Translation-Invariant *p*-Adic Gibbs Measures for the Potts Model on the Cayley Tree of Order Four

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Abstract. Different topological structure between real and *p*-adic fields provides a distinct condition for solution of equations or system of equations. For example, the equation $x^2+1=0$ does not have solution over real field but it has solution over *p*-adic field for $p \equiv 1(\bmod 4)$. Meanwhile, the equation $x^3=p$ has solution in real field but not in *p*-adic field. It is convenience to investigate the translation-invariant *p*-adic Gibbs measures of Potts model on Cayley trees in terms of zeros of a certain polynomial. The translation-invariant *p*-adic Gibbs measures of Potts model on Cayley trees of order two and three was described with respect to some respective conditions on the coefficient of certain quadratic and cubic polynomials. In this paper, the set of *p*-adic Gibbs measures of *p*-adic Potts model on the Cayley tree of order four is considered. For this case, it is possible to associate the existence of the translation-invariant *p*-adic Gibbs measures with zeros of quartic polynomial over *p*-adic field.

Keywords: p-adic field · p-adic Gibbs measure · p-adic Potts model · translation-invariant · Cayley trees

1 Introduction

The completion of rational field gives rise to the p-adic field where non-Archimedean norm on metric spaces is used instead of Archimedean norm [1]. A non-Archimedean quantum physics problem involving wave functions with p-adic values can be analysed statistically using p-adic probability theory, where the values of p-adic number determine the probabilities [2]. An abstract p-adic probability theory can be created through the application of the non-Archimedean measures [3]. The authors in [4] proved the non-Archimedean equivalent of the Kolmogorov theorem to study the models of the statistical mechanics models over p-adic field.

The Gibbs measures, originate from Boltzmann and Gibbs, is a measure in probability theory and statistical mechanics in which a number is allocate to each acceptable attribute of a system, which indicates the outcome of the system's study [5]. It is a probability measure that is associated with the system's Hamiltonian where it gives a state of the

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system. Since the system's state depends on Gibbs measures, therefore the state of the system is unchanged in case of uniqueness of the Gibbs measures. Contrarily, for the case of non-uniqueness of the Gibbs measures, the state of the system is changing and known as phase transition. The focus was given on determining all of the possible extremal Gibbs measures to predict the phase transition [6]. The Potts model, developed by Renfrey B. Potts in 1952, was used to explore the behaviour of systems having multiple states and is now known as the q-state Potts model. The energy of the configurations of the spins that defines this model is known as Hamiltonian and it takes one of q possible values on a lattice's vertices [7].

There are several methods that can be used to describe the Gibbs measures on Cayley Trees, such as the method of Markov random field theory and the recurrent equations, node-weighted random walks, information flows and contour methods on trees, group theory, and non-linear analysis [8]. Based on the result obtained, which shows the existence of a relationship between the ultrametricity with the structure of correlation functions for spin glasses, the p-adic approaches had been applied to study the statistical mechanics.

Recently, the involvement of p-adic properties to investigate the Gibbs measures for the q-state Potts model had been considered and is referred as p-adic Gibbs measures which shows examples of p-adic valued process. For case of p-adic Potts model on Cayley trees, the method of p-adic probability theory can be used to investigate the $q + 1$ state nearest-neighbour [11–13]. There exists a phase transition through the construction of the infinite volume p-adic Gibbs measures for the p-adic Potts model. Further development on the study give rise to some alternative form known as p-adic quasi-Gibbs measures where the change of state of the model been studied based on the related dynamical system perspective [14, 15]. In case of Cayley trees of order two and three, the roots of certain polynomial equation, respective to the order of the Cayley trees, represent all possible form of the translation-invariant. The translation invariant indicates the phase transition of the p-adic Gibbs measures of the p-adic Potts model [9, 10]. Generally, in the field of p-adic number and real number, the same study of roots of polynomial equations will give a different result since both fields are different topologically.

2 Preliminaries

The foundation for this paper is based on Gibbs measures of Potts model on Cayley trees in p-adic field. This section will show the crucial part that build the study of the problem which involved the p-adic numbers and measures, Cayley trees, p-adic Potts model and lastly, the p-adic Gibbs measures.

