Numerical Approaches for Solving Mixed Volterra-Fredholm Fractional Integro-Differential Equations

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Abstract. In this paper, an approximate solution for solving nonlinear mixed Volterra-Fredholm fractional integro-differential equations is presented. The fractional derivative is defined in terms of Caputo type. Two methods are suggested: Adomin Decomposition Method (ADM) and Residual Power Series Method (RPSM). In these methods, Adomian polynomials and residual function are derived. The fractional Volterra-Fredholm integro-differential equation is reduced to a recurrence formula, in which it can be solved rather straightforward. Numerical examples demonstrate the efficiency and accuracy of ADM over RPSM.

Keywords: Fractional integro-differential equation · Caputo derivatives · Adomian polynomial · Residual function

1 Introduction

Fractional calculus plays a great role for describing the natural phenomena in different fields such as biology, electrochemistry, control theory, viscoelasticity, and others [1–4]. However, the difficulty of finding the exact solution for many classes of these equations enforce researchers to solve them approximately using different numerical methods. For example, they use the Polynomial Least Squares Method [5], the Reproducing Kernel Method [6, 7], Fractional Power Series Method [8], Haar wavelet [9], Laplace Adomian decomposition method [10], Homotopy perturbation method [11].

Adomian decomposition method (ADM) and Residual power series (RPSM) method were utilized to solve different problems of fractional integro-differential equations. For example, Ale'damat et al. [12] used (RPSM) to solve a certain class of nonlinear fractional integro-differential equations of Volterra type, and Hamoud et al. [13] used (ADM) to solve the Caputo fractional Volterra-Fredholm integro-differential equations. Momani & Aslam Noor [14] used ADM for solving fourth-order fractional integro-differential equations. Alaroud et al. [15] obtained the approximate solution of fuzzy fractional integro-differential equations utilizing RPSM.

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In this paper, we will study the nonlinear fractional mixed Volterra-Fredholm integro-differential equation of the form:

$$ {}^{c}D^{\alpha}_{0+}u(t) = \varphi(t) + \lambda \int_0^t \int_0^T K(x, s)F(u(s))dsdx \tag{1}$$

subject to the initial conditions

$$u(0) = u_0, u'(0) = u_1 \tag{2}$$

where $\alpha \in (1, 2]$, $0 \leq t, x \leq T$, $\varphi : [0, T] \to R$, is continuous function and $K(x, s)$ is a continuous arbitrary kernel functions, $F(u(s))$ is a function contain linear and nonlinear parts, $u(t)$ is an unknown function, and D^{α} is Caputo fractional derivative.

The paper is organized as follows. We present the necessary basic definitions and theories in fractional calculus in Sect. 2. In Sect. 3 and 4, we explain the Adomian decomposition method and the Residual power series method, respectively. In Sect. 5, we present numerical examples to demonstrate the efficiency of the two proposed methods. In Sect. 6, our conclusion is presented.

2 Preliminaries and Basic Definitions

In this section we will introduce some basic definitions and theorems in fractional calculus.

Definition 2.1 [16]. The Riemann-Liouville fractional integral of real order $\alpha > 0$ of a function $f(t)$ is given by

$$D^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau)d\tau$$

where Γ is Euler's Gamma function.

Definition 2.2 [16]. For any positive real $\alpha > 0$ the Caputo fractional derivative of order α of a continuous function $f(t)$ is defined by

$$ {}^{c}D^{\alpha}_{a} + f(t) = J^{n-\alpha}_{a^+}\left(\frac{d^n}{dt^n} f(t)\right) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n-k^{\sim}-1} f^{(n)}(\tau)d\tau, \alpha > 0$$

where $m = [\alpha] + 1$ and Γ represents gamma function.

