A New Stochastic Response Surface Method in Spatial Variability Slope Stability Analysis

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Abstract. Spatial variability is becoming more and more common in limit equilibrium slope stability analysis. With the recent advances in the abilities for limit equilibrium slope stability analysis to handle complicated models, it is more important than ever to have a fast and accurate method to perform slope stability analysis in spatially varying soils. The use of response surfaces is widely adopted in the literature to deliver efficient stochastic analyses. However, their implementation is usually deeply correlated with the choice of a field generation algorithm. This makes the response surface method inflexible since it sometimes requires the field generation algorithm to be completely rewritten. Finely discretized spatially varying random fields can have many random variables which make the response surface difficult to determine. To solve this, the authors have proposed a modified response surface guided approach to slope reliability analysis in spatially varying soils which is independent of the choice of field generation algorithm. This approach was evaluated using a two-dimensional problem, and validated by comparing its accuracy and speed against the traditional Monte Carlo simulation.

1 Introduction

A probabilistic analysis is one where variability in the parameters is accounted for by considering them as random variables. In contrast, an analysis with constant input parameters is called deterministic analysis. A deterministic and probabilistic analysis should be used together to consider different sources of uncertainty in the analysis and design of slopes (Javankhoshdel and Bathurst 2014, 2016).

In the probabilistic analysis, a distribution (e.g. normal or lognormal) is defined for the random variable. Traditionally, the input parameter distributions are sampled N times – once per parameter, per simulation. In each simulation a limit equilibrium (LE) analysis is performed to determine the factor of safety (FS). The probability of failure (PF) is defined as follows:

$$PF = \frac{\text{Number simulations with FS} < 1}{\text{Total number of simulations}} \times 100\% \qquad (1)$$

Monte Carlo sampling (MC) or Latin Hypercube (LH) are traditionally used for sampling of random variables in the probabilistic analysis (Hicks and Spencer 2010; Nuttall 2011; Ji and Chan 2014). The number of samples required to produce an accurate estimate of the PF depends on the complexity of the model, number of random variables and/or whether the spatial variability of soil properties is considered. The computational time required for such a probabilistic analysis is generally proportional to N.

Spatial variability of soil properties is due to the sedimentation process and chemical composition of the medium during formation. Several researchers in literature have investigated the impact of the spatial variability of soil strength and stiffness parameters on the stability of geo-structures (Javankhoshdel et al. 2022). The importance of spatial variability of soil properties has been investigated through several studies of slope stability analysis (Javankhoshdel and Bathurst 2014; Luo et al. 2016; Javankhoshdel and Bathurst 2016, 2017; Javankhoshdel et al. 2017; Jamshidi Chenari and Izadi 2019; Jamshidi Chenari et al. 2020; Shah Malekpour et al. 2020; Javankhoshdel et al. 2020; Mafi et al. 2020).

The Random Limit Equilibrium Method (RLEM) has recently been implemented commercially in the Slide2 Rocscience software (Rocscience 2022) which allows for stochastic slope stability analyses. Javankhoshdel and Bathurst (2014, 2016), Javankhoshdel et al. (2017), and Jamshidi Chenari and Izadi (2019) are examples of studies considering only circular slip surface assumptions in their RLEM analyses. Lately, with the introduction of optimization techniques in geotechnical field such as Cuckoo search and Particle Swarm Optimization (PSO), which are global optimization techniques together with the Surface Altering Optimization (SAO) which is a local optimization technique (as introduced by Mafi et al. 2020), Tabarroki et al. (2013), Javankhoshdel et al. (2017) and Cami et al. (2018) put into practice the implementation of non-circular failure surface in their slope stability analyses. Owing to their improved efficiency and effectiveness at identifying critical slip surfaces, these non-circular search methods have become more popular where spatial variability of soil strength parameters is considered (Javankhoshdel et al. 2022).

Stochastic response surface (SRS) is a method that can be used to reduce the computation time of the complicated probabilistic analyses. However, SRS is sensitive to the number of random variables in the analysis. In the spatial variability analysis, each cell of spatial variable material is a variable itself so the original SRS approach is may not always be suitable for such problems.

