Prime Power Noncoprime Graph and Probability for Some Finite Groups

Nurfarah Zulkifli(✉) and Nor Muhainiah Mohd Ali

Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, UTM, 81310 Johor Bahru, Johor, Malaysia
nurfarah3@graduate.utm.my

Abstract. The study of coprime probabilities and graphs have its own uniqueness that produces a particular pattern according to its variabilities. Some obvious results can be seen from previous research where the domination number will always be equal to one and the types of graphs that can be formed are either star, planar or r-partite graph depending on certain cases. For the probability, the results vary according to the groups and certain cases need to be considered. The noncoprime graph has been introduced and it is defined as a graph associated to the group G with vertex set $G\backslash\{e\}$ such that it is possible that two separate vertices are adjacent when the orders are relatively noncoprime. However, in probability theory, the study of noncoprime probability of a group has not been introduced yet. Hence, a thorough study has been conducted where the goal of this research is to introduce a newly defined graph and probability which are the prime power noncoprime graph and prime power noncoprime probability of a group. The focus of this approach is that the greatest common divisor of the order of x and y, where x and y are in G, is equal to a power of prime number. In this paper, the scope of the group is mainly focused on some dihedral groups, quasi-dihedral groups, and some generalized quaternion groups. Some invariants, which are the diameter, girth, clique number, chromatic number, domination number, and independence number of prime power noncoprime graph are found. Additionally, the generalization of the prime power noncoprime probability are also obtained.

Keywords: Prime Power Noncoprime Probability · Prime Power Noncoprime Graph · Dihedral Groups · Quasi-dihedral Groups · Generalized Quaternion Groups

1 Introduction

The study of graphs, which are made up of vertex connected by edges, is known as graph theory. Graph theory is significant and can be used in a wide variety of fields of study, including mathematics, operational research, computer science, chemistry, and social sciences. Over the years, many graphs theory studies have been conducted using various approaches, and in this paper, the research focuses exclusively on the extension of noncoprime graphs.

N. Wahi et al. (Eds.): ICMSS 2022, ACSR 98, pp. 11–23, 2023.
https://doi.org/10.2991/978-94-6463-014-5_3

Various types of graphs have been defined over time. Williams [1] discovered the prime graph of a group for the first time in 1981. The prime graph of a group G is a graph in which the vertices are the prime numbers dividing the order of G and two vertices x and y are joined by an edge if and only if G contains an element of order xy. The prime graph study has increased rapidly in recent years, with researchers extending the prime graph using various groups to learn more about this topic.

After several years, Ma et al. [2] introduced another graph called the coprime graph of a group, which is expressed as given below.

Definition 1: [2] **Coprime graph of a group**

The coprime graph of a group G, Γ_G, is a graph where vertices are elements of G, while two distinct vertices x and y are adjacent if and only if $\gcd(|x|, |y|) = 1$.

The characteristics such as the diameter, partition, clique number and planarity of a coprime graph, along with the graph types, for example complete, regular, star, planar and so on were determine in the study. In the same year, Dorbidi [3] pursued the research done by [2] and proved that the chromatic number of the coprime graph of G equals its clique number.

The study of the coprime graph has grown substantially interesting, and researchers have begun to dig deeper into the subject by considering specific situations involving the pair of elements x and y in which x and y in G, are not equal to 1. It is referred to as a noncoprime graph, where the definition is given below.

Definition 2: [4] **Noncoprime Graph**

Let G be a group. The noncoprime graph of G is a graph having vertex set G\e with two distinct vertices x and y joined by an edge when $\gcd(|x|, |y|) \neq 1$. The noncoprime graph of G is denoted by Π_G.

Mansoori et al. [4] discussed the topic of the noncoprime graph specifically on the properties of the graph, for example, the diameter, girth, domination number, and chromatic numbers as well as independence number, for some finite groups. Simultaneously, the authors also proved that G is planar if and only if G is isomorphic to one of the groups $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_5$, $\mathbb{Z}_6$ or S_3. For nilpotent groups, G is regular if and only if G is a p-group, in which p is prime number. Apart from that, they also discussed on the relation gained between prime and noncoprime graph.

