Lie Symmetries, Optimal System, and Invariant Solutions of the Generalized Cox-Ingersoll-Ross Equation

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Abstract. The Cox-Ingersoll-Ross (CIR) model is a short-rate model and is widely used in the finance field to predict the movement of interest rates in bond pricing models. This paper exploited Lie symmetry analysis to solve the generalized CIR model by determining the infinitesimal generators. Lie symmetry is one of the powerful tools to solve the partial differential equation (PDE) analytically by reducing the PDE into a lower form. Besides, an optimal system of one-dimensional subalgebras is constructed and then used to reduce the generalized CIR equation by introducing the similarity variables. Lastly, the invariant solutions are obtained by solving the reduced equation.

Keywords: Cox-Ingersoll-Ross (CIR) model · Lie symmetry analysis · Optimal system · invariant solutions

1 Introduction

In modern financial analysis, partial differential equations (PDE) are often applied to model a real-world problem. In 1973, Black and Scholes [1] introduced the option pricing model by partial differential equation also known as the Black-Scholes equation. Vasicek [2] in 1977, described the movement of interest rates in bond pricing models as a PDE. However, there is a limitation of the Vasicek model which allow negative interest rate in the calculation which is an unforeseen situation in any economic field.

The Cox-Ingersoll-Ross (CIR) model [3] was derived to cover the shortcoming of the Vasicek model. The CIR model only allows positive interest rates in the calculation. The function $u(x, t)$ define as the zero-coupon bond price and the PDE is given

$$u_t + \frac{1}{2}\sigma^2 x u_{xx} + \kappa(\theta - x)u_x - xu = 0, \tag{1}$$

where x is the interest rate, t is denoted as time. The volatility (σ), rate of mean reversion (κ) and long-term mean variance (θ) are real and positive constants. Khalique and Motsepa [4] proposed new group invariant solutions of the generalized Vasicek model

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by changing $\alpha = \frac{\sigma^2}{2}$ and $\gamma = -1$ in the Vasicek equation through the Lie symmetry method. This paper will study the solutions of the generalized CIR equation. According to the change of variables in [4], the generalized CIR equation is given as

$$u_t + \alpha x u_{xx} + \kappa(\theta - x)u_x + \gamma x u = 0. \tag{2}$$

There are many methods to solve PDE. Unfortunately, not all methods can obtain the analytical solution of PDE. In this paper, Lie symmetry analysis is chosen to obtain the exact solutions of Eq. (2). Lie symmetry is one of the most powerful methods to solve ordinary differential equations (ODE) and PDE by reducing the differential equation into a lower order [5, 6]. Plenty of researchers from various fields applied the symmetry method to solve a particular differential equation and the results are convincing. The Lie symmetry analysis was first introduced in the finance field to solve the Black-Scholes (BS) equation and transform the BS equation into heat equations [7]. Liu and Wang [8] applied the symmetry method to solve the BS equation with dividend yield. The solutions of the Asian option which satisfies the terminal condition has been obtained via Lie symmetry analysis [9]. In recent years, Kaibe and O'Hara [10] studied the zero-coupon pricing equation to determine the symmetries point to reduce the equation into the ODE and hence, the solutions are obtained. Some works and examples of Lie symmetry to solve nonlinear PDE and system of PDE can be obtained from [11–14].

The paper is organized as follows; In Sect. 2, the steps of the symmetry method and symmetries point of the generalized CIR equation are presented. The derivation of the optimal system of the generalized CIR equation by the symmetries point is discussed in Sect. 3. In Sect. 4, the group invariant solutions of the generalized CIR equation are obtained. Finally, Sect. 5 contains the conclusion.

