Simultaneous Algorithms for Solving Split Equality Fixed Point Problems for Demicontractive Mappings

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Abstract. In this paper, we introduced new algorithms for solving split equality fixed point problems for the class of demicontractive mappings in Hilbert spaces and proved the convergence results of the proposed algorithms. The results presented in this paper generalized a number of well-known results announced.

Keywords: Iterative Algorithm · Nonlinear Mappings · Fixed Point and Weak Convergence

1 Introduction

Throughout this paper, we consider $\langle ., .\rangle$ to be inner product and $||\,.\,||$ as its corresponding norm, H_1, H_2 and H_3 are Hilbert spaces, while C and D are nonempty, closed and convex subset of H_1 and H_2, respectively.

Let $S : H_1 \rightarrow H_1$ be a mapping. S is called quasi nonexpansive if $||\mathrm{Sx} - \mathrm{y}|| \leq ||\mathrm{x} - \mathrm{y}||$, for all $x \in H_1$ and $\mathrm{y} \in \mathrm{Fix(S)}$, where Fix(S) denote the fixed point set of S i.e. $\mathrm{Fix(S)} = \{\mathrm{y} \in \mathrm{H_1}\text{: } \mathrm{Sy} = \mathrm{y}\}$. S is called demicontractive if there exist a constant $k \in [0, 1)$ such that $||Sx - y||^2 \leq ||x - y||^2 + ||x - Sx||^2$, for all $y \in Fix(S)$ and $x \in H_1$.

The split feasibility problem (SFP) in finite-dimensional Hilbert space was introduced by Censor *et al.* [1] and is entailed as follows:

$$\text{Find } \overline{\mathrm{y}} \in \mathrm{C} \text{ such that } \mathrm{A}\overline{\mathrm{y}} \in \mathrm{D}, \tag{1}$$

where C and D are nonempty, closed and convex subset of H_1 and H_2, respectively and $A : H_1 \rightarrow H_2$ is a bounded linear operator. The SFP had received a lot of attention due to its applications in many real lives problems, such as in image reconstruction, phase retrievals, signal processing, see for example; [2, 3] and references therein.

Since every nonempty, closed and convex subset of Hilbert space can be seen as a fixed point of its associating projection, therefore, problem (1) reduces to the following:

$$\text{Find } \overline{\mathrm{y}} \in \mathrm{Fix(S)} \text{ such that } \mathrm{A}\overline{\mathrm{y}} \in \mathrm{Fix(T)}, \tag{2}$$

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where $S : C \to H_1$ and $T : D \to H_2$, respectively. Problem (2) is known as split fixed point problem (SFPP), see [4].

Iterative methods for solving the SFP and its related problems can be found in [5–8]. Related to the SFP, we have split equality problem (SEP), see Moudafi and Al-Shemas [9]. The SEP is defined as follows:

$$\text{Find } \bar{x} \in C \text{ and } \bar{y} \in D \text{ such that } A\bar{x} = B\bar{y}, \tag{3}$$

where C and D are nonempty, closed and convex subsets of H_1 and H_2, respectively, and $A : H_1 \to H_3, B : H_2 \to H_3$ are bounded linear operators.

Recently, Moudafi [10] also introduced another problem namely; split equality fixed point problem (SEFPP) which is defined as follows:

$$\text{Find } \bar{x} \in \mathrm{Fix}(T_1) \text{ and } \bar{y} \in \mathrm{Fix}(T_2) \ni A\bar{x} = B\bar{y}, \tag{4}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators, $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ are two nonlinear mappings with $\mathrm{Fix}(T_1) \neq \emptyset$ and $\mathrm{Fix}(T_2) \neq \emptyset$. He further obtained a weak convergence results for the SEFPP involving firmly quasi-nonexpensive mappings.

Very recently, Padcharoen *et al.* [11] studied the SEFPP by proposing the following iterative methods involving the class of demicontractive mappings.

$$\begin{cases} x_1 \in H_1; \\ v_n = (1-\alpha_n)x_n; \\ y_n = v_n + \tau_n A^*(S-I)Av_n; \\ x_{n+1} = (1-\gamma\tau_n)y_n + \gamma\tau_n T y_n, \end{cases} \tag{5}$$

where $\rho, \mu, \gamma, \alpha_n \in (0,1), \theta > 0$ and $\tau_n = \theta\rho^{l_n}$ with l_n being the smallest non-negative integer such that $\tau_n||A^*(S-I)Ay_n - A^*(S-I)Av_n|| \leq \mu||y_n - v_n||$.