2.1 p-Adic Numbers

For any prime number $p \in \mathbb{N}$, the mapping of a p-adic norm $|\cdot|_p$ on the field of rationals $\mathbb{Q}$ leads to its completion, $\mathbb{Q}_p$, which is the field of p-adic numbers. The mapping can be defined by

$$|x|_p = \begin{cases} p^{-k}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where for $k, m \in \mathbb{Z}$ and $n \in \mathbb{N}$, there exist $x = p^k\ (m/n)$, such that $(m, p) = (n, p) = 1$. The value of k is known as p-order of x and denoted by $k = \text{ord}_p\ (x)$ for $x \in \mathbb{Q}_p$.

The set of p-adic integers and p-adic unit denoted by $\mathbb{Z}_\text{p}$ and $\mathbb{Z}_\text{p}^*$ respectively as follow,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\},$$

$$\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}.$$

For any p-adic number $x \neq 0$, they can be represented by $x = x^*/|x|_p$ where $x^* \in \mathbb{Z}_p^*$. The canonical expansion for the p-adic number, x and p-adic unit, x^* are given by (2.1) and (2.2) below,

$$x = p^k(x_0 + x_1 p + x_2 p^2 + \ldots), \tag{2.1}$$

$$x^* = x_0 + x_1 p + x_2 p^2 + \ldots, \tag{2.2}$$

where $x_0 \in \{1, 2, \ldots, p-1\}$ and $x_i \in \{0, 1, 2, \ldots, p-1\}$ for i $\in \mathbb{N}$ [16, 17].

For a given centre $a \in \mathbb{Q}_p$ and radius $r > 0$, an open ball $B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}$ is also closed due to the non-Archimedean property. The p-adic logarithm $\log_p (\cdot) : B(1,1) \to B(0,1)$ is defined by,

$$\log_p(x) = \log_p(1 + (x-1)) = \sum\nolimits_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$$

The p-adic exponential $\exp_p(\cdot) : B(0, p^{-1/(p-1)}) \to B(1,1)$ is defined by,

$$\exp_p(x) = \sum\nolimits_{n=0}^{\infty} \frac{x^n}{n!}.$$

Let $x \in B(0, p^{-1/(p-1)})$, then the following can be obtained

1. $\exp_p(\log_p(x+1)) = x + 1$;
2. $\log_p(\exp_p(x)) = x$;
3. $|\exp_p(x)|_p = 1$;
4. $|\exp_p(x) - 1|_p = |x|_p < 1$;
5. $|\log_p(x+1)|_p = |x|_p < p^{-1/(p-1)}$.

Let a group under multiplication $\mathscr{E}_p = \{x \in \mathbb{Q}_p : |x - 1|_p < p^{-1/(p-1)}\}$. For $a, b \in \mathscr{E}_p$ and $h \in B(0, p^{-1/(p-1)})$,

$$|a-b|_p < 1,\ |a-b|_p = \begin{cases} \frac{1}{2}, & \text{if } p = 2 \\ 1, & \text{if } p \neq 2 \end{cases},\ a = \exp_p(h).$$

For more explanation, the reader could refer to [1].

2.2 p-Adic Measure

A p-adic measure can be defined by a function μ: $\mathscr{B} \to \mathbb{Q}_p$ for a measurable space (X, $\mathscr{B}$) if for any $A_1, \ldots, A_n \in \mathscr{B}$ such that $A_i \cap A_j = \emptyset$ and $i \neq j$,

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu\left(A_j\right),$$

where $\mathscr{B}$ is algebra of subsets of X. It is a probability measure when the p-adic measure, $\mu(X) = 1$, and μ is bounded if $\sup\{|\mu(A)|_p : A \in \mathscr{B}\} < \infty$. The details on p-adic probability measure can be obtained in [3, 18].

2.3 Cayley Tree

A semi-infinite Cayley Tree, Γ_+^k of order $k > 1$, is built up by sets of vertices V, and edges L, and can be define as $\Gamma_+^k(V, L)$. . It begins from a single vertex (root) $x^0 \in V$ and expands to $k + 1$ edges $l \in L$. Each edge built up of two vertices, called as nearest neighbour, x and y, where the edges can be denoted by $l = x, y$. The path from x to y is the total of nearest neighbour $x, x_1, x_1, x_2, \ldots, x_{d-1}, y$ where d is the distance of Cayley Tree, d(x, y) = n, and n is the total edges from x to y. Hence for fixed root $x^0 \in V$,

$$W_n = \left\{x \in V : d\left(x, x^0\right) = n\right\}, \qquad V_n = \bigcup_{m=0}^{n} W_m,$$

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\},$$

where for $\forall x \in W_n$, S(x) is the set of direct successors of x.

