Induction and Reflection in Linear Algebra Teaching

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Abstarct. Based on the characteristics and teaching practice of the course of "Linear Algebra", this paper introduces how to induction, summarize and reflection in the course of "Linear Algebra" through rich examples, so as to make the students better understand the concept and fully develop students' ability of problem analysis, problem-solving and innovation, and their mathematical literacy.

Keywords: Induction · Reflection · Linear algebra · Teaching strategy

Induction, summary and reflection are the process of deepening the learning, from cognition to understanding, from assimilation to absorption. After completing the learning of a section, it is recommended that one should take a systematic review of the contents, summarize the concepts, theories, and methods, and sum up the relationship between them. It would be helpful for the comprehensive and systematic grasp and understanding of the knowledge.

In this paper, on the basis of the teaching practices, and in combination with the characteristics of the "linear algebra" curriculum content, a lot of examples are presented to show how to timely induction and summarize in linear algebra teaching, so as to enable students to deepen the understanding of the concepts, fully cultivate students' ability to analyze problems, solve problems and innovate, and then improve their mathematical literacy.

According to the characteristics of the content, we mainly adopt the following induction methods in the teaching of linear algebra.

1 Induction and Summary of the Scattered Knowledge

In linear algebra, there are many different concepts interconnected, but the learning of these concepts occurs in different learning processes. If we can point out a relationship between a certain concept and other concepts at a certain stage, it will be easier for the students to understand and grasp the concepts. Therefore, we should timely summarize the scattered knowledge to associate them with other concepts, so that the students can have a comprehensive understanding of what they are learning.

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Table 1. The operational properties of the matrix

	addition	number multiplication	multiplication	inverse	transposition	companion matrix
transposition	$(A+B)^T = A^T + B^T$	$(kA)^T = kA^T$	$(AB)^T = B^T A^T$	$(A^T)^{-1} = (A^{-1})^T$	$(A^T)^T = A$	$(A^T)^* = (A^*)^T$
determinant	$\lvert A+B\rvert \neq \lvert A\rvert + \lvert B\rvert$	$\lvert kA\rvert = k^n \lvert A\rvert$	$\lvert AB\rvert = \lvert A\rvert\lvert B\rvert$	$\left\lvert A^{-1}\right\rvert = \lvert A\rvert^{-1}$	$\left\lvert A^T\right\rvert = \lvert A\rvert$	$\lvert A^*\rvert = \lvert A\rvert^{n-1}$
inverse	$(A+B)^{-1} \neq A^{-1} + B^{-1}$	$(kA)^{-1} = \frac{1}{k}A^{-1}$	$(AB)^{-1} = B^{-1}A^{-1}$	$(A^{-1})^{-1} = A$	$(A^{-1})^T = (A^T)^{-1}$	$(A^{-1})^* = (A^*)^{-1}$
companion matrix	$(A+B)^* \neq A^* + B^*$	$(kA)^* = k^{n-1}A^*$	$(AB)^* = B^* A^*$	$(A^*)^{-1} = (A^{-1})^*$	$(A^*)^T = (A^T)^*$	$(A^*)^* = \lvert A\rvert^{n-1} A$

1.1 The Operational Properties of the Matrix [1]

Common operations of the matrix include addition, number multiplication, multiplication, inverse, transposition, etc., which are interrelated and different with each other. The operations can be listed as Table 1 (the symbols and conditions mentioned in the table are all meaningful) in teaching,which will help the students to sort out and improve the knowledge understanding.

1.2 Equivalent Scale of the Matrix Rank [2]

The rank of a matrix is an important notion in theory of the matrix that reflects the essential properties of the matrix. We can scale the rank of a matrix from various angles of determinants, equivalent normal form, vectors, linear space, linear equations, linear transformation, matrix decomposition, etc.

Definition 1.1 Assume that $A \in F^{m\times n}$, then the maximum order r of nonzero sub-determinants of A is called the rank of A, denoted by $r(A) = r$.

Proposition 1.1. Assume that $A \in F^{m\times n}$, then $r(A) = r$ if and only if there is a r order nonzero sub-determinant of A, but all $r + 1$ order sub-determinants are all 0.

Proposition 1.2. Assume that $A \in F^{m\times n}$, then $r(A) = r$ if and only if there is at least one nonzero sub-determinant D of A with order r, and any sub-determinant of order $r+1$ including D as a sub-determinant has value zero.