It is known that computation of the norm of bounded linear operator is not an easy task, this was why Chang *et al.* [12] had considered the following algorithms for solving the SEFPP involving two quasi pseudocontractive mappings and proved the convergence results of the algorithms:

$$\begin{cases} \text{choose } x_0 \in H_1, y_0 \in H_2, \text{ arbitrary}, \\ x_{n+1} = x_n - \rho_n\big[x_n - Ux_n + A^*\big(Ax_n - By_n\big)\big], \\ y_{n+1} = y_n - \rho_n\big[y_n - Vx_n + B^*\big(By_n - Ax_n\big)\big], \end{cases} \tag{6}$$

where A and B are two bounded linear operators on H_1 and H_2, respectively, U and V are quasi pseudocontractive mappings.

Motivated by these results, in this paper, we proposed new iterative algorithms for solving the SEFPP involving the class of demicontractive mappings in Hilbert spaces and proved the convergence results of the proposed algorithms.

2 Preliminaries

In sequel, we shall make use of the following lemmas in proving our main results:

Lemma 2.1. (Opial [13]) Let $\{x_n\} \subseteq H$, such that there exist a nonempty set $\Omega \subseteq H$ such that the following conditions are satisfied:

i. For each $x \in \Omega$, $\lim_{n \to \infty} ||x_n - x||$ exists,

ii. Any weak-cluster point of the sequence $\{x_n\}$ belongs to Ω.

Then, there exists $y \in \Omega$ such that $\{x_n\}$ converges weakly to y.

Lemma 2.2. For x, y ∈ H, the following result hold:

i. $||x + y||^2 = ||x||^2 + 2x, y + ||y||^2, \forall x, y \in H$,
ii. $||x - y||^2 = ||x||^2 - 2x, y + ||y||^2, \forall x, y \in H$,
iii. $||\alpha x + (1 - \alpha)y||^2 = \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)||x - y||^2$ $\forall x, y \in H$, and $\alpha \in \mathbb{R}$.

3 Main Results

Let $A : H_1 \to H_2$ and $B : H_2 \to H_3$ be two bounded linear operators, and $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ be two non-linear mappings with Fix $(T_1) \neq \emptyset$ and Fix $(T_2) \neq \emptyset$. The SEFPP is defined as follows:

$$\text{Find } x^* \in \mathrm{Fix}(T_1) \text{ and } y^* \in \mathrm{Fix}(T_2) \ni Ax^* = By^* \tag{7}$$

In what follows, we donate the solution set of the SEFPP (7) by

$$\Gamma = \left\{\left(x^*, y^*\right) : x^* \in \mathrm{Fix}(T_1) \text{ and } y^* \in \mathrm{Fix}(T_2) \ni Ax^*By^*\right\}. \tag{8}$$

To approximate the solution of problem (8), we make the following assumptions:
(C1) H_1, H_2 and H_3 are Hilbert spaces, C and D are two nonempty, convex and closed subset of H_1 and H_2, respectively.
(C2) $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ are two demicontractive mappings with $\mathrm{Fix}(T_1) \neq \emptyset$ and $\mathrm{Fix}(T_2) \neq \emptyset$.
(C3) $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators with their adjoints
A^* and B^*, respectively.
(C4) $(T_1 - I)$ and $(T_2 - I)$ are demiclosed at zero.
(C5) **Algorithm:** Let $x_n \in H_1$ and $y_n \in H_2$, where $x_0 \in H_1$ and $y_0 \in H_2$ are chosen arbitrary.

Step 1. Compute

$$\begin{cases} x_{n+1} = \left(1-\gamma\tau_n\right)v_n + \gamma\tau_n T_1 v_n, \\ y_{n+1} = (1-\delta\tau_n)u_n + \delta\tau_n T_2 u_n, \end{cases} \tag{9}$$

Step 2. Choses v_n and u_n as follows:

$$\begin{cases} v_n = x_n + \tau_n A^*\left(By_n - Ax_n\right), \\ u_n = y_n + \tau_n B^*\left(Ax_n - By_n\right), \end{cases}$$

where $\tau_n \in \left(0, \frac{2}{L_1+L_2}\right)$, where $L_1 = ||A^*A||$ and $L_2 = ||B^*B||$, respectively.