Vertex x^0 and vertex $x \neq x^0$ has the coordinate ($\emptyset$) and $(i_1, \ldots, i_n)$ respectively, where $i_m \in \{1, \ldots, k\}$, for $1 \leq m \leq n$. The coordinate ($\emptyset$) indicates level 0 of the Cayley tree while the coordinate $(i_1, \ldots, i_n)$ is level n of V from vertex x^0. So, for any $x = (i_1, \ldots, i_n) \in V$, $S(x) = \{(x, i) : 1 \leq i \leq k\}$ where (x, i) means $(i_1, \ldots, i_n, i)$. For any two elements, x and y, which have a respective coordinate of $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_m)$, a binary operation $\circ : V \times V \to V$ is defined by $x \circ y = (i_1, \ldots, i_n) \circ (j_1, \ldots, j_m) = (i_1, \ldots, i_n, j_1, \ldots, j_m)$ and $y \circ x = (j_1, \ldots, j_m) \circ (i_1, \ldots, i_n) = (j_1, \ldots, j_m, i_1, \ldots, i_n)$. Then, $(V, \circ)$ with the unit $x^0 = (\emptyset)$ is a noncommutative semigroup.

A translation $\tau_g : V \to V$ for $g \in V$ can be defined as $\tau_g(x) = g \circ x$ for any $x \in V$. Let $G \subset V$ be a sub-semigroup of V and $h : V \to Y$ be a Y–valued function. A function h is said to be G–periodic if $h\left(\tau_g(x)\right) = h(x)$ for all $g \in G$ and $x \in V$. A V–periodic function is called translation-invariant [19, 20].

2.4 p-Adic Potts Model

For a finite set $\Phi = \{1, 2, \ldots, q\}$, a configuration is a function $\sigma : V \to \Phi$. The configurations of the finite-volume and the boundary are the functions $\sigma_n : V_n \to \Phi$ and $\sigma^{(n)} : W_n \to \Phi$ respectively. The set of all configurations is denoted by Ω while Ω_{V_n} for the set of all finite-volume configurations and Ω_{W_n} for the set of boundary configurations.

For a given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\sigma^{(n)} \in \Omega_{W_n}$ the concatenation is a finite-volume configuration $\sigma_{n-1} \vee \sigma^{(n)} \in \Omega_{V_n}$ such that,

$$\left(\sigma_{n-1} \vee \sigma^{(n)}\right)(v) = \begin{cases} \sigma_{n-1} \text{ if } v \in V_{n-1}, \\ \sigma^{(n)} \text{ if } v \in W_n. \end{cases}$$

In case of p-adic Potts model, the Hamiltonian for all $\sigma_n \in \Omega_{V_n}$ and $n \in \mathbb{N}$, with the spin value set Φ on the finite volume configuration is defined by,

$$H_n(\sigma_n) = J \sum_{x,y \in L_n} \delta_{\sigma_n(x)\sigma_n(y)},$$

where $J \in B(0, p^{-1/(p-1)})$ is a coupling constant, $\langle x, y \rangle$ stands for nearest neighbour vertices and δ is Kronecker's delta symbol such that $\delta_{ij} = 0$ if $i = j$ and $\delta_{ij} = 1$ if $i \neq j$.

2.5 p-Adic Gibbs Measure

The finite-dimensional distribution of a p-adic probability measure μ in the volume V_n is defined by,

$$\mu_{\tilde{h}}^{(n)}(\sigma_n) = \frac{1}{Z_{\tilde{h}}^{(n)}} \exp_p\{H_n(\sigma_n)\} \prod\nolimits_{x \in W_n} \tilde{z}_x^{(\sigma_n(x))}, \quad (2.3)$$

where $\tilde{\mathbf{h}}(x) = \left(\tilde{z}_x^{(1)}, \ldots, \tilde{z}_x^{(q)}\right) \in \mathbb{Q}_p^q, x \in V$ is $\mathbb{Q}_p^q$ – valued function and $Z_{\tilde{\mathbf{h}}}^{(n)}$ is the partition function defined by,

$$Z_{\tilde{h}}^{(n)} = \sum\nolimits_{\sigma_n \in \Omega_{V_n}} \exp_p\{H_n(\sigma_n)\} \prod\nolimits_{x \in W_n} \tilde{z}_x^{(\sigma_n(x))}. \quad (2.4)$$

The p-adic probability measure given in Eq. (2.3) are said to be compatible if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$, we have

$$\sum_{\sigma^{(n)} \in \Omega_{W_n}} \mu_{\tilde{\mathbf{h}}}^{(n)}\left(\sigma_{n-1} \vee \sigma^{(n)}\right) = \mu_{\tilde{\mathbf{h}}}^{(n-1)}(\sigma_{n-1}), \quad (2.5)$$

where $\sigma_{n-1} \vee \sigma^{(n)}$ is the concatenation of the configurations.