Proposition 1.3. Assume that $A \in F^{m\times n}$, then $r(A) = r$ if and only if there is two inversible matrices $P \in F^{m\times m}$, $Q \in F^{n\times n}$, such that $PAQ = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$.

Corollary 1.4. Assume that $A \in F^{m\times n}$, then $r(A) = r$ if and only if $A \cong \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$.

Proposition 1.5. Assume that $A \in F^{m\times n}$, then $r(A) = r$ if and only if the number of vectors contained in an extremely irrelevant set of row vectors of A is r.

Proposition 1.6. Assume that $A \in F^{m\times n}$, then $r(A) = r$ if and only if the number of vectors contained in an extremely irrelevant set of column vectors of A is r.

Corollary 1.7. Assume that $A \in F^{m\times n}$, if $r(A) = r$, then both row rank and column rank of A are also equal to r.

Proposition 1.8. Assume that $A \in F^{m\times n}$, then $r(A) = r$ if and only if the dimension of the solution space of the system of linear equations $AX = 0$ is equal to $n - r$.

Proposition 1.9. Assume that $A \in F^{m\times n}$, then $r(A) = r$ if and only if the system of linear equations has r independent equations, and the rest are linear combinations of these equations.

Proposition 1.10. Let a basis $\alpha_1, \alpha_2, \ldots, \alpha_n$ of n-dimensional linear space, a basis $\beta_1, \beta_2, \ldots, \beta_m$ of m-dimensional linear space, the linear map σ corresponds $A \in F^{m\times n}$, i. e

$\sigma(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1, \beta_2, \ldots, \beta_m)A.$
Then $r(A) = r$ if and only if the dimension of $Im\sigma$ is equal to r.

Proposition 1.11. Assume that $A \in F^{m\times n}$ has a linear mapping $\sigma : F^n \to F^m, X \mapsto AX$, then $r(A) = r$ if and only if $dim(\mathrm{Im}\sigma) = r$.

Proposition 1.12. Assume that $A \in F^{m\times n}$, then $r(A) = r$ if and only if there are two matrices $P \in F^{m\times r}$, $Q \in F^{r\times n}$ and $r(P) = r(Q) = r$ such that $A = PQ$.

Proposition 1.13. Assume that $A \in F^{m\times n}$, then $r(A) = r$ if and only if there are r linear-independent vectors $\alpha_1, \alpha_2, \ldots, \alpha_\mathrm{r} \in F^{1\times n}$ and r linear-independent vectors $\beta_1, \beta_2, \ldots, \beta_r \in F^{m\times 1}$ such that $A = \beta_1\alpha_1 + \beta_2\alpha_2 + \ldots + \beta_r\alpha_r$.

The rank of the matrix is simple in definition, but is not easy to be understood and mastered. Only by correctly understanding the definition of the rank of the matrix, mastering the carving of the matrix rank, and organically combining with the knowledge in mathematics, can the rank of the matrix be better mastered.

1.3 The Induction of the Invertible Matrix [3]

In the teaching of linear algebra, the invertible array is closely related to the rank of the vector group matrix, the determinant of matrix, the linear correlation, the identification of the solutions of linear equations, the eigenvalues of the matrix, etc.

Proposition 1.14. Let A be a n-order phalanx, then

(1) A is an invertible matrix if and only if A is a full-rank matrix.
(2) A is an invertible matrix if and only if A is a non-singular matrix.

(3) A is an invertible matrix if and only if the determinant of A is not equal to zero.
(4) A is an invertible matrix if and only if A is a product of finite elementary matrices.
(5) A is an invertible matrix if and only if the row vector group of A is linearly independent.
(6) A is an invertible matrix if and only if all the eigenvalues of A are not equal to zero.
(7) A is an invertible matrix if and only if Equation $AX = b$ has a unique solution.

1.4 Application of the Elementary Transformation [4]

The elementary transformation of a matrix is a major tool for linear algebra problem solving. Here nine main applications of the matrix elementary transformation are summarized.

(1) Seek the rank of the matrix or the set of vectors;
(2) Seek the inverse matrix A^{-1} of the invertible matrix A;
(3) Seek the solution of matrix equation $AX = A + X$;
(4) The linear correlation of a set of vectors is determined, find the maximally independent group and represent the remaining vectors linearly by the maximally independent group;
(5) Determine whether the two sets of vectors are equivalent;
(6) Find the underlying solution system of homogeneous linear equations and the solution of the system of inhomogeneous equations;
(7) Quadratic form to standard form;
(8) Seek the maximum factor of integer and the maximum factor of polynomials;
(9) Solving the linear indefinite equations.

