Teaching Reform of Singular Value Decomposition Based on the Exploratory Typical Case

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ABSTRACT

Singular value decomposition is an important way of matrix decomposition, which has many important applications in the fields of machine learning, signal processing and statistics. However, the singular value decomposition in the matrix computations course focuses on the theoretical teaching of basic concepts and calculation methods. Students can't understand the importance of singular value decomposition and how to use singular value decomposition. Therefore, an exploratory typical case is designed and implemented in this paper based on singular value decomposition theorem. The typical case is about image compression and student-friendly, which mainly includes gray-image compression matrix model, color-image compression matrix model and color-image compression tensor model. The realization of gray-image compression matrix model is helpful for students to understand and master the significance and application of the singular value decomposition. The analysis of the two color-image compression models is helpful for students to explore the similarities and difference between tensor singular value decomposition and matrix singular value decomposition. In the teaching process of singular value decomposition, discussion teaching is adopted to implement this teaching case, which has received good teaching feedback, mobilized students' enthusiasm in learning mathematics, and improved the teaching effect of matrix computations course.

Keywords- *matrix; tensor; singular value decomposition; typical case*

1. Introduction

Matrix computation aims to use computer as a tool to study and design algorithms to solve various mathematical models quickly, effectively and accurately. It is an indispensable tool in big data processing. Among them, singular value decomposition is the core content of matrix calculation. In the field of machine learning, there are many applications related to singular value decomposition, such as principal component analysis for feature dimensionality reduction, image data compression algorithms, and late semantic indexing for semantic hierarchical retrieval of search engine. In order to arouse the students' enthusiasm in the teaching of matrix computations for the second grade, it is necessary to introduce the applications of matrix decomposition so as to further mobilize the students' enthusiasm in the teaching of mathematics. At the same time, it is the trend of the development of modern science and technology to make students realize that various disciplines intersect and penetrate each other [1][2].

Case teaching is a student-centered teaching method with a specific real situation as the background, aiming at improving students' ability of independent analysis and solving practical problems, and strengthening the cognition and mastery of professional knowledge and theory [3]. Case teaching is focus on cultivating students' ability to use theoretical analysis and solve complex practical problems, as well as students' ability to think rationally and make forward-looking judgments in practical innovation. It plays an important role in guiding the development direction of high-level applied talents [4]. It can make students become learning initiators, aspirants, knowledge explorers and skill masters. This paper is devoted to design and implement a typical case of singular value decomposition in matrix computation, since exploring the case teaching mode suitable for matrix calculation is inseparable from excellent teaching cases.

2. Case Design

In the era of information explosion, there are more and more image information in daily work and life, but it

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will face the problems of image communication and storage. Due to the fact that images occupy a large storage space, it brings great difficulties to image transmission efficiency and image preservation. Digital images need fast transmission efficiency and great space reserve, which has become the biggest obstacle to the promotion of digital images. How to solve this problem with the help of singular value decomposition, inspire students to think and debate about practical problems and further explore theoretical knowledge is the starting point and goal of the case design.

2.1. Design Basis

According to the SVD theorem of singular value decomposition in [5], we can make a low rank matrix to approximate the original matrix by means of the largest k singular values and the corresponding left and right singular vectors of the original matrix. Since real images are stored as matrices and almost all images are approximately low rank, image compression can be realized by singular value decomposition. For example, for a $m \times n$ matrix, the original matrix needs $m \times n$ bytes of storage space if we assume that a pixel occupies 1 byte, while only $k \times (m + n + 1)$ pixel are needed for the low rank approximation matrix.

2.2. Design Framework

Note that a gray image is usually stored as a matrix and a color image is stored as a third-order tensor which includes three matrices representing red, green and blue channels, respectively. Singular value decomposition can be applied to each matrix to get the gray-image compression matrix model and color-image compression matrix model. Meanwhile, color-image compression tensor model is constructed by applying t-SVD in [6] to the tensor data. The framework is shown in Fig. 1.

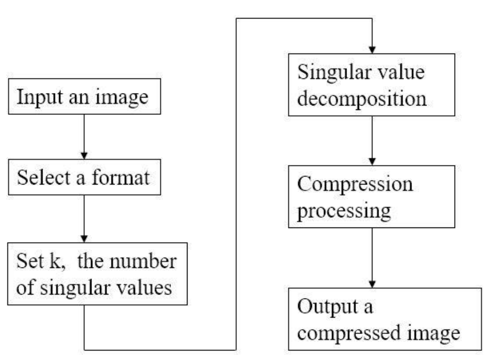


Figure 1. The framework of image compression

Since a $m \times n$ matrix can be seen as a third-order tensor with $m \times n \times 1$. Let a_{ijt} and $\tilde{a}_{ijt}$ be the pixel of the input image and the output image respectively. There are two evaluation measures to reflect the effect of compression:

- Mean square error method

$$MSE = \frac{1}{m\times n\times c}\sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{t=1}^{c} (a_{ijt} - \tilde{a}_{ijt})^2}$$

The smaller the value of MSE, the more similar the reconstructed image is to the original image.