Due to the Kolmogorov extension theorem [21–23] of the p-adic probability measure given in Eq. (2.3), there exist a unique p-adic measure $\mu_{\tilde{\mathbf{h}}}$ on $\Omega = \Phi^V$, such that for all n and $\sigma_n \in \Omega_{V_n}$,

$$\mu_{\tilde{\mathbf{h}}} = \left(\left\{\sigma|_{V_n} = \sigma_n\right\}\right) = \mu_{\tilde{\mathbf{h}}}^{(n)}(\sigma_n).$$

Such a measure is known as p-adic Gibbs measure. The condition on the function $\tilde{\mathrm{h}}$ that satisfied the compatibility condition given in Eq. (2.5) is described in the following theorem.

Theorem 2.1. [11, 12] *For* $i = 1, \ldots, q-1$, let $\tilde{\mathbf{h}} : V \to \mathbb{Q}_p^q$, $\tilde{\mathbf{h}}(x) = \left(\tilde{z}_x^{(1)}, \ldots, \tilde{z}_x^{(q)}\right)$ *and* $\mathbf{h} : V \to \mathbb{Q}_p^{q-1}$, $\mathbf{h}(x) = \left(z_x^{(1)}, \ldots, z_x^{(q-1)}\right)$ *be some functions defined as* $z_x^{(i)} = \tilde{z}_x^{(i)}/\tilde{z}_x^{(q)}$*.The p-adic probability distribution* $\left\{\mu_{\tilde{\mathbf{h}}}^{(n)}\right\}_{n\in\mathbb{N}}$ *is compatible if and only if,*

$$\mathbf{h}(x) = \prod\nolimits_{y\in S(x)} \mathbf{F}(\mathbf{h}(y)), \tag{2.6}$$

for all $x \in V \setminus \{x^0\}$*, where* $S(x)$ *in* (2.6) *is the set of direct successors of x. For* $\mathbf{h} = \left(z_1, \ldots, z_{q-1}\right)$ *and the mapping* $\mathbf{F} : \mathbb{Q}_p^{q-1} \to \mathbb{Q}_p^{q-1}$*, the function* $\mathbf{F}(\mathbf{h}) = \left(F_1, \ldots, F_{q-1}\right)$ *is defined by*

$$F_i = \frac{(\theta-1)z_i + \sum_{j=1}^{q-1} z_j + 1}{\theta + \sum_{j=1}^{q-1} z_j}, \qquad \theta = \exp_p(J).$$

Remark: The multiplication in

(2.6) *means the multiplication coordinate-wise, i.e. for* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$, $xy = (x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_1y_1, \ldots, x_ny_n)$.

3 Existence of the Translation-Invariant *P*-Adic Gibbs Measures

The translation-invariant p-adic Gibbs measure exist if and only if the function $\tilde{\mathbf{h}}(x) = \tilde{\mathbf{h}}$ for any $x \in V$, which means that $\tilde{\mathbf{h}} : V \to \mathbb{Q}_p^q$, is constant. Hence, in case of translation-invariant p-adic Gibbs measure, the compatibility condition (2.6) takes the form

$$z_i = \left(\frac{(\theta-1)z_i + \sum_{j=1}^{q-1} z_j + 1}{\theta + \sum_{j=1}^{q-1} z_j}\right)^k, i = 1, \ldots, q-1. \tag{3.1}$$