2 Lie Symmetry of the Generalized CIR Equation

To solve Eq. (2) by Lie symmetry method, one must determine the symmetries point of the Eq. (2). Finding the symmetries point is similar as determining the infinitesimal generators of the equation. The general infinitesimal generators are given by

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}, \tag{3}$$

with infinitesimals ξ, τ and η are functions of variables x and t. These infinitesimal generators (3) are obtained if and only if it satisfies the Lie's invariance condition

$$\Gamma^{(2)}\Delta\,|_{\Delta=0} = 0, \tag{4}$$

where Δ is the PDE and $\Gamma^{(2)}$ is defined as

$$\Gamma^{(2)} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{tt}\frac{\partial}{\partial u_{tt}} + \eta^{xt}\frac{\partial}{\partial u_{xt}},$$

and the η^x, η^t, η^{xx}, η^{tt} and η^{xt} represent the second prolongation

$$\eta^x = D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau),$$

$$\eta^t = D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau),$$

$$\eta^{xx} = D_x\left(\eta^x\right) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau),$$

$$\eta^{tt} = D_t\left(\eta^t\right) - u_{xt} D_t(\xi) - u_{tt} D_t(\tau),$$

$$\eta^{xt} = D_x\left(\eta^t\right) - u_{xt} D_x(\xi) - u_{tt} D_x(\tau),$$

with the total differential operators

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t},$$
$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_x}.$$

Substituting all the prolongations formula in the Eq. (4) and comparing the coefficients of u will lead to a system of determining equations. Solving the determining equations will yield the infinitesimal generators.

The manual calculation of calculating the infinitesimals ξ, τ and η in Eq. (3) are difficult and require much works. Some researchers had proposed some packages to compute the symmetries point for differential equations through various mathematics software, see [15–18]. In this paper, the *MathLie* in Mathematica software [16] is utilized to determine the infinitesimal generators of Eq. (2). The infinitesimal generators are

$$\begin{aligned} \xi &= x\left(e^{t\beta} c_1 + e^{-t\beta} c_2\right), \\ \tau &= c_3 + \frac{e^{t\beta} c_1}{\beta} - \frac{e^{-t\beta} c_2}{\beta}, \\ \eta &= u\left(c_4 + e^{t\beta} c_1(\mathcal{P}x + \mathcal{Q}) + e^{-t\beta} c_2(\mathcal{R}x + \mathcal{S})\right) + \phi(x, t) \end{aligned} \tag{5}$$

where c_1, c_2, c_3, c_4 are any random constants, ϕ is any function that satisfies Eq. (2) and

$$\beta = \sqrt{\kappa^2 - 4\alpha\gamma}, \qquad \mathcal{P} = \frac{\kappa + \beta}{2\alpha},$$
$$\mathcal{Q} = \frac{-\kappa\theta(\kappa + \beta)}{2\alpha\beta}, \qquad \mathcal{R} = \frac{\beta - \kappa}{-2\alpha},$$
$$\mathcal{S} = \frac{\kappa\theta(\kappa - \beta)}{2\alpha\beta}. \tag{6}$$

Separating constants c_1 to c_4 and rearrange Eq. (5) into the form of Eq. (3) gives the following generators,

$$X_1 = e^{t\beta}\left(x\partial x + \frac{1}{\beta}\partial t + u(\mathcal{P}x + \mathcal{Q})\partial u\right),$$

$$X_2 = e^{-t\beta}\left(x\partial x - \frac{1}{\beta}\partial t + u(\mathcal{R}x + \mathcal{S})\partial u\right),$$

$$X_3 = \partial t,$$

$$X_4 = u\partial u,$$

$$X_\phi = \partial u. \tag{7}$$

3 Optimal System of the Generalized CIR Equation

Different linear combinations of Eq. (7) will give various solutions and it is impossible to list down all the group invariant solutions. Several works of constructing an optimal system of one-dimensional subalgebras by Olver's method [6] can be read from [4, 19–21]. These involve calculating the commutator and adjoint representations tables of the differential equation.

3.1 Computation of Commutator Table

A commutator of symmetries group is also known as Lie bracket. Consider two symmetries group X_i and X_j for a one-dimensional PDE, where $i, j = 1, 2, 3, \ldots$. The formula of computing the commutator provided by Olver [6] is given by

$$\left[X_i, X_j\right] = \left(X_i\xi_j - X_j\xi_i\right)\partial x + \left(X_i\tau_j - X_j\tau_i\right)\partial t + \left(X_i\eta_j - X_j\eta_i\right)\partial u,$$

where $\xi_i, \xi_j, \tau_i, \tau_j, \eta_i$ and η_j are derivatives of the infinitesimals. The Lie bracket has some properties such that it is skew-symmetric, $\left[X_i, X_j\right] = -[X_j, X_i]$ and the diagonal elements in the commutator table are all zero, $[X_i, X_i] = 0$. The commutator table of Eq. (2) is shown in Table 1.