- Correct rate method

$$Ture\_rate = \frac{\text{the number of correct } \tilde{a}_{ijt}}{m\times n\times c}$$

where the number of correct $\tilde{a}_{ijt}$ means $\tilde{a}_{ijt}$ lies in the neighborhood of a_{ijt} with radius 0.05.

2.3. Experiments

Select an image and use the matrix form or tensor form to represent the image, then perform singular value decomposition on the matrix or tensor, and select the appropriate number of k to compress the image data without losing the quality of the original image as much as possible.

1) Gray-image compression matrix model: Given a color image, convert it to a gray image with 3024×4032 by the Matlab function rgb2gray, and then we get a matrix whose entries are the pixels of the obtained gray image. The values of MSE and True_rate with different k are list in Table 1. It is clear that with the increase of k the value of MSE decreases and the value of True_rate increases, which coincides with the theorem of singular value decomposition for matrix. To further illustrate the effect of approximating the original gray image, the compressed image obtained by setting k be 80 is shown in Fig. 2.

TABLE I. THE RESULTS OF MSE AND TRUE_RATE FOR

k	5	10	20	40	80
MSE	5.53e-6	3.22e-6	2.11e-6	1.38e-6	8.81e-7
True_rate	0.6902	0.7738	0.8211	0.8613	0.8940

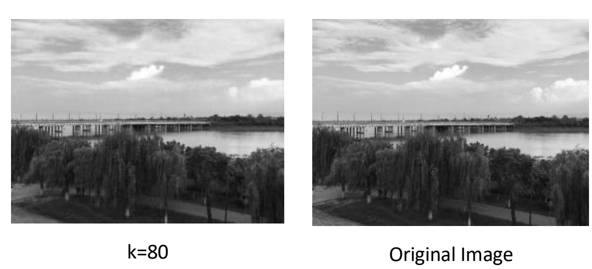
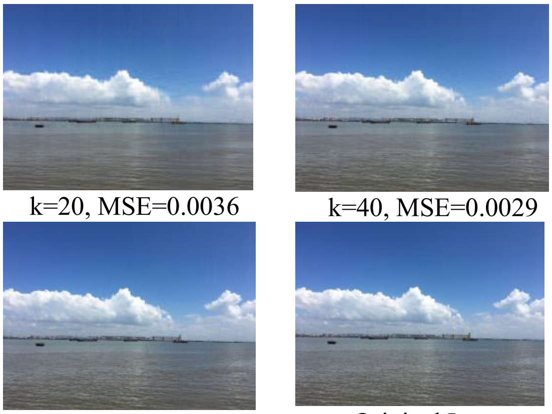


Figure 2. The performance of gray image compression (photo credit: original)

2) Color-image compression matrix model: It is natural from design basis to get the fact that a large image can be approximated by a low-rank matrix, which reduces the data storage in image transmission. However, a new question appears, how to deal with color image which isn't stored as a matrix.



k=20, MSE=0.0036 k=40, MSE=0.0029

k=80, MSE=0.0022 Original Image

Figure 3. The performance of color image compression (photo credit: original)

Given a color image, which is stored as a 1706 × 1279 × 3 tensor A in Matlab. Three slices of the tensor A, A(:, :, 1), A(:, :, 2) and A(:, :, 3) are three 1706 × 1279 matrices which are red, green and blue channels, respectively. For every channel, we apply the same operation as the gray image to get an approximate matrix, and then get the compressed image by reconstructing a tensor with the same size 1706 × 1279 × 3 by means of the approximate matrices. The value of MSE with different k are shown in Fig. 3.

3) Color-image compression tensor model: A color image is stored as a third-order tensor in Matlab. Tensor is a higher-order generalization of vector and matrix, and it can well characterize the natural structure of multidimensional data. In color-image compression matrix model, the tensor is treated as three matrices which destories the structure of the third-order tensor.

