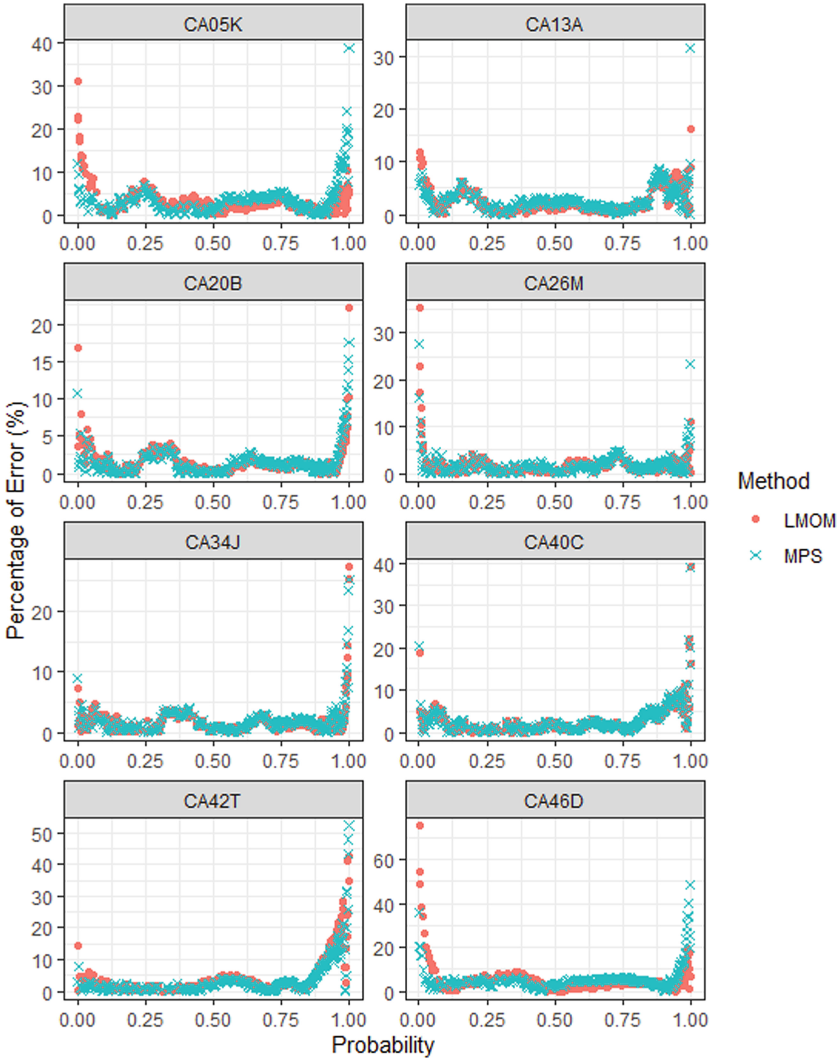

Fig. 3. Percentage of error for observed values of PM_{10} and values of PM_{10} obtained from GEV distribution with parameters estimated using LMOM and MPS method.

Since the Malaysian Ambient Air Quality Guidelines (MAAQG) states that the standard limit for the average level of PM_{10} is 150 $\mu g/m^3$ [26], hence the return period for PM_{10} to exceed 150 $\mu g/m^3$ is of great concern. According to Fig. 4, the chances for the concentration of PM_{10} exceeding 150 $\mu g/m^3$ in any given year is highly unlikely for four stations located in the central and southern region of Peninsular Malaysia. The return period for PM_{10} concentration to be 150 $\mu g/m^3$ is more than 500 years at these stations. Meanwhile, at three other stations, the return period for a 150 $\mu g/m^3$ concentration of PM_{10} is smaller based on the curves obtained using MPS method