Table 1. Commutator table of the generalized CIR equation

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$\frac{2}{\beta}X_3 - \frac{\kappa^2\theta}{\alpha\beta}X_4$	$-\beta X_1$	0
X_2	$-\frac{2}{\beta}X_3 + \frac{\kappa^2\theta}{\alpha\beta}X_4$	0	βX_2	0
X_3	βX_1	$-\beta X_2$	0	0
X_4	0	0	0	0

3.2 Adjoint Representations

The adjoint representation of the Eq. (2) is given by

$$Ad(\exp(\varepsilon X_i))X_j = \sum\nolimits_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(adX_i)^n\left(X_j\right)$$

$$= X_j - \varepsilon\left[X_i, X_j\right] + \frac{\varepsilon^2}{2!}\left[X_i, \left[X_i, X_j\right]\right] - \ldots,$$

The adjoint representation table of Eq. (2) is displayed in Table 2.

3.3 Construction of the Optimal System

To construct the optimal system set of invariant solutions, let the linear combinations of the infinitesimal generators (5) be

$$X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4, \tag{8}$$

with λ_1 to λ_4 are any random coefficients. Follow Olver's method [6], in order to obtain the optimal system of Eq. (2), the coefficients in Eq. (8) must be simplify as much as possible.

Case 1: Assume $\lambda_1 \neq 0$, and let $\lambda_1 = 1$.
Equation (8) becomes

$$X = X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4. \tag{9}$$

Referring to Table 2, the generator X_3 can be vanished by solving $X^{'} = Ad(\varepsilon X_2)X$. After simplified the generator X_3, it gives

$$X^{'} = X_1 + \lambda_2^{'} X_2 + \lambda_4^{'} X_4,$$

where X', λ' are the different versions of X and λ. Continue to act on X' by $X'' = Ad(\varepsilon X_3)X'$ and lead to

$$X^{''} = X_1 + e^{2\varepsilon\beta}\lambda_2^{''} X_2 + e^{\varepsilon\beta}\lambda_4^{''} X_4.$$

Table 2. Adjoint representation table of the generalized CIR equation

$Ad[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	X_1	$X_2 - \varepsilon^2 X_1 - \varepsilon\left(\frac{2}{\beta}X_3 - \frac{\kappa^2\theta}{\alpha\beta}X_4\right)$	$X_3 + \varepsilon\beta X_1$	X_4
X_2	$X_1 - \varepsilon^2 X_2 - \varepsilon\left(-\frac{2}{\beta}X_3 + \frac{\kappa^2\theta}{\alpha\beta}X_4\right)$	X_2	$X_3 - \varepsilon\beta X_2$	X_4
X_3	$e^{-\varepsilon\beta}X_1$	$e^{\varepsilon\beta}X_2$	X_3	X_4
X_4	X_1	X_2	X_3	X_4

Taking the coefficients of X_2 and X_4 as ± 1 and hence Eq. (9) can be simplified to either

$$X_1 + X_2 + X_4, X_1 + X_2 - X_4,$$

$$X_1 - X_2 + X_4, X_1 - X_2 - X_4. \tag{10}$$

No further simplification can be made for $\lambda_1 \neq 0$.

Case 2: Assume $\lambda_1 = 0$, $\lambda_2 \neq 0$, and take $\lambda_2 = 1$.
The linear combination generators are given by

$$X = X_2 + \lambda_3 X_3 + \lambda_4 X_4.$$

The generator X_2 can be simplified and yield.

$$X^{'} = \lambda_3 X_3 + \lambda_4 X_4.$$

It is also equivalent to the scalar multiple of

$$X^{'} = aX_3 + bX_4, \tag{11}$$

where a and b are any real constants. No further simplifications are possible for second case.

Case 3: Assume $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$, and let $\lambda_3 = 1$.
Referring to Table 2, no further simplification, Eq. (8) becomes

$$X = X_3 + \lambda_4 X_4,$$

and it is also equal to

$$X = X_3 + aX_4, \tag{12}$$

where a is any random constant. Note that the scalar multiple of $aX_3 + bX_4$ and $X_3 + aX_4$ are similar by taking 1 as the coefficient of X_3.