Next, Rilwan et al. [5] analyzed some invariants such as the girth, circumference, clique, and chromatic number of the noncoprime graph of integers. In their paper, they also proved that the bounds of the domination number, independence number, and independent domination number are sharp. In addition to the research on noncoprime graph, other interesting results also have been obtained by Misuki et al. [6] and in their paper, a different scope of group has been used, which is the dihedral group, D_n, in which n is a prime power. The results were separated into two cases. Firstly, if n is a power of even prime. Then a complete graph is generated. On the other hand, when n is a power of odd prime, then the types of graphs obtained are partitioned into two complete graphs.

The study of the noncoprime graph was later continued by Aghababaei et al. [7], who studied the concept of a noncoprime graph of a finite group with regards to a subgroup. They explored the invariants of the graph for Mathieu and nilpotent groups, as well as the isomorphisms for nilpotent groups. The results stated that the noncoprime graph for finite nilpotent groups, G_1 and G_2, are isomorphic if and only if the order of those finite nilpotent groups is the same.

The extension of the coprime graph, which has been extensively investigated in various domains for many groups, inspired the idea of the coprime probability. In 2018, Abd Rhani [8] introduced the concept of coprime probability of a group and it is defined in Definition 3. In the research, the author determined the coprime probability generalization of p-groups and dihedral groups, D_n in which n is odd as stated below.

Definition 3 [8] **: Coprime Probability of *G***

Let G be a finite group. For any $x, y \in G$, the coprime probability of G expressed as $P_{copr}(G)$, is defined as

$$P_{copr}(G) = \frac{|\{(x, y) \in G \times G : (|x|, |y|) = 1\}|}{|G|^2}.$$

Proposition 1 [8]: Let G be a finite p-groups of order p^n, in which $n \geq 1$. It follows that

$$P_{copr}(G) = \frac{2p^n - 1}{2p^n}.$$

Proposition 2 [8]: Let D_n be a dihedral group of order $2n$, in which $n \geq 3$ and n is odd. Then

$$P_{copr}(D_n) = \frac{2n^2 + 2n - 1}{4n^2}.$$

Zulkifli and Mohd Ali [9] conducted an extensive study in this part which they determined the coprime probability for nonabelian metabelian groups of order less than 24. Here, it may be discovered that if G has the same order, then the coprime probability of nonabelian metabelian groups of order less than 24 are the same. This only happens with certain orders. Following from there, similar approach is employed, except that [10] seeks to compute the coprime probability for nonabelian metabelian groups of order 24. The obtained results revealed that the coprime probability varies for nonabelian metabelian groups of order 24.

The coprime probability was further extended to the relative coprime probability by Zulkifli and Mohd Ali [11], who were interested in studying and understanding the pattern that may possibly be identified within a group. The following is the definition of the relative coprime probability of G.

Definition 4 [11] **: Relative Coprime Probability of *G***

Provided that G be a finite group while H be a subgroup of G. Here, the relative coprime probability of G is defined as given below.

$$P_{copr}(H, G) = \frac{|\{(h, g) \in H \times G : (|h|, |g|) = 1\}|}{|H||G|}.$$

In the study, the target group was the nonabelian metabelian groups of order less than 24. As a result of the findings, the relative coprime probability of nonabelian metabelian groups of order less than 24 are the same if G has the same order. This case happened only to certain orders.

Another paper has been published on this subject too, but the scope of study is targeting on the dihedral group, D_n in which n is an odd prime, [12]. In the study, H is set to be a cyclic subgroup of D_n. If $|H| = 1$, it follows that $P_{copr}(H, D_n) = 1$ whereas if $|H| = 2$ then $P_{copr}(H, D_n) = \frac{3}{4}$. Finally, when $|H| = n$, it follows that $P_{copr}(H, D_n) = \frac{n^2+2n-1}{2n^2}$.

Based on the graphs discussed earlier, namely prime graph, coprime graph, and noncoprime graph, these three topics are finally combined to form a new graph related to prime number and greatest common divisor(gcd). In this research, the new graph emphasizes on the $\gcd(|x|, |y|) = p^s$ where p is prime, $s \in \mathbb{N}$ and it is denoted as prime power noncoprime graph. Inspired by the study of the noncoprime graph, a new probability is also introduced in this research which is the prime power noncoprime probability of a group. The generalization of the prime power noncoprime probability and prime power noncoprime graph together with some properties which include the types of graphs, the diameter, clique number, independence number, domination number, and chromatic number are also gained. On top of everything, throughout this research, prime numbers will often be mentioned and used in many cases, so in this study, the prime number is understood to be prime.