In class, if teachers can summarize the knowledge related to invertible array, matrix rank and the application of primary transformation in different chapters, it will help not only with the students' overall understanding of relevant knowledge, but also the application of this knowledge, and then expand students' ideas of solving problems. If the students are assigned to do such a summary work by themselves, it will help them to cultivate the ability to learn independently, and also improve the interest in learning mathematics.

2 Refining Summary of the Knowledge

Although some knowledge in Linear Algebra are in one chapter, in order to make students better understand the conclusions of propositions, the textbooks often prove or explain the conditions that need to be paid attention to, so they tend to be longer. For students of engineering, the task is to apply these theories as tools, so it is more important to remember the conditions and conclusions of these propositions. Therefore, it is necessary to refine and summarize these propositions to facilitate the learning of the students.

2.1 Identification Theorem for the Linear Correlations of Vector Groups [5]

Proposition 2.1. The set of vectors containing a zero vector has a certain linear dependent.

Proposition 2.2. For a set of vectors containing only one vector α, if $\alpha = 0$, then α is linear dependent, if $\alpha \neq 0$, then α is linearly independent.

Proposition 2.3. For a set of vectors containing only two vectors, If the two vectors are linearly dependent, then the corresponding components of the two vectors are proportional.

Proposition 2.4. A set of vectors with more than two vectors are linear dependent if and only if at least one vector can be linearly represented by the rest.

Proposition 2.5. A set of linear-dependent vectors adding some vectors is still linearly dependent.

Proposition 2.6. A set of linear-independent vector groups removing a number of vectors is still linearly independent.

Proposition 2.7. The new set of vectors obtained after adding component to each vector of a set of independent vectors is still linearly independent.

Proposition 2.8. A set of orthogonal vectors is linearly independent.

Proposition 2.9 If the rank r of a set of vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ is less than the number n of vectors, then the set of vectors are linearly dependent. If r equal to n, then the set of vectors are linearly independent.

Proposition 2.10. The set of n-dimensional vectors made up of $n+1$ vectors are linearly dependent.

Proposition 2.11. The set of n-dimensional vectors made up of n vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly dependent if and only if $\det(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$.

Proposition 2.12 If the set of vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent, but the set of vectors $\alpha_1, \alpha_2 \ldots, \alpha_n, \beta$ are linearly dependent, Then β can be represented linearly by $\alpha_1, \alpha_2, \ldots, \alpha_n$, and the representation method is unique.

2.2 The Properties of the Determinant [5]

(1) The determinant of the transpose of a square matrix equals the determinant of the original matrix.
(2) If any two rows (columns) of a determinant are interchanged, then the resulting determinant is the negation of the original determinant.
(3) If a determinant D has two identical rows (columns), then its determinant is zero.

(4) Multiplying a determinant by a scalar k is same as multiplying every entries of a row(column) of the determinant by k.
(5) The common factors of a row(column) of a determinant can be factored out of the determinant.
(6) In a determinant, if one row(column) is a constant multiple of another row(column), then the value of the determinant is zero.
(7) If each element of a row (column) of a determinant D is the sum of two numbers, it can be expressed as the sum of two determinants D_1 and D_2.
(8) In a determinant, adding k multiple of one column(row) to another, does not change the value of the determinant.

2.3 The Properties of Inverse Matrices [5]

(1) If A is invertible, then its inverse is unique.
(2) Suppose A is invertible. Then A^{-1} is invertible too, and $(A^{-1})^{-1} = A$.
(3) Suppose A is invertible and λ is any nonzero scalar. Then λA is invertible too, and $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$.
(4) Suppose A and B are invertible and have same dimension. Then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
(5) Matrix A is invertible if and only if det(A)$\neq$ 0, and $A^{-1} = \frac{1}{|A|}A^*$.
(6) If matrix A is invertible, then $\left|A^{-1}\right| = \frac{1}{|A|}$.

With these properties summed up, the following example is easy to solve.

Example 2.1. Suppose A is a 3-order matrix, and$|A| = \frac{1}{2}$, Compute the value $\left|(3A)^{-1} - 2A^*\right|$.