In this paper the original SRS algorithm that is well-suited for non-spatial analyses is presented first. Then a new algorithm is developed to be able to use SRS for the spatial variability analyses as well. The proposed SRS approach is more suitable for spatially variable problems than the original SRS method.

2   Original Stochastic Response Surface Method

The original SRS method first employs a small sample of strategically selected computations. These are used to generate a polynomial-based response surface of FS as a function of the input parameters. The response surface can then predict the FS value of every simulation, and hence provide an estimation of the PF. As expected, this method

Table 1. Examples of transformation equations from three common distributions to standard normal distributions (Li et al., 2011)

Distribution	Probability density function, $f(x)$	Transformation, $x = f(U)$
Uniform	$f(x) = \frac{1}{b-a}$	$x = \Phi(U)(b-a) + a$
Normal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2)$	$x = \mu + \sigma U$
Lognormal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma x}\exp(-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2)$	$x = \exp(\mu + \sigma U)$

See Li et al. (2011) *for more details on the included parameters.*

saves a significant amount of time when compared to traditional methods. However, it also preserves a very reasonable degree of accuracy. The steps are outlined below (Isukapalli 1999, Cami et al., 2021).

Step 1: Convert all random variables to standard normal. The initial random variables are first converted to standard normal random variables. Standard normal random variables are defined as having a mean of 0 and standard deviation of 1. This is done with a series of transformation functions, some of which are listed in Table 1 as examples (Li et al., 2011).

Step 2: Represent FS in polynomial form. Once all the variables have been normalized, FS can be expressed as a function of all the normalized variables. The function of choice can be anything but based on previous studies (Isukapalli 1999, Cami et al., 2021) this study has adopted the 3rd order Hermite polynomial chaos expansion, shown in Eq. 2. The coefficients will be determined by the sample of initial simulations.

$$F(U_1, \ldots U_n) = a_0 + \sum_{i=1}^{n} a_i \Gamma_1(U_i) + \sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij}\Gamma_2(U_i, U_j)$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} a_{ijk}\,\Gamma_3(U_i, U_j, U_k) \qquad (2)$$

Note that in Eq. 2, F is the factor of safety and U_i is the value of parameter i. Γ_p is the pth degree Hermite polynomial. Its definition is found in Eq. 3.

$$\Gamma_p(U_1, \ldots, U_p) = (-1)^p e^{0.5 U^T U} \frac{\delta^p}{\delta U_1, \ldots, \delta U_p} e^{-0.5 U^T U} \qquad (3)$$

Step 3: Determine the coefficients. The number of simulations required to be computed (N) is calculated as shown in Eq. 4. These are the number of simulations that must be pre-computed in order to determine the coefficients of the response surface.

$$N = 2\left(1 + 3n + \frac{3n(n-1)}{2} + \frac{n(n-1)(n-2)}{6}\right) \qquad (4)$$

The most challenging part of the SRS method is picking the best collocation points. Various studies have determined various different algorithms for picking out the best

collocation points. These are important in that they are used to train the model – they determine the extent of the information fed into the response surface to train it. Several method were compared in this study, and in the end Latin Hypercube was used to determine the collocation samples (Choi et al., 2004). Latin-Hypercube ensures that the solution space has been equally sampled, hence not ignoring any key solution regions. Once the collocation samples have been gathered, the FS for each one is computed, and they are used to solve for the coefficients in Eq. 2, *a*.

Step 4: Estimate PF. Once *a* has been determined, Eq. 2 is complete as a predicting function. The desired number of samples is now collected, and each one is fed through the equation, resulting in the desired number of predicted FS values. These are then used to compute the predicted PF value, using Eq. 1.

3   Sparse Quadratic Response Surface

The shortfall of the original SRS method is that it is not suitable for spatially variable materials, where the number of random variables can become very large (one variable per cell). With so many variables, overfitting in the model can occur, and the number of parameters in the response surface function becomes unreasonably large (Zhou et al. 2021).