The following properties are well known in fractional calculus. Let $\alpha > 0$ and $\beta > 0$, and let $f \in L^1[a, b]$. Then,

$$J^{\alpha}_{a^+} J^{\beta}_{a^+} f(t) = J^{\beta}_{a^+} J^{\alpha}_{a^+} f(t) = J^{\alpha+\beta}_{a^+} f(t).$$

$$ {}^{c}D^{\alpha}_{a^+}\left[J^{\alpha}_{a^+} f(t)\right] = f(t).$$

$$J_{a+}^{\alpha}\left[{}^{c}D_{a+}^{\alpha}f(t)\right] = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^{k} for n - 1 < \alpha \leq n.$$

Also, the fractional integral acts on a power function according to the following formula:

$$J_{a^+}^{\beta}(t-a)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\beta+\mu+1)}(t-a)^{\beta+\mu}, \mu > -1.$$

Definition 2.3 [8]. A power series expansion of the form

$$\sum_{m=0}^{\infty} c_m(x-x_0)^{m\alpha} = c_0 + c_1(x-x_0)^{\alpha} + c_2(x-x_0)^{2\alpha} + \cdots,$$

where $0 \leq m - 1 < \alpha \leq m$, , is called fractional power series (FPS) about $x = x_0$.

Theorem 2.1 [12]. Suppose that f has a fractional FPS representation at $x = x_0$ of the form.

$$g(x) = \sum_{m=0}^{\infty} c_m(x-x_0)^{m\alpha}, x_0 \leq x < x_0 + \beta$$

If $D^{m\alpha}g(x), m = 0, 1, 2, \ldots$ are continuous on R, then $c_m = D^{m\alpha}g(x_0))/\Gamma(1+m\alpha)$.

Theorem 2.2. Let $u(x) \in C([x_0, x_0 + R))$ and $D^{i\alpha}u(x) \in C((x_0, x_0 + R))$ for $i = 0, 1, \ldots, m+ 1$, where $0 \leq m - 1 < \alpha \leq m$. Then,

$$I^{(m+1)\alpha}D^{(m+1)\alpha}u(x) = \frac{D^{(m+1)\alpha}(\omega)}{\Gamma((m+1)\alpha+1)}(x-x_0)^{(m+1)\alpha+1}$$

where $x_0 \leq \omega \leq x < x_0 + R$

3 Adomian Decomposition Method (ADM)

In this Section, we apply ADM for solving nonlinear fractional integro-differential with mixed Volterra-Fredholm type Eqs. (1)-(2). Note that the function $F(u(s)) = [Ru(s)+ Nu(s)]$ where $Ru(s)$ is the linear part, and $Nu(s)$ is the nonlinear part. The integral part of Eq. (1) can be written

$$\int_0^t\int_0^T K(x,s)F(u(s))dxds = \int_0^t\int_0^T K(x,s)[Ru(s)+Nu(s)]dsdx. \tag{3}$$

Firstly, substitute Eq. (3) in Eq. (1), then we apply the Riemann integral operator J^{α} to both sides yield

$$u(t) = \sum_{k=0}^{m-1} u^k(0)\frac{t^k}{k!} = \hat{j^k}\varphi(t) + \lambda J^{\alpha}\left(\int_0^t\int_0^T K(x,s)[Ru(s)+Nu(s)]dsdx\right). \tag{4}$$

Consider the unknown solution $u(t)$ in terms of an infinite series as

$$u(t) = \sum_{i=0}^{\infty} u_i(t) \tag{5}$$

where $u_i(t)$ for $i = 0, 1, 2, \ldots$ are evaluated recursively. For the non-linear term $Nu(t)$ will be decomposed in terms of Adomian polynomial p_n in the form

$$Nu(t) = \sum\nolimits_{n=0}^{\infty} p_n(t).$$

where $P_n, n = 0, 1, 2, \ldots$ is defined by

$$P_n(t) = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum\nolimits_{i=0}^{n} \lambda^i u_i(t)\right)\Big|_{\lambda=0} \tag{6}$$