Thus, the first part of this paper covers the research's introduction whereas the second part lays out the fundamental concepts and results for both groups and graphs theory which are essential in this study. Interesting results and conclusion for both the prime power noncoprime probability and the prime power noncoprime graph for some finite groups are discussed in the third and fourth sections, respectively.

2 Preliminaries

Some fundamental concepts about groups and graphs theory are stated in this section and will be used throughout this research.

Definition 5 [13] Dihedral Groups of Degree n

Dihedral Groups, D_n in which the order of D_n is $2n$ for every $n \in \mathbb{Z}$ as well as $n \geq 3$, is expressed as the set of symmetries of a regular n-gon. Here, the dihedral groups, D_n can be represented in the following representation:

$$D_n = \left\langle a, b : a^n = b^2 = e, ba = a^{-1}b \right\rangle.$$

Definition 6 [14] Quasi-dihedral Groups

The quasi-dihedral groups, QD_{2^n}, of order 2^n where $n \geq 4$ is generated by two elements a and b. The quasi-dihedral groups, QD_{2^n} can be represented in the following representation:

$$QD_{2^n} = \left\langle a, b : a^{2n-1} = b^2 = e, ba = a^{2^{n-2}-1}b \right\rangle.$$

Definition 7 [15] Generalized Quaternion Groups

The group of generalized quaternions, Q_{4n}, of order $4n$ where $n \geq 1$ is generated by two elements a and b. The generalized quaternion group, Q_{4n}, can be represented in the following representation:

$$Q_{4n} = \left\langle a, b : a^n = b^2, a^{2n} = e, b^{-1}ab = a^{-1} \right\rangle.$$

Definition 8 [16] Complete Graph

A graph in which each ordered pair of distinct vertices are adjacent is called a complete graph, K_n, and in this case, n is the number of connected vertices.

Definition 9 [16] Diameter

The diameter. $diam(\Gamma)$, of a connected graph Γ denotes the greatest distance between all pairs of the vertices of Γ.

Definition 10 [16] Chromatic Number

The chromatic number of Γ, denoted as $\chi(\Gamma)$, is defined as the smallest number of colors required to color the vertices of Γ provided that no two adjacent vertices have the same color.

Definition 11 [16] Clique Number

The clique number of Γ, denoted by $\omega(\Gamma)$, is the size of the largest complete subgraph in Γ.

Definition 12 [16] Domination Number

The domination set $X \subseteq V(\Gamma)$ is a set where for each y outside X, there exists $x \in X$ such that y is adjacent to x. The minimum size of X is called the domination number and it is denoted by $\gamma(\Gamma)$.

Definition 13 [16] Independence Number

A non-empty set S of $V(\Gamma)$ is known as an independent set of Γ provided there do not exists any adjacent vertices between two elements of $S \in \Gamma$. Thus, the independence number denotes the number of vertices in the maximum independent set, which is expressed as $\alpha(\Gamma)$.

3 Main Results

This section is split into two parts. The first section introduces a newly defined probability known as the prime power noncoprime probability, while the second part introduces a newly defined graph known as the prime power noncoprime graph. The results for both the probability and the graph are generalized, as well as the graph's properties. Throughout this research, some dihedral groups, quasi-dihedral groups, and generalized quaternion groups are the targeted group in this investigation. Simultaneously, an example is also provided to enhance the comprehensiveness of this research.

3.1 Prime Power Noncoprime Probability of a Group

Previous research on the coprime probability and its extension have focused on the coprime, which also means the greatest common divisor (gcd) of two numbers is equal to one. In this research, another aspect of the coprime has been examined, namely when the gcd is not equal to one. To be more specific, this research discussed and explained the case when the gcd is equal to p^s, in which p is prime and $s \in \mathbb{N}$. The formal definition is given below, along with an example for a better understanding of the concept.