Case 4: Assume $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 \neq 0$, and take $\lambda_4 = 1$.
Equation (8) can be simplified to

$$X = X_4. \tag{13}$$

To sum up, the optimal system of Eq. (2) is given by

$$\{X_1 + X_2 + X_4, \quad X_1 + X_2 - X_4, \quad X_1 - X_2 + X_4,$$

$$X_1 - X_2 - X_4, X_3 + aX_4, X_4\}, \qquad \text{where a } \in \mathbb{R}.$$

4 Symmetry Reduction and Group Invariant Solutions of the Generalized CIR Equation

Lie symmetry is widely applied is due to it can reduce the order of the original equation by introducing the similarity variables via the symmetries point and hence make the equation easier to solve. A function of PDE $u = u(x, t)$ is invariant under its symmetries group, if and only if it satisfies the invariant surface condition

$$\xi(x, t, u)u_x + \tau(x, t, u)u_t = \eta(x, t, u). \tag{14}$$

Taking ξ and τ are not both zero, Eq. (14) can be solved through the characteristic method,

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}, \tag{15}$$

which will introduce the similarity variables and lead to a reduction of the PDE to an ODE,

$$u = F(r), \tag{16}$$

where r is a function of x and t. Solving the function $F(r)$ will give the invariant solutions of the PDE. In this section, we give some invariant solutions of the generalized CIR equation.

Case 1: $X_1 + X_2 + X_4$.

Recalling the generators in Eq. (7), the linear combination generators of $X_1+X_2+X_4$ is given by

$$\begin{aligned} X =& x\left(e^{t\beta} + e^{-t\beta}\right)\frac{\partial}{\partial x} + \frac{1}{\beta}\left(e^{t\beta} - e^{-t\beta}\right)\frac{\partial}{\partial t} \\ &+ u\left(e^{t\beta}(\mathcal{P}x + \mathcal{Q}) + e^{-t\beta}(\mathcal{R}x + \mathcal{S}) + 1\right)\frac{\partial}{\partial u}, \end{aligned}$$

and its characteristics equation are

$$\frac{dx}{x\left(e^{t\beta} + e^{-t\beta}\right)} = \frac{dt}{\frac{1}{\beta}\left(e^{t\beta} - e^{-t\beta}\right)} = \frac{du}{u\left(e^{t\beta}(\mathcal{P}x + \mathcal{Q}) + e^{-t\beta}(\mathcal{R}x + \mathcal{S}) + 1\right)},$$

provides the similarity variables and the similarity equation,

$$u = e^{\frac{-\mathcal{R}x - \mathcal{Q}t\beta + e^{2t\beta}(\mathcal{P}x + \mathcal{Q}t\beta)}{-1+e^{2t\beta}}}\left(1 - e^{-t\beta}\right)^{\frac{1}{2}(1+\mathcal{Q}+\mathcal{S})}\left(1 + e^{-t\beta}\right)^{\frac{1}{2}(-1+\mathcal{Q}+\mathcal{S})}F(\omega), \tag{17}$$

where

$$\omega = \frac{x}{e^{-t\beta} - e^{t\beta}}. \tag{18}$$

Differentiate and substitute Eq. (17) into Eq. (2) gives the reduced equation

$$\alpha^2\beta^2\omega F''(\omega) + \kappa\alpha\theta\beta^2 F'(\omega) + \left(\omega\left(-\kappa^4 + 8\kappa^2\alpha\gamma - 16\alpha^2\gamma^2\right) - \alpha\beta^3\right)F(\omega) = 0.$$

Solving the reduced equation and substitute into Eq. (17) lead to the invariant solution for $X_1 + X_2 + X_4$,

$$u(x,t) = e^{\frac{\frac{x\beta\left(1+e^{2t\beta}\right)}{-1+e^{2t\beta}} - \kappa^2 t\theta + \kappa(x - t\beta\theta)}{2\alpha}} \left(1 - e^{-t\beta}\right)^{\frac{\alpha-\kappa\theta}{2\alpha}} \left(1 + e^{-t\beta}\right)^{-\frac{\alpha+\kappa\theta}{2\alpha}}$$