Step 3. If $x_{n+1} = v_n$ and $y_{n+1} = u_n$, stop, otherwise, compute Step 1 to 2.

Theorem 3.1. Suppose that assumptions (C1)–(C5) are satisfied and $\boldsymbol{\Gamma} \neq \emptyset$. Then, the sequence $\left\{\left(x_n, y_n\right)\right\}$ generated by algorithm (9) converges weakly to the solution of problem (8).

Proof: Let $(x^*, y^*) \in \Gamma$, by algorithm (9) and the fact that T_1 and T_2 are demicontractive mappings, we have

$$\begin{aligned} \left\|x_{n+1}-x^*\right\|^2 &= \left\|(1-\gamma\tau_n)v_n + \gamma\tau_n T_1 v_n - x^*\right\|^2 \\ &= \left\|(1-\gamma\tau_n)\left(v_n - x^*\right) + \gamma\tau_n\left(T_1 v_n - x^*\right)\right\|^2 \\ &= (1-\gamma\tau_n)||v_n - x^*||^2 + \gamma\tau_n||T_1 v_n - x^*||^2 - \gamma\tau_n(1-\gamma\tau_n)||T_1 - v_n||^2 \\ &\le \left\|v_n - x^*\right\|^2 - \gamma\tau_n(1-k-\gamma\tau_n)\|T_1 v_n - v_n\|^2 \\ &= \left\|x_n + \tau_n A^*\left(By_n - Ax_n\right) - x^*\right\|^2 - \gamma\tau_n(1-k-\gamma\tau_n)\|T_1 v_n - v_n\|^2 \\ &= \left|\left|x_n - x^*\right|\right|^2 + 2\tau_n x_n - x^*, A^*\left(By_n - Ax_n\right) + \left|\left|\tau_n A^*\left(By_n - Ax_n\right)\right|\right|^2 - \gamma\tau_n\left(1-k-\gamma\tau_n\right)||T_1 v_n - v_n||^2 \\ &= \left|\left|x_n - x^*\right|\right|^2 + 2\tau_n Ax_n - Ax^*, By_n - Ax_n + \left|\left|\tau_n A^*\left(By_n - Ax_n\right)\right|\right|^2 - \gamma\tau_n\left(1-k-\gamma\tau_n\right)||T_1 v_n - v_n||^2. \end{aligned} \tag{10}$$

Thus,

$$\left|\left|x_{n+1}-x^*\right|\right|^2 \le \left|\left|x_n - x^*\right|\right| + 2\tau_n Ax_n - Ax^*, By_n - Ax_n + \tau_n^2||AA^*||||By_n - Ax_n||^2 - \gamma\tau_n\left(1-k-\gamma\tau_n\right)||T_1 v_n - v_n||.^2 \tag{11}$$

On the other hand,

$$|\,|y_{n+1}-x^*|\,|^2 \le \left|\left|y_n - x^*\right|\right|^2 + 2\tau_n y_n - By^*,\ Ax_n - By_n + \tau_n^2||BB^*||\,|By_n - Ax_n|\,|^2 - \delta\tau_n(1-k-\delta\tau_n)||T_2 u_n - u_n||^2 \tag{12}$$

From Eq. (11), Eq. (12) and the fact that $Ax^* = By^*$, we have that

$$\begin{aligned} &\left|\left|x_{n+1}-x^*\right|\right|^2 + \left|\,|y_{n+1}-x^*|\,\right|^2 \le \left|\left|x_n - x^*\right|\right| + \left|\left|y_n - x^*\right|\right|^2 - \tau_n(2-\tau_n(L_1+L_2))||By_n - Ax_n||^2 \\ &-\gamma\tau_n(1-k-\gamma\tau_n)||T_1 v_n - v_n||^2 - \delta\tau_n(1-k-\delta\tau_n)||T_2 u_n - u_n||^2. \end{aligned} \tag{13}$$