Definition 14: Prime Power Noncoprime Probability of a Group

Let G be a finite group. Here, for any $x, y \in G$, p is prime as well as $s \in \mathbb{N}$, the prime power noncoprime probability of G, denoted by $P_{noncopr}(G)$, is expressed as

$$P_{noncopr}(G) = \frac{|\{(x, y) \in G \times G : (|x|, |y|) = p^s\}|}{|G|^2}.$$

Example 1

Let $G = D_3 = \{e, a, a^2, b, ab, a^2b\}$ while each element's order is, $|e| = 1$, $|a| = |a^2| = 3$, and $|b| = |ab| = |a^2b| = 2$. Let $W = \{(x, y) \in G \times G : (|x|, |y|) = p^s\}$ where p is prime and $s \in \mathbb{N}$. In order to determine the prime power noncoprime probability of D_3, several circumstances must be considered.

Case 1:

Let $x, y \in G$. If $x = y = e$, then $|x| = |y| = 1$. It is obvious that x and y are not a noncoprime since $\gcd(|x|, |y|) = \gcd(1, 1) = 1 \neq p^s$.

Case 2:

Let $x, y \in G \backslash e$ and $|x| = |y|$. If $|x| = |y| = 2$, then the $\gcd(|x|, |y|) = \gcd(2, 2) = 2$. Therefore, $W_1 = \{(x, y) \in G \times G : (|x|, |y|) = 2\}$, where x and y are either b, ab or a^2b. This implies $|W_1| = 9$. Also, If $|x| = |y| = 3$, then the $\gcd(|x|, |y|) = \gcd(3, 3) = 3$. Therefore, $W_2 = \{(x, y) \in G \times G : (|x|, |y|) = 3\}$, where x and y are either a or a^2. This implies $|W_2| = 4$.

Case 3:

If $x = e$, then $|x| = 1$ and for $y \in G$, then $|y| = 1, 2$ or 3. So, the $\gcd(1, |y|) = 1$, and 1 is not a prime number. Same goes to if $y = e$, then $|y| = 1$ and for $x \in G$, then $|x| = 1, 2$ or 3. So, the $\gcd(|x|, 1) = 1$. Hence, x and y are relatively prime which is also not a noncoprime. Now, let $x, y \in G \backslash e$. If $|x| = 2$ and $|y| = 3$, then the $\gcd(|x|, |y|) = \gcd(2, 3) = 1$. Same goes to when $|x| = 3$ and $|y| = 2$. Hence, x and y are relatively prime which is also not a noncoprime.

Hence,

$$\begin{aligned} P_{noncopr}(G) &= \frac{|W_1| + |W_2|}{|G|^2} \\ &= \frac{9 + 4}{36} \\ &= \frac{13}{36}. \end{aligned}$$

Finally, the generalization of the prime power noncoprime probability for some dihedral groups, D_n are discussed in Theorem 1 and 2.

Theorem 1: Let G be a dihedral group, D_n where n is prime. Then $P_{noncopr}(G) = \frac{1}{2} + \frac{1-2n}{4n^2}$.

Proof: Let G be a dihedral group, D_n in which n is prime and $D_n = \langle a, b \mid a^n = b^2 = e, bab = a^{-1} \rangle = \{e, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$. Also, each element's order in G is either 1, 2 or n. Now, let $W = \{(x, y) \in G \times G : (|x|, |y|) = p^s\}$ where p is prime and $s \in \mathbb{N}$. Following that, some circumstances must be taken into account in order to determine the prime power noncoprime probability for D_n, where n is prime.

Case 1:

For $x, y \in G$, if $x = y = e$ then it is obvious that $(|x|, |y|) = 1 \neq p^s$. Now, let $x = e$, and $y \in G \backslash e$, then all elements in $y \in G \backslash e$ are not noncoprime to x since $(|e|, |y|) = 1 \neq p^s$. Same goes to when $y = e$ and $x \in G \backslash e$. Therefore, x and y are not noncoprime.

Case 2:

Let $x, y \in G \backslash e$. If x and y are two non-identical elements of G, then $|x|$ and $|y|$ are either 2 or n. Now, if $|x| \neq |y|$, then, $(|x|, |y|) = 1 \neq p^s$. Therefore, x and y are not noncoprime. But if $|x| = |y|$, then $(|x|, |y|) = p^s$ such that $(2, 2) = 2$ and $(n, n) = n$. Finally, the total pair of elements of x and y that are prime power noncoprime is $2n^2 - 2n + 1$.