Then, substitute $Ru(t), Nu(t)$, and $u(t)$ in Eq. (4), we have

$$\sum\nolimits_{i=0}^{\infty} u_i(t) = \sum\nolimits_{k=0}^{m-1} u^k(0)\tfrac{t^k}{k!} + J^\alpha \varphi(t) + \lambda J^\alpha \left(\int_0^t \int_0^T K(x,s)\left[\sum\nolimits_{i=0}^{\infty} u_i(s) + \sum\nolimits_{i=0}^{\infty} p_i(s)\right] ds dx\right).$$

with u_0 identified as all terms out of the integral sign. Consequently, the components $u_i, i \geq 1$ of the unknown function $u(t)$ are completely determined in a recurrent manner if we set

$$u_0(t) = J^\alpha \varphi(t) + \sum\nolimits_{k=0}^{m-1} u^k(0)\frac{t^k}{k!} \tag{7}$$

And

$$u_{i+t}(t) = \lambda J^\alpha \left(\int_0^t \int_0^T K(x,s)\left[u_i(s) + p_i(s)\right] ds\, dx\right), i = 0, 1, 2 \ldots \tag{8}$$

where $p_i(s)$ are given as in Eq. (6).

As a result, the solution u(x) of Eq. (1) is obtained by using the series Eq. (5). Hence, the Adomian decomposition method converts the fractional Volterra-Fredholm integro-differential equation into a recursion formula with easily computations. Previous studies investigated the convergence of the decomposition series $u_n(t), n > 0,$, and demonstrated that if the exact solution exists for our problem, then the obtained series $u_n(t)$ converge rapidly to that solution. And the accuracy increased by increasing n-th iteration.

4 Residual Power Series Method

In this section, we will construct the residual power series method to solve fractional integro-differential equation with mixed Volterra-Fredholm Eqs. (1)-(2). The solution can be written in fractional power series form as

$$u(t) = \sum\nolimits_{n=0}^{\infty} c_n \frac{(t-a)^{n\alpha}}{\Gamma(n\alpha+1)}$$

when $a = 0$, then the previous expression become

$$u(t) = \sum\nolimits_{n=0}^{\infty} c_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{9}$$

To obtain the approximate values of the above series (Eq. (9)), the k-th truncated series $u_k(x)$ is written in the form

$$u_k(t) = \sum\nolimits_{n=0}^{k} c_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{10}$$

Since $u(a) = u_0 = c_0$, we rewrite Eq. (10) as

$$u_k(x) = c_0 + \sum\nolimits_{n=1}^{k} c_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, k = 1, 2, \ldots \tag{11}$$

Define the residual power series of our problem as

$$Res(t) = {}^{c}D_{0+}^{\alpha} u(t) - \varphi(t) - \lambda \int_0^t \int_0^T K(x,s) F(u(s)) ds dx.$$

where the k-th residual function is defined by

$$Res_k(t) = {}^{c}D_{0+}^{\alpha} u_k(t) - \varphi(t) - \lambda \int_0^t \int_0^T K(x,s) F(u_k(s)) ds dx. \tag{12}$$

Substitute Eq. (11) in Eq. (12), we get

$$\begin{aligned} Res_k(t) = {}& {}^{c}D_{0+}^{\alpha}\left(c_0 + \textstyle\sum_{n=1}^{k} c_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right) - \varphi(t) \\ & -\lambda \textstyle\int_0^t \int_0^T K(x,s) F\left(c_0 + \sum_{n=1}^{k} c_n \frac{s^{n\alpha}}{\Gamma(n\alpha+1)}\right) ds dx. \end{aligned}$$

Many references mentioned the important properties of residual function which help us to applying the method (see: [12]):

$lim_{k\to\infty} Res_k(t) = Res(t) = 0$, for each $t \in (0, 1)$, and $D_{0+}^{n\alpha} Res(0) = D_{0+}^{n\alpha} Res_k(0)$, for each $n = 0, 1, 2, \ldots k$.

To find the coefficients c_n for $n = 1, 2, 3, \cdots, k$, we solve $D^{(n-1)\alpha} Res_n(t)|_{t=0} = 0, n = 1, 2, 3, \ldots, k$, where $D^{n\alpha} = D^{\alpha} \cdot D^{\alpha} \cdots D^{\alpha} (n - times)$.