To resolves these issues, the variational autoencoder (VAE) approach was used to discretize the spatially varying fields. VAE's (Kingma and Welling 2014) provide efficient inference and learning for the given probabilistic model. Furthermore, the VAE was used instead of other classical approaches because the variables in the compressed representation had two desirable properties: the distribution was normal (desirable for the Response Surface function), and the variables in the latent space were disentangled. In statistics, latent random variables not directly observed, but are rather inferred from observed variables. The resulting random field has latent random variables which closely follow a standard normal distribution.

Afterwards, a preliminary slope reliability analysis was performed by training a sparse quadratic response surface on the learned features. The nature of the FS computation is expensive, which means that to generate the response surface we require generalization from only a few training samples - a few hundred computations in our case. As such, this problem is in the area of few-shot learning (FSL), which is described in Wang et al. (2019). One approach to this type of problem is called self-supervised learning, in which the model learns from both a small number of labelled data (fields which we have found the FS for), and many unlabeled data (fields with unknown FS). In this modality of machine learning, the aim is to help the downstream task (determining the factor of safety) by first pertaining a model in the upstream task (compressing the random fields).

If it is desired, the resulting response surface can be further iteratively corrected to an unbiased estimation of a limit equilibrium target analysis by performing subset simulations near the failure domain. Results indicate that variational autoencoders can successfully be applied to discretize random fields, allowing for response surfaces to be implemented independently of the field generation method.

4   Example: 2D

Figure 1 shows the example model used in this study to compare the results of the probabilistic analysis using LHS method and the Stochastic Response Surface algorithm presented in this study.

In this model, the green region is a weak layer as it can be seen in Table 2.

The cohesion of the "Soil" material and friction angle of the "Weak Layer" material are assumed to have spatial variations. The statistical properties including the spatial correlation length of these properties are presented in Table 3.

Figure 2 and Fig. 3 present an example of the random field generated for cohesion of the Soil material and friction angle of the Weak Layer material, respectively.

1000 samples were generated for the probabilistic analysis using both methods. The Spencer LEM method together with the PSO and Surface Altering Optimizations are used to calculate FS values with the non-circular failure mechanisms.

The surface results of the LHS method are presented in Fig. 4 and the results of the Stochastic Response Surface method are presented in Fig. 5. The critical slip surface shown in both figures is the deterministic factor of safety. It can be seen in Figs. 4 and 5
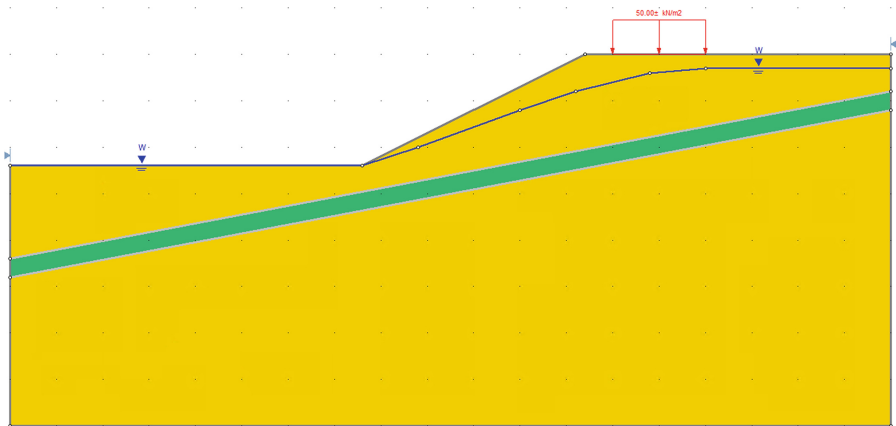

Fig. 1. Example model with a weak layer.

Table 2. Material Properties table.

Material Name	Color	Unit Weight (kN/m3)	Strength Type	Cohesion (kPa)	Phi (deg)	Water Surface	Hu Type	Hu
Soil		19	Mohr-Coulomb	12	25	Water Surface	Custom	1
weak layer		18.5	Mohr-Coulomb	0	30	Water Surface	Custom	1

Table 3. Statistical parameters.