Let $\mathbf{I}_{q-1} = \{1, \ldots, q-1\}$. For $j \in \mathbf{I}_{q-1}$ and δ_{ij} is the Kronecker's delta symbol, let $\mathbf{e}_j = \left(\delta_{1j}, \delta_{2j}, \ldots, \delta_{q-1j}\right) \in \mathbb{Q}_p^{q-1}$ be vectors. Let $\mathbf{e}_\alpha = \sum_{j\in\alpha} \mathbf{e}_j$ and $\mathbf{h} = \left(z_1, \ldots, z_{q-1}\right) \in \mathbb{Q}_p^{q-1}$ be any vector. Let $\left\{\alpha_j(\mathbf{h})\right\}_{j=1}^d$ be a disjoint partition of the index set $\mathbf{I}_{q-1}$, i.e., $\bigcup_{j=1}^d \alpha_j(\mathbf{h}) = \mathbf{I}_{q-1}$, $\alpha_{j_1}(\mathbf{h}) \cap \alpha_{j_2}(\mathbf{h}) \neq \emptyset$ for $\mathrm{j}_1 \neq \mathrm{j}_2$ such that $\mathrm{z}_{\mathrm{i}_1} = \mathrm{z}_{\mathrm{i}_2}$ for all $i_1, i_2 \in \alpha_j(\mathbf{h})$ and $z_{i_1} \neq z_{i_2}$ for all $i_1 \in \alpha_{j_1}(\mathbf{h})$, $i_2 \in \alpha_{j_2}(\mathbf{h})$. Then, $\mathbf{h}$ can be expressed as follow

$$\mathbf{h} = \sum\nolimits_{j=1}^d z_j^{\mathrm{o}} \mathbf{e}_{\alpha_j(h)}, \tag{3.2}$$

where $z_i = z_j^{\mathrm{o}}$ for any $i \in \alpha_j(\mathbf{h})$ and $j = 1, \ldots, d$.

Theorem 3.1. [10] *If* $\mathbf{h}$ *is a solution of the system given in Eq.* (3.1) *then* $1 \leq d \leq k$.

Proof. Let $\mathbf{h} = \left(z_1, \ldots, z_{q-1}\right)$ be a solution of the system (3.1). The case $q-1 \leq k$ is trivial. Suppose that $q > k+1$. Then we can write $\mathbf{h}$ in the form (3.2). Since $\mathbf{h}$ is a fixed solution, $S(h) = \sum_{j=1}^{q-1} z_j = \sum_{j=1}^{q-1} \left|\alpha_j(h)\right| z_j^{\mathrm{o}}$ is also a fixed number where $z_i = z_j^{\mathrm{o}}$ for any $i \in \alpha_j(\mathbf{h})$ and $j = 1, \ldots, d$.

In the case of order $k = 2$ and 3, the descriptions of all translation-invariant p-adic Gibbs measures of the system given in Eq. (3.1) were shown in [9, 10]. Next, the study will be focus on system of equation of order $k = 4$.

Lemma 3.2. *Let* $\mathbf{h} = \left(z_1, \ldots, z_{q-1}\right)$ *be a solution of the system given in Eq.* (3.1)*. Then* $\mathbf{h} = \sum_{j=1}^{d} z_j^{\mathrm{o}} \mathbf{e}_{\alpha_j(h)}$, $S(\mathbf{h}) = \sum_{j=1}^{q-1} \left|\alpha_j(\mathbf{h})\right| z_j^{\mathrm{o}}$ *and for* $1 \le \mathrm{j} \le \mathrm{d}$,

$$z_j^{\mathrm{o}}(\theta + S(\mathbf{h}))^4 = \left((\theta - 1)z_j^{\mathrm{o}} + S(\mathbf{h}) + 1\right)^4. \tag{3.3}$$

Proof. Let $k = 4$, $z_i = z_j^o$ and $\mathbf{h} = \left(z_1, \ldots, z_{q-1}\right)$ be a solution of the system given in Eq. (3.1). Substitute $S(\mathbf{h}) = \sum_{j=1}^{q-1} z_j = \sum_{j=1}^{q-1} \left|\alpha_j(\mathbf{h})\right| z_j^o$ into Eq. (3.1) as follow,

$$z_j^o = \left(\frac{(\theta - 1)z_j^o + S(\mathbf{h}) + 1}{\theta + S(\mathbf{h})}\right)^4.$$

Rearrange the equation, we obtain Eq. (3.3) as below,

$$z_j^o = \frac{\left((\theta - 1)z_j^o + S(\mathbf{h}) + 1\right)^4}{(\theta + S(\mathbf{h}))^4},$$

$$z_j^o(\theta + S(\mathbf{h}))^4 = \left((\theta - 1)z_j^o + S(\mathbf{h}) + 1\right)^4.$$