There are a lot of research on tensor and its applications [7]. Among them, tensor singular value decomposition is very important since the singular value decomposition of matrix plays an important role in matrix research and tensor is a higher- order generalization of vector and matrix. For tensor singular value decomposition, it is inevitable to utilize the tensor rank. However, the definition of tensor rank is not unique[7-11], which is different from matrix rank. Here, we select t-SVD [6] which is easy to understand and program.

k=5, MSE=0.0106 k=10, MSE=0.0080

k=20, MSE=0.0063 k=40, MSE=0.0052

k=80, MSE=0.0041 Original Image

Figure 4. Compressed images with t-SVD for a color image (photo credit: original)

An example can be found in Fig. 4. For the input original image A with size 1706 × 1280 × 3, we first use Afft = fft(A, [ ], 3) to transform the tensor A in time domain into the same dimension tensor Afft in frequency domain, and then apply the color-image compression matrix model to the tensor Afft in frequency domain to get a new tensor Bfft. In the end, we transform Bfft into the time domain by means of B = fft(Bfft, [ ], 3) and get the output image B.

3. Implementation and Evaluation

Excellent teaching cases need to be equipped with appropriate teaching means and forms in order to express the charm of case *teaching*. This typical case designed in section II has been applied in matrix computations teaching and in three classes, about 200 undergraduates learned this case. According to the cognitive law and acceptance characteristics of sophomores, after comprehensively considering the advantages and disadvantages of online and offline teaching, we learn from each other and then adopt the discussion teaching mode with teaching students according to their aptitude, in order to promote the communication and interaction between teachers and students.

Matrix computing is a mathematics course, which covers concept definitions, theorems and algorithms. The course itself is abstract and monotonous. Singular value decomposition is designed as a typical case, is not only suitable for traditional offline teaching reducing the boredom caused by PPT expression. It also improves the communication between teachers and students, and deepens students' understanding of high-dimensional data structure by the visual display. The discussion teaching mode includes many good questions, not limited to the listed below:

1) From the display of the typical case, please think about the format in which the data of gray image is stored, the format in which the data of color image is stored, the difference of the formats of gray image and color image.

2) Please think about how to store a gray video or a color video.

3) Please think about the difference of the two models designed to deal with color images, how to select the definition of singular value decomposition for a tensor.

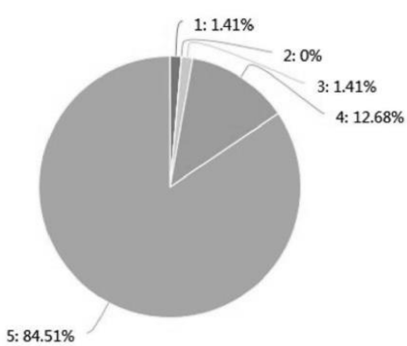


Figure 5. Score proportion of satisfaction survey

Answering these questions can help students further understand abstract mathematical concepts，which can help teachers free themselves from traditional and inefficient teaching work and pay more personalized attention to students' growth.

In order to evaluate the teaching effect, we conducted a satisfaction survey and set a total of 5 points. One point is dissatisfied and five points are satisfied. The higher the score, the higher the satisfaction. A total of 172 students participated in the satisfaction survey, specific score proportion is shown in Fig. 5. 84.51% of the students are satisfied with the teaching case and only 1.41% of the students are not satisfied with the teaching case. About 14.09% of the students are basically satisfied with the case. The survey result shows that the design of the case is reasonable and the implementation of the case is effective.

Through the classroom demonstration and discussion of teaching cases, students not only understand the application of singular value decomposition, but also master the storage mode of digital images. By combining abstract mathematical concepts with vivid digital images, the classroom atmosphere is activated and the enthusiasm of active learning is mobilized. The discussion on the two compression methods of color images directly leads to the concept of tensor and expands students' understanding of high-dimensional data space. The comparison of the results of numerical experiments of the two modes leads to the discussion of tensor rank. How to define tensor rank is still an open problem, which stimulates students' curiosity in unknown fields.

4. CONCLUSION

The current era is an era of information explosion. More and more multimedia information is filled with our life and work. Because the image has good singular value properties, singular value decomposition has been widely used in image feature extraction, image retrieval and matching, digital watermarking, video compression and other fields. The discussion of these fields needs to be further studied, and then a more comprehensive and timely teaching case is designed to improve the teaching effectiveness. Singular value decomposition is one of the basic tools to study matrix decomposition. Although this paper discusses image compression based on singular value decomposition for gray images and color images, the discussion in this field is not very in-depth and comprehensive. In the follow-up research, we can further explore other applications of singular value decomposition, such as applications in signal denoising, latent semantic index and recommendation system.

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