Therefore,

$$
\begin{aligned}
P_{noncopr}(G) =& \frac{2n^2 - 2n + 1}{4n^2} \\
=& \frac{1}{2} + \frac{1 - 2n}{4n^2}.
\end{aligned}
$$

Lemma 1: Let x be an element of a finite group G. If $|x| = 1$ or 2^i where $1 \leq i \leq m-1$ and $m \geq 5$, then $P_{copr}(G) = \frac{2^{m+1}-1}{2^{2m}}$.

Proof: Let x be an element of a finite group. For $|x| = 1$ or 2^i where $1 \leq i \leq m-1$, when $|x|$ divides $|G|$, the possible order of G is 2^m. In this case, G is a 2-group. Hence, by Proposition 1,

$$
\begin{aligned}
P_{copr}(G) =& \frac{2p^m - 1}{p^{2m}} \\
=& \frac{2(2)^m - 1}{(2)^{2m}} \\
=& \frac{2^{m+1} - 1}{2^{2m}}.
\end{aligned}
$$

Theorem 2: Let G be a dihedral group, D_n, where $n = 2^{m-1}$ and $m \geq 5$. Then $P_{noncopr}(G) = 1 - \frac{4n-1}{4n^2}$.

Proof: Let G be a dihedral group, D_n, where $n = 2^{m-1}$ and $m \geq 5$. Now, let $x \in G$. When $|x|$ divides $|G| = 2n$, then order for each element in G are either 1 or 2^i, where $1 \leq i \leq m-1$. So, the possible cases that can be considered in the case of $(|x|, |y|)$ are when $|x| = |y| = 1$ and $|x| \neq |y|$. Hence, by Lemma 1, $P_{copr}(G) = \frac{2^{m+1}-1}{2^{2m}}$. There is another case that needs to be considered and that is when $x, y \in G \backslash e$, $|x|$ and $|y|$ are 2^i, where $1 \leq i \leq m-1$. In this case, x and y are noncoprime since $(|x|, |y|) \neq 1$ but $(|x|, |y|) = 2^i$. Therefore,

$$\begin{aligned} P_{copr}(G) + P_{noncopr}(G) =& 1 \\ =& 1 - P_{copr}(G) \\ =& 1 - \frac{2p^m - 1}{p^{2m}} \\ =& 1 - \frac{2(2)^m - 1}{(2)^{2m}} \\ =& 1 - \frac{2^{m+1} - 1}{2^{2m}} \\ =& 1 - \frac{4n - 1}{4n^2}. \end{aligned}$$

Proposition 3: For quasi-dihedral group, QD_{2n} and generalized quaternion group, Q_{2n}, where $n = 2^{m-1}$ and $m \geq 5$, if $|D_n| = |QD_{2n}| = |Q_{2n}|$, then $P_{noncopr}(D_n) = P_{noncopr}(QD_{2n}) = P_{noncopr}(Q_{2n}) = 1 - \frac{4n-1}{4n^2}$.

Proof: For quasi-dihedral group, QD_{2n} and generalized quaternion group, Q_{2n}, where $n = 2^{m-1}$ and $m \geq 5$, let x be an element of this group, so $|x|$ are either 1 or 2^i, where $1 \leq i \leq m-1$. By Lemma 1, $P_{copr}(QD_{2n}) = P_{copr}(Q_{2n}) = \frac{2^{m+1}-1}{2^{2m}} = \frac{4n-1}{4n^2}$. In this circumstances, these two groups exhibits the same pattern as the dihedral group, D_n, where $n = 2^{m-1}$ and $m \geq 5$, such that the order of these three groups are the same that are 1 and 2^i, where $1 \leq i \leq m-1$. As a result, the numbers on its (x, y) are the same. Hence, by Theorem 2, $P_{noncopr}(D_n) = P_{noncopr}(QD_{2n}) = P_{noncopr}(Q_{2n}) = 1 - \frac{4n-1}{4n^2}$.

Next, the introduction of a newly defined graph named the prime power noncoprime graph for some dihedral groups, D_n are discussed.

3.2 Prime Power Noncoprime Graph of a Group

Noncoprime is understood as greatest common divisor (gcd) of order of two elements is not equal to one. Moreover, in this case, several approaches can be made in this situation because a pair of elements x and y may be equal to any number except 1. As a result, particular attention has been given in this study to the condition where gcd of x and y is equal to p^s, in which p is prime and $s \in \mathbb{N}$. It is also known as the prime power noncoprime graph. The definition of the prime power noncoprime graph is given in Definition 15 and at the same time, graph's characteristics such as the clique number, independence number, domination number, diameter and chromatic number are also discussed in this research. Throughout this research, the target is to study the prime power noncoprime graph for some dihedral groups.