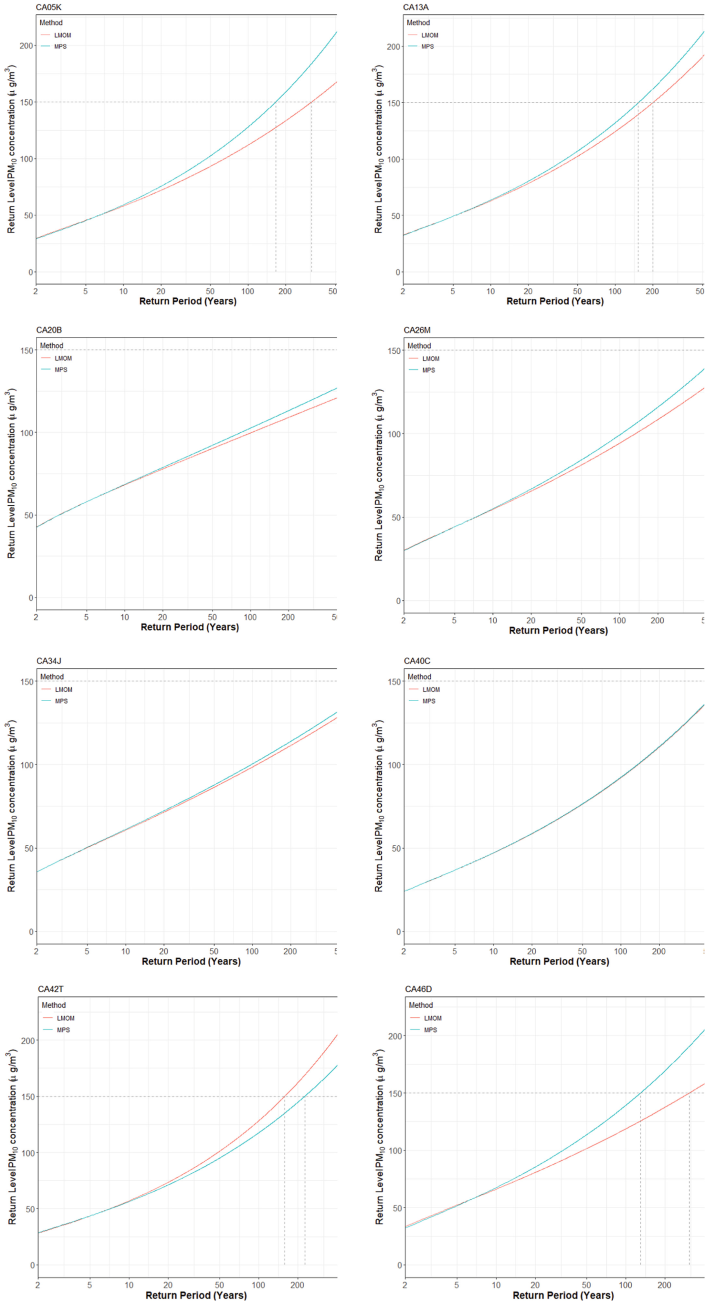

Fig. 4. The return level versus return period plot for daily maximum PM_{10} concentration based on the GEV distribution with estimated parameters from LMOM and MPS method respectively.

Table 3. Values for the estimated parameters and RMSE found from both the LMOM and MPS methods