Solution $\left|(3A)^{-1} - 2A^*\right| \overset{(1)}{=} \left|\frac{1}{3}A^{-1} - 2A^*\right| \overset{(2)}{=} \left|\frac{1}{3}A^{-1} - 2 \times \frac{1}{2}A^{-1}\right| \overset{(3)}{=} \left|-\frac{2}{3}A^{-1}\right| \overset{(4)}{=} \left(-\frac{2}{3}\right)^3 \left|A^{-1}\right| \overset{(5)}{=} -\frac{8}{27} \times 2 = -\frac{16}{27}$.

The reason why this problem is favored by many textbooks is that it examines matrix inverse and many other knowledge related with matrix. For example, the (1) equal sign uses the knowledge of property 3, the (2) equal sign uses the property 5, the (3) equal sign uses the linear operation of matrix, the (4) equal sign uses the determinant property of square matrix, the (5) equal sign uses the property 6.

2.4 The Relationship Between the Solution of Linear Equations and the Linear Dependence of Vectors

The linear dependent of set vectors is combined with the identification of solution of equations, and the identification of determinant and rank is integrated into it, which can be summarized as follows.

For $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, there are the following conclusions:

(1) The linear system $AX = 0\begin{cases} has\ a\ non-zero\ solution. \\ only\ has\ zero\ solutions. \end{cases}$

$\Leftrightarrow \alpha_1, \alpha_2, \ldots, \alpha_n \begin{cases} are\ linear\ dependent. \\ are\ linear\ independent. \end{cases}$

$\Leftrightarrow \begin{cases} \textit{There are numbers} k_1, k_2, \ldots, k_n \textit{that are not all zero,} \\ \textit{There are no numbers } k_1, k_2, \ldots, k_n \textit{that are not all zero,} \end{cases}$ such that $k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n = 0. \Leftrightarrow \begin{cases} r(\alpha_1, \alpha_2, \ldots, \alpha_n) < n \\ r(\alpha_1, \alpha_2, \ldots, \alpha_n) = n \end{cases} \Leftrightarrow \begin{cases} r(A) < n \\ r(A) = n \end{cases} \Leftrightarrow \begin{cases} |A| = 0 \\ |A| \neq 0 \end{cases} A$

(2) The linear system $AX = \beta \begin{cases} \textit{has infinitely many solutions.} \\ \textit{has a unique solution.} \\ \textit{has no solution.} \end{cases}$

$\begin{cases} r(A) = r(A, \beta) < n \\ r(A) = r(A, \beta) = n \\ r(A) < r(A, \beta) \end{cases} \Leftrightarrow$

$$\begin{cases} \beta \textit{ can be linearly represented by} \\ \alpha_1, \alpha_2, \ldots, \alpha_n, \textit{ and the representation is not unique.} \\ \beta \textit{ can be linearly represented by} \\ \alpha_1, \alpha_2, \ldots, \alpha_n, \textit{ and the representation is unique.} \\ \beta \textit{ can't be linearly represented by} \\ \alpha_1, \alpha_2, \ldots, \alpha_n. \end{cases}$$

3 Generalization From Special to General [6]

A method of reasoning by which a general conclusion is deduced from a particular premise. Facts first, conclusions later. This is working from the special to the general, to seek the universal characteristics of the knowledge method. For example, it is easy to know that second-order and third-order determinants are defined as follows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = \sum_{p_i} (-1)^{\tau(p_1p_2)} a_{1p_1}a_{2p_2};$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{p_i} (-1)^{\tau(p_1p_2p_3)} a_{1p_1}a_{2p_2}a_{3p_3}.$$

So we naturally get

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{p_i} (-1)^{\tau(p_1p_2\cdots p_n)} a_{1p_1}a_{2p_2}\cdots a_{np_n}.$$

From the algebraic expressions of the 2 and 3-order determinants, we com to the conclusion that the n-order determinant for n > 3 is also similar to the expressions of the 2 and 3-order determinants, which is the algebraic sum of the product of all the n elements taken from each row and column.

4 The Comparison and Induction of Different Knowledge Points

In teaching, students often encounter a lot of similar concepts or knowledge of similar properties that can easily cause confusion. We can make their similarities and differences clear by comparison and induction.

For example, in linear algebra, there are often equivalence, similarity and congruence between matrices. These three kinds of relations have similarities, but also have their own characteristics. Students often mix up the concepts and properties of these three kinds of relations. Here, we can compare and summarize them from the definition, relationship, property and equivalent classification.