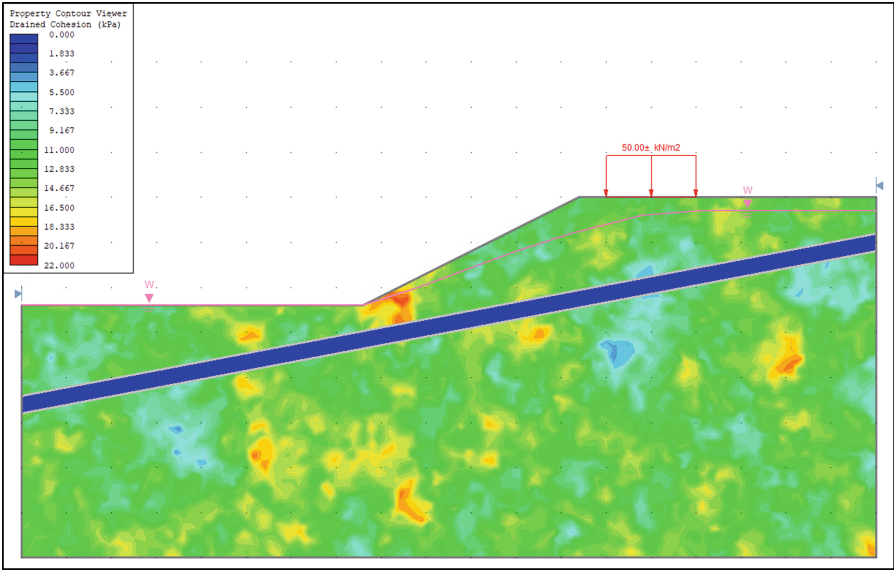| Material Name	Random parameter	Distribution type	COV	Horizontal Correlation length	Vertical Correlation length
Soil	Cohesion	Normal	0.25	5	5
Weak Layer	Friction angle	Lognormal	0.23	1	1

Fig. 2. An example random field for cohesion in the Soil material.

that, a smaller number of samples required in the Stochastic Response Surface Method (a smaller number of surfaces in Fig. 5). In other words, to train the stochastic response surface, we require only a portion of the 1000 samples, i.e., the computation time for the Stochastic Response Surface method is less that the LHS method.

Also, the mean FS value computed using LHS method (not shown in these figures) was 1.065 and the mean FS for the Stochastic Response Surface method was 1.07 which are in a good agreement.
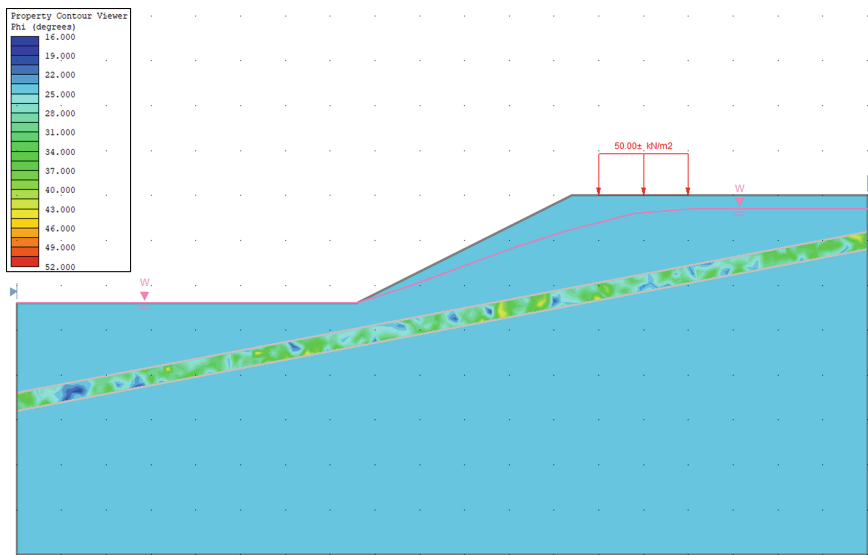
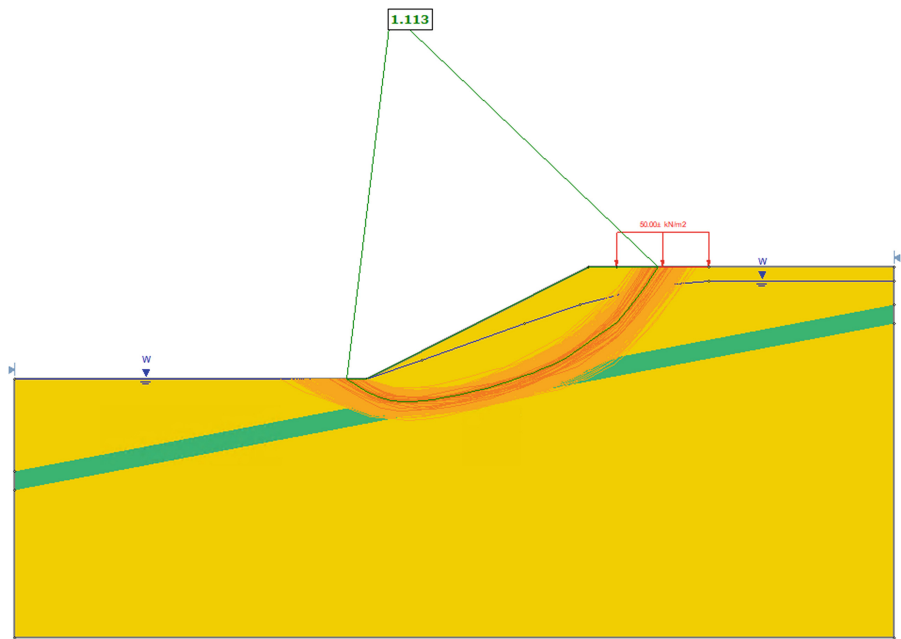**Fig. 3.** An example random field model for the friction angle of the Weak Layer material.