Definition 15: Prime Power Noncoprime Graph of a Group

The prime power noncoprime graph of a group G denoted as $\Pi_{noncopr}(G)$ is a graph whose vertices are elements of $G\backslash e$ and two distinct vertices x and y in G are adjacent if and only if $(|x|, |y|) = p^s$ in which $x, y \in G\backslash e$, p is prime and $s \in \mathbb{N}$.

Throughout this research, the element e is excluded. Next, the types of graphs for the prime power noncoprime graph for some dihedral groups, D_n are discussed in Theorem 3 and 4.

Theorem 3: Let G be a dihedral group, D_n where n is prime. Then $\Pi_{noncopr}(G)$ is a disconnected graph with $K_n \cup K_{n-1}$.

Proof: Let $G = D_n = \langle a, b | a^n = b^2 = e, bab = a^{-1} \rangle = \{e, aa^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$. For $g \in G$, each element's order is either 1, 2 or n. By the Definition of $\Pi_{noncopr}(G)$, $(|x|, |y|) = p^s$ where $x, y \in G$, p is prime and $s \in \mathbb{N}$. So, there are two cases that need to be considered.

Case 1:

Let $x, y \in G\backslash e$, and $|x| = |y|$. If the order of x and y are either 2 or n,, then $(|x|, |y|) = (2, 2) = 2$ and $(|x|, |y|) = (n, n) = n$. Therefore, adjacencies exist between each vertex of the same order. This implies that two components exist through the difference between the order of x which is 2 and n. Hence, a complete subgraph are formed in each component.

Case 2: Let $x, y \in G$ and $|x| \neq |y|$. Given that the order of each element of G is either 2 or n. So, there is no adjacencies between two elements x and y since $(|x|, |y|) = 1 \neq p^s$.

Hence, it can be concluded that $\Pi_{noncopr}(G)$ is a disconnected graph with two components since they are separated according to the elements' order in G given by 2 or n. Also, each component produces a complete graph and it is proven by Case 1. Hence, $\Pi_{noncopr}(G) = K_n \cup K_{n-1}$.

Theorem 4: Let G be a dihedral group, D_n, where $n = 2^{m-1}$ and $m \geq 5$. Then the $\Pi_{noncopr}(G)$ is a complete graph with $K_{|G|-1}$.

Proof: Let G be a dihedral group, D_n, where $n = 2^{m-1}$ and $m \geq 5$. Let $x, y \in G\backslash e$ and their orders are 2^i where $1 \leq i \leq m-1$. Hence, $(|x|, |y|) = 2^i$ which means there are edges incident to every vertex in a graph. $\Pi_{noncopr}(G)$ is a complete graph with $K_{|G|-1}$.

Next, the properties such as the diameter, clique number, chromatic number, domination number, and independence number of the prime power noncoprime graphs for some dihedral groups are discussed in Theorem 5 to 11.

Theorem 5: Let G be a dihedral group, D_n where n is prime. Then the $diam(\Pi_{noncopr}(G))$ is infinite.

Proof: By Theorem 3, $\Pi_{\theta copr}(G)$ is a disconnected graph. By the definition of diameter, it is obvious that the $diam(\Pi_{noncopr}(G))$ is infinite.

Theorem 6: Let G be a dihedral group, D_n, where $n = 2^{m-1}$ and $m \geq 5$. Then the $diam\big(\Pi_{noncopr}(G)\big) = 1$.

Proof: By the definition of a diameter of a graph, the longest chain in this graph that connects from one vertex to another vertex is one. This is because each vertex has adjacent to all other vertex since $\Pi_{noncopr}(G)$ is a complete graph by Theorem 4. Hence, the $diam\big(\Pi_{noncopr}(G)\big) = 1$.

Theorem 7: Let G be a dihedral group, D_n where n is prime. Then $\omega\big(\Pi_{noncopr}(G)\big) = \chi\left(\Pi_{noncopr}(G)\right) = n$.

