A Study of the Rainbow Antimagic Coloring of Double Wheel and Parachute Graphs

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Abstract. In this study, we explore the idea of rainbow antimagic coloring, which combines the notions of antimagic labeling with rainbow coloring. We consider a connected and simple graph $G(V, E)$ with a vertex set $V(G)$ and an edge set $E(G)$. A vertex labeling of graph G is defined as a bijective function $f : V(G) \rightarrow \{1, 2, 3, ..., k\}$, where the resulting weight for an edge $uv \in E(G)$ is determined by the function $f(u)+f(v) = w_f(uv)$. A set of edge weight functions $(W(G))$ is obtained from the edge weight function $(wf(uv))$, and is determined by the cardinality of the set $(|W(G)|)$. A function f of graph G is considered a rainbow antimagic labeling if the weight of any two edges uv and $u'v'$ on the path $u - v$ are distinct. The minimum amount of hues necessary for a rainbow coloring that emerges from a rainbow antimagic labeling of graph G, denoted as $rac(G)$, is referred to as the rainbow antimagic connection number. Our study has led to the discovery of the precise numerical values for $rac(G)$ of both a double wheel graph and a parachute graph. Specifically, $rac(DW_n) = 2n$ and $rac(PC_n) = n + 2$.

Keywords: rainbow antimagic connection number · double wheel graph · parachute graph · rainbow connection number

1 Introduction

A graph G is a finite set of ordered pairs (V, E), where V represents a set of vertices that is not empty and E represents a set of edges (which may be empty) consisting of unordered pairs of two vertices (v_1, v_2) where $v_1, v_2 \in V$, referred to as edges. In a graph G, the collection of vertices is represented by $V(G) = \{v_1, v_2, v_3, ..., v_n\}$, while the set of edges in a graph G is represented by

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Dafik (Ed.): ICCGANT 2022, APR 6, pp. 120–132, 2023.
https://doi.org/10.2991/978-94-6463-138-8_11

$E(G) = \{e_1, e_2, e_3, ..., e_n\}$. It should be noted that a graph G may not have any edges, but must have at least one vertex [1]. The order of a graph G is defined as the number of vertices, and is denoted by $|V(G)|$, whereas the size of a graph G is defined as the number of edges, and is denoted by $|E(G)|$ [4].

One of the significant areas of study in graph theory is graph coloring. Initially presented by Chartrand *et al.* in their publication "Introduction to Graph Theory 4th ed" [2], the concept of graph coloring includes the term rainbow coloring of graphs. In a graph G that is connected and has no loops or multiple edges, edge coloring is defined as a function $c : E(G) \rightarrow \{1, 2, 3, ..., |E(G)|\}$, where edges that are adjacent may have the same color. A path in which no two consecutive edges possess identical colors is referred to as a rainbow path. A graph G is considered to have the property of rainbow connection if there exists a path between any two vertices u and v that is composed of edges with different colors. The coloring of the edges of graph G that results in this property is known as rainbow coloring. The minimum quantity of hues necessary for a connected graph G to possess the property of rainbow connection, denoted as $rc(G)$, is referred to as the rainbow connection number of the graph. Subsequently, this concept has been further developed by various researchers; Agustin *et al.* [14] obtained rc for specific graphs and obtained a sharp lower bound, while Hasan *et al.* [15] obtained rainbow connection numbers for several shackle graphs and showed that the boundaries are sharp.

Another crucial area of study within the field of graph theory is the concept of graph labeling, including the term antimagic labeling. The introduction of this concept was first presented by Hartsfield and Ringel in their publication "Pearls in Graph Theory: A Comprehensive Introduction 3rd ed" [3]. A graph is considered labeled antimagic if there is a bijective function $f : E(G) \rightarrow \{1, 2, 3, ..., |E(G)|\}$ such that the weight of all vertices are distinct. The weight of a vertex u is represented by $w(u)$, which is number of the labels from each edge associated with u and can be expressed as $w(u) = \Sigma_{e \in E(u)} f(e)$. In this context, f is referred to as antimagic labeling.

In this study, we investigate the concept of rainbow antimagic coloring, which is an idea developed by Dafik *et al.* [10] which combines the notions of antimagic labeling with rainbow coloring. We consider a connected and simple graph $G(V, E)$ with a vertex set $V(G)$ and an edge set $E(G)$. A vertex labeling of graph G is defined as a bijective function $f : V(G) \rightarrow \{1, 2, 3, ..., k\}$, where the resulting weight for an edge $uv \in E(G)$ is determined by the function $f(u) + f(v) = w_f(uv)$. A set of edge weight functions $(W(G))$ is obtained from the edge weight function $(w_f(uv))$, and is determined by the cardinality of the set $(|W(G)|)$. A function f of graph G is considered a rainbow antimagic labeling if the weight of any two edges uv and $u'v'$ on the path $u - v$ are distinct. The rainbow antimagic connection number, represented by $rac(G)$, is established as the least quantity of colors necessary for a rainbow coloring obtained through a labeling of the graph G that is both rainbow and antimagic.