4.1 The Definition of Matrix Equivalence, Similarity and Congruence is Compared and Summarized

Definition 4.1. Let A and B be two the same type of matrix. A is said to be equivalent to B, if there exist two invertible matrices P,Q such that $PAQ = B$.

Definition 4.2. Let A and B be two n-order square matrices. A is said to be similar to B, if there exists an non-singular matrix P such that $P^{-1}AP = B$.

Definition 4.3. Let A and B be two n-order square matrices. A and B are said to be congruent if there exists an invertible matrix P, such that $P^TAP = B$.

4.2 The Relation of Matrix Equivalence, Similarity, Congruence are Compared and Summarized (Fig. 1)

4.3 The Properties of Matrix Equivalence, Similarity and Congruence are Compared and Summarized (Table 2)

4.4 Equivalence Classes of Matrix Equivalence, Similarity and Congruence are Compared and Summarized

Proposition 4.1. $m \times n$ matrix can be divided into k class under the equivalence relation, where $k = min\{m, n\}$.

Proposition 4.2. For all the n-order square matrices, there are infinitely many equivalence classes under similarity relation.

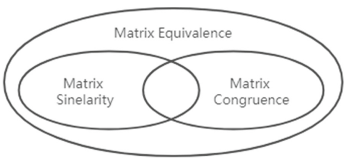

Fig. 1. The relation of matrix equivalence, similarity and congruence.

Table 2. The properties of matrix equivalence, similarity and congruence

relation	rank	determinant	eigenvalue	standard	symmetry
equivalence	same	---	---	same	---
similarity	same	same	same	---	---
congruence	same	same positive and negative	same positive and negative	same	same

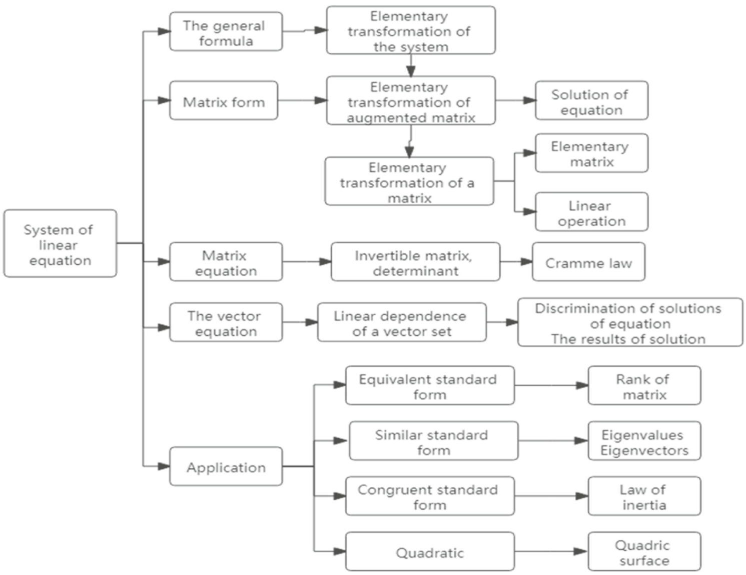

Fig. 2. Knowledge structure system of “linear algebra”

Proposition 4.3. In the complex field, all the n-order symmetric matrices can be divided into $n + 1$ classes under the consistency relation (Fig. 2).

Proposition 4.4. In the real field, all the n-order symmetric matrices can be divided into $\frac{1}{2}(n + 1)(n + 2)$ classes under the consistency relation.

4.5 Refinement of the Main Line of Knowledge [3]

The extraction of the so-called knowledge mainline refers to categorizing the teaching content, connecting the most important concepts, laws and principles hidden in the complex contents, establishing knowledge chains or summarizing chapter and unit knowledge by using diagrams, and constructing knowledge system or knowledge network.

Linear equations are the main line of knowledge throughout linear Algebra. This induction method can not only screen out the main contents, but also make students clear about the relationship between each knowledge point and the status and role in the whole knowledge system, deepen the understanding of knowledge, and is more conducive to students' memory and transfer of the knowledge.

5 Conclusion

Induction and reflection are effective teaching methods. In teaching, the scientific and appropriate application of induction and reflection can not only make the knowledge system clear, reduce the difficulty of teaching, deepen students' understanding of knowledge, improve students' interest in learning, but also cultivate students' ability to think and innovate, hence improve students' scientific literacy.

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