Proof: By Theorem 3, $\Pi_{noncopr}(G)$ is a disconnected graph with $K_n \cup K_{n-1}$. Since each component is a complete graph, hence there are edges incident to every vertex of each component. By Definition 10, the largest complete subgraph determined the minimum colors need to color the vertex of the subgraph such that no two adjacent have the same color. Therefore,

$$\begin{aligned}\omega\big(\Pi_{noncopr}(G)\big) =&\chi\left(\Pi_{noncopr}(G)\right)\\ =&\max\{V(K_n), V(K_{n-1})\}\\ =&\max\{n, n-1\}\\ =&n.\end{aligned}$$

Theorem 8: Let G be a dihedral group, D_n, where $n = 2^{m-1}$ and $m \geq 5$. Then the $\chi\left(\Pi_{noncopr}(G)\right) = \omega\big(\Pi_{noncopr}(G)\big) = |G| - 1$.

Proof: By Theorem 4, a complete graph with $K_{|G|-1}$ is obtained. Therefore, the size of the largest complete subgraph that can be obtained is $|G| - 1$. Simultaneously, the minimum colors required to color the vertex of a graph such that no two adjacent vertices have the same color is also $|G| - 1$. This is because each vertex is adjacent to every other vertex. Hence, $\chi\left(\Pi_{noncopr}(G)\right) = \chi\left(\Pi_{noncopr}(G)\right) = \omega\big(\Pi_{\theta noncopr}(G)\big) = |G| - 1$.

Theorem 9: Let G be a dihedral group, D_n where n is prime. Then $\gamma\left(\Gamma_{noncopr}(G)\right) = 2$.

Proof: By Theorem 3, $\Pi_{noncopr}(G)$ is a disconnected graph with $K_n \cup K_{n-1}$. Since each component is a complete graph, then the minimum vertex in each component needed to form an adjacency to all other element is one. Since there are two components in $\Pi_{noncopr}(G)$, therefore, $\gamma\left(\Pi_{noncopr}(G)\right) = 2$.

Theorem 10: Let G be a dihedral group, D_n where n is prime. Then $\alpha\big(\Pi_{noncopr}(G)\big) = 2$.

Proof: By the definition of independence number, $\alpha\big(\Pi_{noncopr}(G)\big) = 2$ since $\Pi_{noncopr}(G)$ formed a disconnected graph with two components and each component is a complete graph with $K_n \cup K_{n-1}$. It is obvious that there are edges incident to every vertex for each component. Therefore, the independent set are formed through the two difference components.

Theorem 11: Let G be a dihedral group, D_n, where $n = 2^{m-1}$ and $m \geq 5$. Then the $\gamma\left(\Pi_{noncopr}(G)\right) = \alpha\left(\Pi_{noncopr}(G)\right) = 1$.

Proof: By Theorem 4, $\Pi_{noncopr}(G)$ is a complete graph with $K_{|G|-1}$. Let $x \in G\backslash e$. So, the maximum independent set that can be obtained is $\{x\}$ since each vertex is adjacent to every other vertex. At the same time, the minimum vertex required to generate an adjacency to all other elements is also one. Hence, the $\gamma\left(\Pi_{noncopr}(G)\right) = \alpha\left(\Pi_{noncopr}(G)\right) = 1$.

Proposition 4: For quasi-dihedral group, QD_{2n} and generalized quaternion group, Q_{2n}, where $n = 2^{m-1}$ and $m \geq 5$, if $|D_n| = |QD_{2n}| = |Q_{2n}|$, then

1) The $\Pi_{noncopr}(G)$ is a complete graph with $K_{|G|-1}$.
2) The $diam\left(\Pi_{noncopr}(G)\right) = 1$.
3) The $\chi\left(\Pi_{noncopr}(G)\right) = \omega\left(\Pi_{noncopr}(G)\right) = |G| - 1$.
4) The $\gamma\left(\Pi_{noncopr}(G)\right) = \alpha\left(\Pi_{noncopr}(G)\right) = 1$.

Proof: For quasi-dihedral group, QD_{2n} and generalized quaternion group, Q_{2n}, where $n = 2^{m-1}$ and $m \geq 5$, let $x \in G$, so $|x|$ are either 1 or 2^i, where $1 \leq i \leq m - 1$. In this circumstances, these two groups exhibits the same pattern as the dihedral group, D_n, where $n = 2^{m-1}$ and $m \geq 5$, in term of the order of the element of the group. Therefore, the types of graphs, diameter, chromatic number, clique number, dominance number, and independence number of a quasi-dihedral group, QD_{2n} and generalized quaternion group, Q_{2n}, where $n = 2^{m-1}$ have the same results as dihedral group, D_n, where $n = 2^{m-1}$ and $m \geq 5$.