This completes the proof. □

The Eq. (3.3) has a trivial root $\mathrm{z_i} = 1$ and can be expand into the following,

$$\begin{aligned}
&((\theta - 1)z_i + S(\mathbf{h}) + 1)^4 - z_i(\theta + S(\mathbf{h}))^4 \\
&\quad = (z_i - 1)\Big[(\theta - 1)^4\left(z_i^3 + z_i^2 + z_i\right) + 4(\theta - 1)^3(S(\mathbf{h}) + 1)\left(z_i^2 + z_i\right) \\
&\quad + 6(\theta - 1)^2(S(\mathbf{h}) + 1)^2 z_i - (S(\mathbf{h}) + 1)^4\Big].
\end{aligned}$$

It means, any root $\mathrm{z_i} \neq 1$ for Eq. (3.3) is the roots of

$$\begin{aligned}
&(\theta - 1)^4\left(z_i^3 + z_i^2 + z_i\right) + 4(\theta - 1)^3(S(\mathbf{h}) + 1)\left(z_i^2 + z_i\right) \\
&\quad + 6(\theta - 1)^2(S(\mathbf{h}) + 1)^2 z_i - (S(\mathbf{h}) + 1)^4.
\end{aligned} \tag{3.4}$$

For simplicity, the term $S(\mathbf{h})$ is denoted with S. The following theorem 3.3 is the main result of this paper which describe the possible forms of the translation-invariant p-adic Gibbs measures (TIpGM) of the p-adic Potts model on the Cayley tree of order four.

Theorem 3.3. (Description of TIpGM, $\mathrm{k} = 4$) *There exists a TIpGM* $\mu_{\tilde{\mathbf{h}}}$ *associated with function* $\tilde{\mathbf{h}}(x) = \left(\tilde{z}_1, \ldots, \tilde{z}_q\right)$ *if and only if* $\tilde{z}_j = \mathbf{h}z_j$ *for all* $j = 1, \ldots, q - 1$ *and* $\tilde{z}_q = \mathbf{h}$ *where h is any p-adic number, and* $\mathbf{h} = \left(\tilde{z}_1, \ldots, \tilde{z}_{q-1}\right)$ *is defined by either one of the following,*

1. $\mathbf{h} = (1, \ldots, 1)$;
2. $\mathbf{h} = (z, \ldots, z)$, *where* z *is the zeros of Eq.* (3.4);
3. $\mathbf{h} = \mathbf{e}_{\alpha_1} + z\mathbf{e}_{\alpha_2}$, *with* $|\alpha_i| = m_i, m_1 + m_2 = q - 1$ *and* $S = m_1 + m_2 z$ *where* z *is the zeros of Eq.* (3.4);
4. $\mathbf{h} = z_1\mathbf{e}_{\alpha_1} + z_2\mathbf{e}_{\alpha_2}$, *with* $|\alpha_i| = m_i, m_1 + m_2 = q - 1$ *and* $S = m_1 z_1 + m_2 z_2$ where z_1, z_2 *are the zeros of the system of equations derived from Eq.* (3.4);
5. $\mathbf{h} = \mathbf{e}_{\alpha_3} + z_1\mathbf{e}_{\alpha_1} + z_2\mathbf{e}_{\alpha_2}$, with $|\alpha_i| = m_i$, $m_1 + m_2 + m_3 = q - 1$ and $S = m_1 z_1 + m_2 z_2 + m_3$ where z_1, z_2 *are the zeros of the system of equations derived from Eq.* (3.4);
6. $\mathbf{h} = z_1\mathbf{e}_{\alpha_1} + z_2\mathbf{e}_{\alpha_2} + z_3\mathbf{e}_{\alpha_3}$, with $|\alpha_i| = m_i$, $m_1 + m_2 + m_3 = q - 1$ and $S = m_1 z_1 + m_2 z_2 + m_3 z_3$ where z_1, z_2, z_3 *are the zeros of the system of equations derived from Eq.* (3.4);
7. $\mathbf{h} = \mathbf{e}_{\alpha_4} + z_1\mathbf{e}_{\alpha_1} + z_2\mathbf{e}_{\alpha_2} + z_3\mathbf{e}_{\alpha_3}$, with $|\alpha_i| = m_i$, $m_1 + m_2 + m_3 + m_4 = q - 1$ and $S = m_1 z_1 + m_2 z_2 + m_3 z_3 + m_4$ where z_1, z_2, z_3 *are the zeros of the system of equations derived from Eq.* (3.4).