5 Illustrative Example

To illustrate the effectiveness of the presented methods, we are applying the two proposed method on next example, then comparing the results.

Example. Consider the following form of the nonlinear fractional integro-differential equation.

$$\begin{aligned} {}^{c}D_0^{\alpha} + u(t) = \varphi(t) + \lambda \textstyle\int_0^t \int_0^1 (x-s) F(u(s)) ds\, dx, u(0) = 1, u'(0) = 0, 1 < \alpha \\ \leq 2, t \in [0, 1] \end{aligned}$$

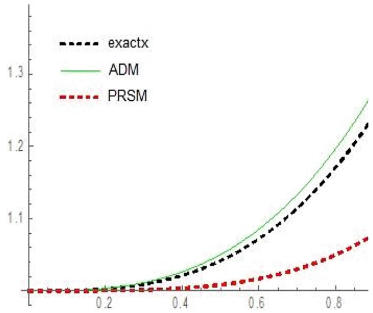
where $\varphi(t) = -\frac{25}{504} t^2 + \frac{749}{360} t \; and \; F(u(s)) = (u(s))^2 - u(s) \; with \; \lambda = 1.$

The exact solution is $u(t) = \frac{1}{3} t^3 + 1$.

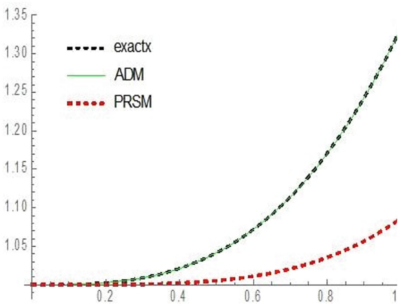
Table 1 shows the comparison between the approximate solution and the absolute error for ADM and RPSM at $\alpha = 1.90$ and $\alpha = 2$, respectively. The results show that ADM gives more accurate results as compared to RPSM. Furthermore, when we increase the values of α, the approximate solution is in agreement with the exact solution. Figure 1 represents the comparison between the two proposed methods at $\alpha = 1.90$ and $\alpha = 2$.

Table 1. Exact solution and Approximate Solutions of ADM and RPSM for $N = 10$ and $\alpha = 1.90$, and $\alpha = 2$.

t	Exact Solution	Approximate Solution at $\alpha = 1.90$		Approximate Solution at $\alpha = 2$	
		ADM	RPSM	ADM	RPSM
0.1	1.0003333333	1.0004718768	1.0000184852	1.0003333333	1.0000086688
0.2	1.0026666667	1.0035230216	1.0002574592	1.0026666667	1.0001386948
0.3	1.009	1.0114203547	1.0012017290	1.009	1.0007020870
0.4	1.0213333333	1.0263088301	1.0035851536	1.0213333333	1.0022186948
0.5	1.0416666667	1.0502620452	1.0083691827	1.0416666667	1.0054159605
0.6	1.072	1.0853032013	1.0167291011	1.072	1.0112285714
0.7	1.1143333333	1.1334168113	1.0300438167	1.1143333333	1.0207980140
0.8	1.1706666667	1.1965561864	1.0498876803	1.1706666667	1.0354720282
0.9	1.243	1.2766486299	1.0780236049	1.243	1.0568039620
1.0	1.3333333333	1.3755992488	1.1163970752	1.3333333333	1.0865520282



(a)



(b)

Fig. 1. Comparison between approximate solution of ADM and RPSM at different values of α, $n = 10$ for (a) $\alpha = 1.90$ and (b) $\alpha = 2$.

6 Conclusion

In this paper, Adomian decomposition method and Residual power series method has been derived to obtain the approximate solution of nonlinear fractional Volterra-Fredholm integro-differential equations. The numerical results showed that ADM method is accurate and effective to solve the nonlinear equations as compared to RPSM.

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