4 Conclusion

In conclusion, prime power noncoprime probability and prime power noncoprime graph are two new definitions introduced in this study. The determination of prime power noncoprime probability for some dihedral groups, quasi-dihedral groups, and some generalized quaternion group, have been made and generalized. On the graph part, this research also covers the types and the characteristics of the prime power noncoprime graph of a group like the chromatic number, diameter, dominance number, and independence number. In this study, some dihedral groups, D_n, quasi-dihedral groups, QD_{2n}, and generalized quaternion group, Q_{2n}, have been selected as the scope groups for this investigation.

For future study, this research can be continued by investigating other invariants of the prime power noncoprime graph of dihedral group, D_n in which n is prime such as the girth and the number of edges of a graph. Additionally, by including vertex with element e in the prime power noncoprime graph of G, the relation between the prime power noncoprime probability and the prime power noncoprime graph for some dihedral group, D_n can also be explored for future research.

Acknowledgments. The authors would like to express their gratitude to Universiti Teknologi Malaysia (UTM) for financial support provided by the UTM Encouragement Research Grant (20J85). Additionally, the first author wishes to express her appreciation to the Ministry of Higher Education (MOHE) for providing financial assistance through the MyBrainSc program.

References

1. Williams, J.: Prime graph components of finite groups. J. Algebra **69**(2), 487–513 (1981)
2. Ma, X., Wei, H., Yang, L.: The co-prime graph of a group. Int. J. Group Theory **3**(3), 13–23 (2014)
3. Dorbidi, H.R.: A note on the co-prime graph of a group. Int. J. Group Theory **5**(4), 17–22 (2016)
4. Mansoori, F., Erfanian, A., Tolue, B.: Non-coprime graph of a finite group. In: AIP Conference Proceedings, p. 050017 (1750). https://doi.org/10.1063/1.4954605
5. Rilwan, N.M., Sameema, M.M., Oli, A.R.: Non-coprime graph of integers. Int. J. Math. Trends Technol. (IJMTT) **66**(2), 116–120 (2020)
6. Misuki, W.U., Wardhana, I.G., Switrayni Irwansyah, N.W.: Some results of non-coprime graph of the dihedral group for a prime power. In: AIP Conference Proceedings, p. 020005 (2021)
7. Aghababaei, G., Ashrafi, A.R., Jafarzadeh, A.: The non-coprime graph of a finite group with respect to a subgroup. Italian J. Pure Appl. Math. **42**, 25–359 (2019)
8. Abd Rhani, N.: Some extensions of the commutativity degree and the relative coprime graph of some finite groups. Ph.D. thesis, Universiti Teknologi Malaysia (2018)
9. Zulkifli, N., Mohd Ali, N.M.: Co-prime probability for nonabelian metabelian groups of order less than 24 and their related graphs. MATEMATIKA: Malaysian J. Ind. Appl. Math. **3**(3), 357–369 (2019)
10. Zulkifli, N., Mohd Ali, N.M.: Co-prime probability for nonabelian metabelian groups of order 24 and their related graphs. Menemui Matematik (Discovering Mathematics) **41**(2), 68–79(2019)
11. Zulkifli, N., Mohd Ali, N.M.: Relative coprime probability for nonabelian metabelian groups of order 24 and their related graphs. In: 8th Graduate Conference on Engineering, Science and Humanities (IGCESH 2020), pp. 195–198 (2020)
12. Zulkifli, N.Z, Mohd Ali, N.M.: Relative coprime probability for cyclic subgroups of some dihedral groups. Open Journal of Science and Technology, vol.3(4), pp. 314–321(2020)
13. Dummit, D.S., Foote, R.M.: Abstract Algebra. Wiley, USA (2004)
14. Jennifer, S., Kathryn, S.: On the structure of symmetric spaces of semidihedral groups. Involve, **10**(4), 665–676 (2017)
15. Rotman, J.J.: Advanced modern algebra. American Mathematical Society (2nd Ed), Rhode Island (2010)
16. Bondy, J.A., Murty, U.S.R.: Graphs Theory (Graduate Texts in Mathematics). Springer, New York (2008)