Station	Method	GEV Parameter Estimates			RMSE
		ξ	α	κ	
CA05K Kulim Hi-Tech, Kedah	LMOM	25.4001 (24.44, 26.21)	11.5149 (10.53, 12.34)	−0.1986 (−0.26, −0.12)	1.385491*
	MPS	24.9523 (23.99, 25.83)	10.8436 (9.98, 11.52)	−0.2845 (−0.35, −0.21)	4.366889
CA13A Seri Manjung, Perak	LMOM	28.1299 (27.00, 29.23)	11.7971 (10.84, 12.67)	−0.2294 (−0.28, −0.15)	2.274520*
	MPS	27.9258 (26.80, 29.04)	11.5887 (10.66, 12.48)	−0.2652 (−0.32, −0.19)	3.102025
CA20B Shah Alam, Selangor	LMOM	37.8065 (36.41, 39.03)	13.5002 (12.50, 14.56)	0 (−0.05, 0)	1.939787*
	MPS	37.6370 (36.31, 38.86)	13.3097 (12.39, 14.26)	−0.0260 (−0.09, 0)	2.005001
CA26M Alor Gajah, Melaka	LMOM	26.0229 (25.23, 26.90)	10.9973 (10.18, 11.86)	−0.1246 (−0.17, −0.05)	0.999383*
	MPS	25.8387 (24.97, 26.73)	10.7544 (9.98, 11.42)	−0.1618 (−0.22, −0.10)	1.726709
CA34J Pasir Gudang, Johor	LMOM	31.2144 (30.18, 32.31)	11.7954 (10.97, 12.65)	−0.0908 (−0.17, −0.03)	3.114310
	MPS	31.1463 (30.19, 32.26)	11.9352 (11.19, 12.63)	−0.0976 (−0.15, −0.04)	2.995499*
CA40C Indera Mahkota, Pahang	LMOM	20.5780 (19.78, 21.48)	9.1505 (8.46, 9.89)	−0.2137 (−0.31, −0.11)	5.124209
	MPS	20.5534 (19.81, 21.45)	9.2158 (8.60, 9.89)	−0.2126 (−0.29, −0.14)	5.113334*
CA42T Kemaman, Terengganu	LMOM	24.4621 (23.64, 25.57)	9.8248 (8.78, 10.78)	−0.3227 (−0.44, −0.15)	11.85039*
	MPS	24.6507 (23.92, 25.62)	10.1931 (9.51, 11.10)	−0.2702 (−0.35, −0.19)	14.11819
CA46D Tanah Merah, Kelantan	LMOM	27.7679 (26.24, 29.05)	14.8092 (13.54, 15.89)	−0.1205 (−0.18, −0.07)	2.363351*
	MPS	27.0110 (25.56, 28.35)	13.6443 (12.47, 14.69)	−0.2325 (−0.30, −0.18)	6.201005

Note: The values in the brackets refer to the 95% confidence interval of the estimated parameters

(130 years to 166 years) compared to the curves found using LMOM method (200 years to 323 years). The opposite is true for CA42T with LMOM and MPS show a return period of 159 years and 226 years respectively for an event with 150 $\mu g/m^3$ concentration of PM_{10}. Nevertheless, both methods indicate that the average time between the extreme event of 150 $\mu g/m^3$ concentration of PM_{10} is more than 130 years which is less than 1% in any given year.

5 Conclusion

The GEV distribution is used in this study to represent the daily maximum of PM_{10} concentrations at eight air quality monitoring stations in Peninsular Malaysia. The GEV distribution is fitted to the observed data at each station by estimating parameters of the GEV distribution using both the LMOM and MPS methods. The suitability of the GEV distribution obtained is checked by using the values of RMSE between the observed values of PM_{10} concentrations and the estimated values of PM_{10} concentrations based on the GEV distribution with the estimated parameters as the goodness-of-fit index. The RMSE values are shown to be very small with respect to the average values of PM_{10} concentrations at all the stations under study. Furthermore, the cumulative distribution plots for the observed values and the estimated values of PM_{10} concentrations from both methods are also shown to be very close to each other. Both the RMSE values and the cumulative plots indicate that the GEV distribution is a good probability distribution for extreme PM_{10} in Peninsular Malaysia. The estimated parameters of GEV distribution found from both LMOM and MPS methods are very similar to each other in term of values and produce small differences in terms of RMSE. This implies that the MPS method is suitable and a good alternative to the popular LMOM method in estimating parameters for extreme distributions. The suitability of the MPS method could be further investigated for other extreme probability distributions in further research.

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