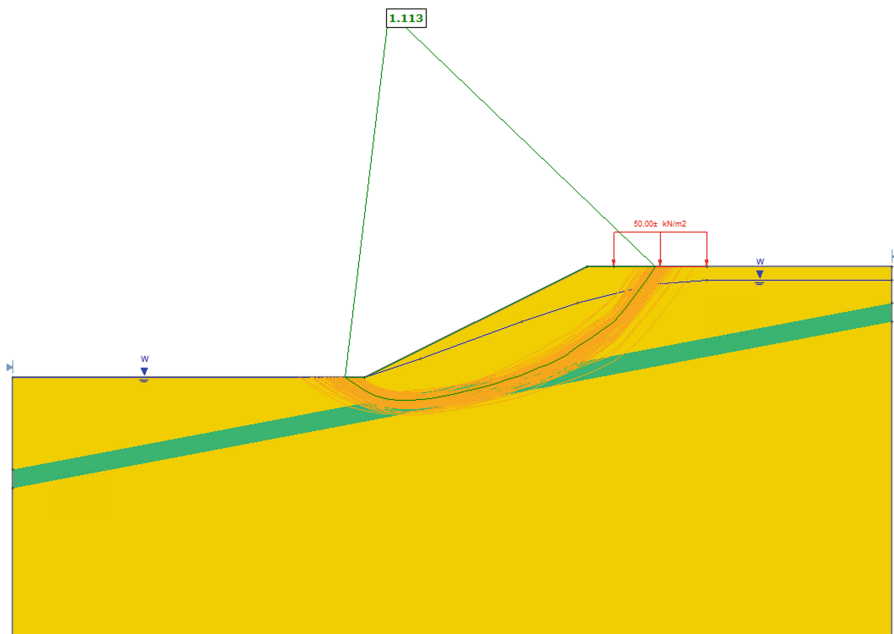**Fig. 4.** Results of the probabilistic analysis in this study using LHS method.

Fig. 5. Results of the probabilistic analysis in this study using the Stochastic Response Surface method.

5   Conclusion

A method for efficiently estimating the probability of failure in slope stability for spatially varying materials is presented in this paper. Stochastic response surfaces (SRS) is a popular method for estimating the probability of failure in a slope without needing to complete a full slope stability analysis for every simulation. However, in the case of spatially variable materials where the properties are assigned using random distributions, traditionally SRS methods fall short due to the large number of spatially distributed variables in the random field. The difficulty of constructing the response surfaces because of the large random fields is overcome in this paper using a sparse quadratic response surface via advanced techniques including variational encoding and few-shot learning. The efficiency of the proposed method is demonstrated via an example.

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