Proof. This theorem is the further description of theorem 2.1. To describe it further, we study the system of Eq. (3.1) to find the solutions. Let $k = 4$, then by theorem 3.1, the solution of Eq. (3.1) has the form as given in Eq. (3.2),

$$\mathbf{h} = \sum_{j=1}^{d} z_j^o \mathbf{e}_{\alpha_j(\mathbf{h})},$$

for $1 \leq d \leq 4$. Moreover, using lemma 3.2, for $z_j^o \neq 1$, we can derive the Eq. (3.4). Since $z_i = z_j^o$ any $i \in \alpha_j(\mathbf{h})$ and $j = 1, \ldots, d$, therefore the possible forms of solution $\mathbf{h}$ are as follow,

1. $\mathbf{h} = \sum_{j=1}^{1} 1 = (1, \ldots, 1)$,
2. $\mathbf{h} = \sum_{j=1}^{1} z = (z, \ldots, z)$, where z is any solutions for Eq. (3.4).
3. $\mathbf{h} = \sum_{j=1}^{2} z_j^o \mathbf{e}_{\alpha_j(\mathbf{h})} = \mathbf{e}_{\alpha_1} + z\mathbf{e}_{\alpha_2}$, with $|\alpha_i| = m_i, m_1 + m_2 = q - 1$and $\mathrm{S} = \mathrm{m}_1 + \mathrm{m}_2 z$ where z is the solution of Eq. (3.4).
4. $\mathbf{h} = \sum_{j=1}^{2} z_j^o \mathbf{e}_{\alpha_j(\mathbf{h})} = z_1\mathbf{e}_{\alpha_1} + z_2\mathbf{e}_{\alpha_2}$, with $|\alpha_i| = m_i$, $m_1 + m_2 = q - 1$ and $S = m_1 z_1 + m_2 z_2$ where z_1, z_2 are the solutions of system of equations derived from Eq. (3.4).
5. $\mathbf{h} = \sum_{j=1}^{3} z_j^o \mathbf{e}_{\alpha_j(h)} = \mathbf{e}_{\alpha_3} + z_1\mathbf{e}_{\alpha_1} + z_2\mathbf{e}_{\alpha_2}$, with $|\alpha_i| = m_i$, $m_1 + m_2 + m_3 = q - 1$ and $S = m_1 z_1 + m_2 z_2 + m_3$ where z_1, z_2 are the solutions of system of equations derived from Eq. (3.4).
6. $\mathbf{h} = \sum_{j=1}^{3} z_j^o \mathbf{e}_{\alpha_j(h)} = z_1\mathbf{e}_{\alpha_1} + z_2\mathbf{e}_{\alpha_2} + z_3\mathbf{e}_{\alpha_3}$, with $|\mathrm{\alpha_i}| = \mathrm{m_i}$, $m_1 + m_2 + m_3 = q - 1$ and $S = m_1 z_1 + m_2 z_2 + m_3 z_3$ where z_1, z_2, z_3 are the solutions of system of equations derived from Eq. (3.4).
7. $\mathbf{h} = \sum_{j=1}^{4} z_j^o \mathbf{e}_{\alpha_j(h)} = \mathbf{e}_{\alpha_4} + z_1\mathbf{e}_{\alpha_1} + z_2\mathbf{e}_{\alpha_2} + z_3\mathbf{e}_{\alpha_3}$, with $|\mathrm{\alpha_i}| = \mathrm{m_i}$, $m_1 + m_2 + m_3 + m_4 = q - 1$ and $S = m_1 z_1 + m_2 z_2 + m_3 z_3 + m_4$ where z_1, z_2, z_3 are the solutions of system of equations derived from Eq. (3.4).

This completes the proof. □

4 Conclusion

This paper focused on listing down all the possible form of the translation-invariant p-adic Gibbs measure of the p-adic Potts model on the Cayley tree of order $k = 4$. The zeros of polynomials equation indicate the translation-invariant of the study and consideration made based on Theorem 3.1, where the possible number of distinct solutions, $d \leq k$, and at least one of the zeros must be equal to 1 when $d = k$. Since this study deals with some polynomial equations of order four, we believe that a better form of polynomial equation can be obtained from this study in the future.