$$\left(\begin{array}{l} c_1 e^{\frac{x\beta}{\alpha\left(e^{-t\beta}-e^{t\beta}\right)}} KummerM\,(m,n,z) \left(\frac{e^{t\beta}x}{1-e^{2t\beta}}\right)^{1-\frac{\kappa\theta}{\alpha}} \\ + c_2 e^{\frac{x\beta}{\alpha\left(e^{-t\beta}-e^{t\beta}\right)}} KummerU\,(m,n,z) \left(\frac{e^{t\beta}x}{1-e^{2t\beta}}\right)^{1-\frac{\kappa\theta}{\alpha}} \end{array} \right), \tag{19}$$

with

$$m = \frac{\alpha - \kappa\theta}{2\alpha}, n = \frac{-\kappa\theta + 2\alpha}{\alpha}, z = \frac{2e^{t\beta}x\beta}{\alpha\left(-1 + e^{2t\beta}\right)}, \tag{20}$$

c_1, c_2 are any arbitrary constants, *KummerM* and *KummerU* are the confluent hypergeometric functions [22] and $\beta = \sqrt{\kappa^2 - 4\alpha\gamma}$.

Case 2: $X_3 + aX_4$.

From the generators (7), the infinitesimals $X_3 + aX_4$ are given by

$$X = \frac{\partial}{\partial t} + au\frac{\partial}{\partial u},$$

of which characteristics equation are

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{au},$$

gives the similarity equation as

$$u = e^{at}F(x). \tag{21}$$

Taking Eq. (21) into Eq. (2) yields an ODE,

$$\alpha x F''(x) + \kappa(\theta - x)F'(x) + a\gamma x F(x) = 0.$$

Solving the ODE, lead to the invariant solution for infinitesimals $X_3 + aX_4$,

$$u(x,t) = e^{at}\left(c_1 e^{-\frac{x(-\kappa+\beta)}{2\alpha}} KummerM\,(m,n,z) x^{\frac{-\kappa\theta+\alpha}{\alpha}} + c_2 e^{-\frac{x(-\kappa+\beta)}{2\alpha}} KummerU\,(m,n,z) x^{\frac{-\kappa\theta+\alpha}{\alpha}}\right), \tag{22}$$

with c_1, c_2 are any real constants and m, n, z are represent

$$m = \frac{-\beta\kappa\theta - \kappa^2\theta + 2\alpha\beta - 2a\alpha}{2\alpha\beta}, n = \frac{-\kappa\theta + 2\alpha}{\alpha}, z = \frac{x\beta}{\alpha},$$

and β is defined in Sect. 2.

Case 3: X_4

Recalling Eq. (7), the infinitesimal X_4 is

$$X = u\frac{\partial}{\partial u},$$

with the transformed ODE

$$x\alpha F''(x) + \kappa(\theta - x)F^{'}(x) + x\gamma F(x) = 0.$$

Integrating the ODE gives the invariant solution of Eq. (2) generated by infinitesimal X_4,

$$\begin{aligned} u(x,t) = & c_1 e^{-\frac{x(\beta-\kappa)}{2\alpha}} KummerM(m,n,z) x^{\frac{-\kappa\theta+\alpha}{\alpha}} \\ & + c_2 e^{-\frac{x(\beta-\kappa)}{2\alpha}} KummerU(m,n,z) x^{\frac{-\kappa\theta+\alpha}{\alpha}}, \end{aligned} \tag{23}$$

where c_1, c_2 are arbitrary constants, β is defined as above and other parameters are

$$m = -\frac{\kappa^2\theta + \beta(\kappa\theta - 2\alpha)}{2\alpha\beta}, n = \frac{-\kappa\theta + 2\alpha}{\alpha}, z = \frac{x\beta}{\alpha}.$$

The infinitesimal generators of $X_1 + X_2 - X_4, X_1 - X_2 + X_4$ and $X_1 - X_2 - X_4$ will generate a much more complicated invariant solution. We are still working on it and the results will be reported later.

5 Conclusion

In conclusion, Lie symmetry is applied to determine the invariant solution of the generalized CIR equation. The Eq. (2) admitted four symmetries point and one infinite-dimensional subalgebra X_ϕ. The symmetries point obtained were then used to compute the commutators and adjoint representation tables to construct the optimal system of Eq. (2). With the optimal system found, the symmetry reduction was performed and hence the new group invariant solutions of Eq. (2) were calculated. Other solutions admitted by other generators will be studied later.

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