The fact that $\tau_n \in \left(0, \frac{2}{L_1+L_2}\right)$, where $L_1 = ||A^*A||$ and $L_2 = ||B^*B||$, Eq. (13) gives

$$\left|\left|x_{n+1}-x^*\right|\right|^2 + \left|\,|y_{n+1}-x^*|\,\right|^2 \le \left|\left|x_n - x^*\right|\right| + \left|\left|y_n - x^*\right|\right|^2. \tag{14}$$

Thus, $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is a monotone decreasing sequence and bounded below by zero, therefore converges. From Eq. (13) and the fact that $\left\{||x_n - x^*||^2 + ||y_n - y^*||^2\right\}$ converges, we deduce that

$$\lim_{n\to\infty} ||Ax_n - By_n|| = 0, \tag{15}$$

and

$$\lim_{n\to\infty} ||T_2By_n - By_n|| = 0, \lim_{n\to\infty} ||T_2y_n - y_n|| \tag{16}$$

Furthermore, since $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ converges, this ensures that both $\{x_n\}$ and y_n are bounded. Thus $x_n \rightharpoonup x^*$ and $y_n \rightharpoonup y^*$, respectively. By Eq. (9), we have that $v_n \rightharpoonup x^*$ and $u_n \rightharpoonup y^*$.

The fact that $x_n \rightharpoonup x^*$, $v_n \rightharpoonup x^*$, and $\lim_{n\to\infty} \left|\left|Ax_n - By_n\right|\right| = 0$ together with Eq. (9), we deduce that $x^* = T_1x^*$.

Similarly, we obtain that $y^* = T_2y^*$.

Now, $x_n \rightharpoonup x^*$ and $\lim_{n\to\infty} ||T_1x_n - x_n|| = 0$ together with the demiclosedness of $(T_1 - I)$ at zero, we deduce that $x^* \in Fix(T_1)$. On the other hand, $y_n \rightharpoonup y^*$ and $\lim_{n\to\infty} \left|\left|T_2y_n - y_n\right|\right| = 0$ together with the demiclosedness of $(T_2 - I)$ at zero, we deduce that $y^* \in Fix(T_2)$.

Since $v_n \rightharpoonup x^*$, $u_n \rightharpoonup y^*$ and the fact that A and B are bounded linear operators, we have

$$Av_n \rightharpoonup Ax^*, \text{ and } Bu_n \rightharpoonup By^*,$$

this implies that

$$Av_n - Bu_n \rightharpoonup Ax^* - By^*,$$

which turn to implies that

$||Ax^* - By^*|| \leq \lim_{n\to\infty} \inf||Av_n - Bu_n|| = 0$, which further implies that $Ax^* = By^*$. Noticing that $x^* \in Fix(T_1)$ and $y^* \in Fix(T_2)$, we conclude that $(x^*, y^*) \in \mathbf{\Gamma}$.

Summing up, we have proved that:

i. For each $(x^*, y^*) \in \mathbf{\Gamma}$, and the $\lim_{n\to\infty} \left(||x_n - x^*||^2 + \left|\left|y_n - y^*\right|\right|^2\right)$ exist
ii. Each weak cluster of the sequence (x_n, y_n) belongs to Γ.

Thus, by Lemma 2.1, we conclude that the sequence (x_n, y_n) converges weakly to $(x^*, y^*) \in \Gamma$. And the proof is complete.

Corollary 3.2. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators with their adjoints $A^* : H_3 \to H_1$ and $B^* : H_3 \to H_2$, respectively. Suppose that $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ are quasinonexpansive mappings and $T_1 - I$ and $T_2 - I$ are demiclosed at 0, then the sequence $\{(x_n, y_n)\}$ generated by algorithm (9) converges weakly to $(x^*, y^*) \in \Gamma$.

4 Conclusion

In this paper, we introduced new algorithms for solving split equality fixed point problems for the class of demicontractive mappings in Hilbert spaces and proved the weak convergence results of the proposed algorithms. However, the strong convergence result can be obtained if semi-compactness type conditions is imposed on our mappings. It is important to note that this semi-compactness condition appeared very strong as many nonlinear mappings are not semi-compact, therefore, a new research can be carried out to prove the strong convergence result without imposing the said condition.

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