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References

1. Katok, S.: p-adic analysis compared with real. Am. Math. Soc., Providence, **37**, 92–98 (2007)
2. Khrennikov, A. Y.: p-Adic probability theory and its applications. The principle of statistical stabilization of frequencies. Theor. Math. Phys. **97**(3), 348–363 (1993). https://doi.org/10.1007/BF01015763
3. Khrennikov, A.Y., Yamada, S., Rooij, A. V.: The measure-theoretical approach to p-adic probability theory. Annal. Math. Blaise Pascal **6**(1), 21–32 (1999). http://www.numdam.org/item/AMBP_1999__6_1_21_0/
4. Mukhamedov, F.: On p-adic quasi Gibbs measures for q+1-state Potts model on the Cayley tree. P-Adic Num. Ultrametr. Anal. Appl. **2**, 241–251 (2010). https://doi.org/10.1134/S2070046610030064
5. Rozikov, U.A.: Gibbs measures of Potts model on Cayley trees: a survey and applications. preprint arXiv:2103.07391v1 [math.PR] (2021). https://doi.org/10.48550/arXiv.2103.07391
6. Georgii, H.O.: Gibbs Measures and Phase Transitions. Walter de Gruyter, Berlin (2011)
7. Beaudin, L.: A review of the Potts model. Rose-Hulman Undergraduate Math. J. **8**(1), 13 (2007). https://scholar.rose-hulman.edu/rhumj/vol8/iss1/13
8. Rozikov, U.A.: Gibbs measures on Cayley trees. World Scientific Publication, Singapore (2013)
9. Rozikov, U. A., Khakimov, O. N.: Description of all translation-invariant p-Adic gibbs measures for the Potts model on a Cayley tree. Markov Process. Relat. Fields **21**(1), 177-204 (2015). https://doi.org/10.48550/arXiv.1402.6150
10. Saburov, M., KhameiniAhmad, M.A.: On Descriptions of all translation invariant $p-$adic gibbs measures for the Potts model on the Cayley tree of order three. Math. Phys. Anal. Geom. **18**(1), 1–33 (2015). https://doi.org/10.1007/s11040-015-9194-5
11. Mukhamedov, F., Rozikov, U.A.: On Gibbs measures of p-adic Potts model on the Cayley tree. Indag. Math. **15**, 85–100 (2004). https://doi.org/10.1016/S0019-3577(04)90007-9
12. Mukhamedov, F., Rozikov, U.A.: On Inhomogeneous p-Adic Potts Model on a Cayley Tree. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **8**(2), 277–290 (2005). https://doi.org/10.1142/S0219025705001974

13. Wu, F.Y.: The Potts model. Rev. Mod. Phys. **54**, 235–268 (1982). https://doi.org/10.1103/RevModPhys.54.235
14. Mukhamedov, F., Rozikov, U.A.: Phase transitions in the Ising model on Z over the p-adic number field. Uzbek. Math. J. **4**, 23–29 (1998)
15. Mukhamedov, F.: Existence of p-adic quasi Gibbs measure for countable state Potts model on the Cayley tree. J. Inequal Appl. **2012**, 104 (2012). https://doi.org/10.1186/1029-242X-2012-104
16. Borevich, Z. I., Shafarevich, I.R.: Number Theory, Acad Press, New York (1966)
17. Koblitz, N.: p-adic Numbers, p-adic Analysis, and Zeta-Functions. Springer, New York (1984). https://doi.org/10.1007/978-1-4612-1112-9
18. Khrennikov, A.Y.: Interpretations of Probability. Walter de Gruyter, Berlin (2009)
19. Mukhamedov, F., Khakimov, O.: On periodic Gibbs measures of p-adic Potts model on a Cayley tree. p-Adic Num. Ultra. Anal. Appl. **8**(3), 225–235 (2016). https://doi.org/10.1134/S2070046616030043
20. Mukhamedov, F., Khakimov, O.: On Julia Set and Chaos in p-adic Ising model on the Cayley tree. Math. Phys. Anal. Geom. **20**(4), 1–14 (2017). https://doi.org/10.1007/s11040-017-9254-0
21. Ganikhodjaev, N., Mukhamedov, F., Rozikov, U.A.: Existence of a Phase Transition for the Potts p-adic Model on the Set $\mathbb{Z}$. Theor. Math. Phys. **130**(3), 425–431 (2002). https://doi.org/10.1023/A:1014723108030
22. Khrennikov, A.Y., Ludkovsky, S.: On infinite products of non-Archimedean measure spaces. Indag. Math. **13**(2), 177–183 (2002). https://doi.org/10.1016/S0019-3577(02)80003-9
23. Ludkovsky, S., Khrennikov, A.Y.: Stochastic processes on non-Archimedean spaces with values in non-Archimedean fields, Markov Process. Relat. Fields **9**, 131–162 (2003). https://doi.org/10.48550/arXiv.math/0110305