The idea of rainbow antimagic coloring has been explored by a number of researchers, such as Sulistiyono *et al.* [6], Jabbar *et al.* [7], Septory *et al.* [8], Budi *et al.* [9], Dafik *et al.* [10], Joedo *et al.* [11], Nisviasari *et al.* [12], and Septory *et al.* [13]. They have established the rainbow antimagic connection numbers on various types of graphs including simple graphs, special graphs, and operational graphs. In order to validate the theorem that has been derived, we have employed the following lemma as a means of support.

Lemma 1. *[5] If G is a connected graph that is not trivial in size and has m elements, the maximum distance between any two vertices in G $(diam(G)) \leq (rc(G)) \leq (src(G)) \leq (m)$.*

Lemma 2. *[8] Supposing G is a connected graph. If G has $\Delta(G)$ and $rc(G)$, then $rac(G) \geq max\{rc(G), \Delta(G)\}$.*

2 Result

Through our research, we have established the values of $rac(DW_n)$ and $rac(PC_n)$ as well as $rc(DW_n)$ and $rc(PC_n)$. The methodology used in this study is based on the techniques previously proposed by Dafik and Septory [8], with the addition of our own methods and techniques.

Theorem 1. *It is established that a double wheel graph with $n \geq 3$ has rainbow connection numbers, namely*

$$rc(DW_n) = \begin{cases} 2, if \;\; n = 3, 4 \\ 3, if \;\; n = 5, 6 \\ 4, if \;\; n \geq 7 \end{cases}$$

Proof. A double wheel graph has a vertex set $V(DW_n) = \{r\} \cup \{s_i; 1 \leq i \leq n\} \cup \{t_i; 1 \leq i \leq n\}$ and an edge set $E(DW_n) = \{rs_i; 1 \leq i \leq n\} \cup \{rt_i; 1 \leq i \leq n\} \cup \{s_i s_{i+1}; 1 \leq i \leq n-1\} \cup \{s_n s_1\} \cup \{t_i t_{i+1}; 1 \leq i \leq n-1\} \cup \{t_n t_1\}$. Therefore, the set of vertices $|V(DW_n)|$ and the set of edges $|E(DW_n)|$ have cardinality of $2n+1$ and $4n$, respectively. The demonstration of the theorem will be separated into three distinct cases.

Case 1. If $n = 3, 4$. According to Lemma 1, $rc(G) \geq diam(G)$, we know that $max\{d(u, v)\}$ for $u = s_i$ and $v = t_i$ then $diam(DW_n) = 2$, so $rc(DW_n) \geq 2$. Next, we will prove $rc(DW_n) \leq 2$, defined a function $c : E(DW_n) \to \{1, 2\}$ as follows:

$$\begin{aligned} c(rs_i) \quad\quad\quad &= 1, if \;\; 1 \leq i \leq n \\ c(s_i s_{i+1}, t_i t_{i+1}) &= 1, if \;\; 1 \leq i \leq n-1 \\ c(rt_i) \quad\quad\quad &= 2, if \;\; 1 \leq i \leq n \\ c(s_n s_1, t_n t_1) \;\; &= 2 \end{aligned}$$

The maximum value of the edge function above is $c = s_n s_1$, so $rc(DW_n) \leq 2$.

Based on the conditions above, we get $2 \geq rc(DW_n) \geq 2$. Thus, it can be concluded that $rc(DW_n) = 2$ for $n = 3, 4$.

The following step is to examine the rainbow path of a double wheel graph as presented in Table 1.

Table 1. The rainbow path that connects u to v in DW_n.

Case	If	x	y	Rainbow Path	Condition
1	$n = 3, 4$	r	s_a	r, s_a	$1 \le a \le n$
2	$n = 3, 4$	r	t_a	r, t_a	$1 \le a \le n$
3	$n = 3, 4$	s_a	t_a	s_a, r, t_a	$1 \le a, a \le n$
4	$n = 3$	s_a	s_b	s_a, s_b	$a \ne b$ $1 \le a, b \le n$
		t_a	t_b	t_a, t_b	
5	$n = 4$	s_a	s_b	s_a, s_b	$a \ne b$ $1 \le a, b \le n$ $a, b \ne odd,$ $a, b \ne even$
		t_a	$1t_b$	t_a, t_b	
6	$n = 4$	s_a	s_b	s_a, s_c, s_b	$a \ne b$ $1 \le b, a \le n$ $a, b = odd,\ c = 4,$ $b, a = even,\ c = 1$
		t_a	t_b	t_a, t_c, t_b	

Case 2. If $n = 5, 6$. According to Lemma 1, $rc(G) \ge diam(G)$, we know that $max\{d(u, v)\}$ for $u = s_i$ and $v = t_i$ then $diam(DW_n) = 2$, so $rc(DW_n) \ge 2$. Assume $rc(DW_n) = 2$, if $c(t_n t_1) = c(s_n s_1) = c(s_i s_{i+1}) = c(t_i t_{i+1}) = 2$ for $i = 3$ then $c(s_2 r) = c(r s_4)$ or $c(s_2 s_3) = c(s_3 s_4)$. It's a contradiction because the rainbow $s_2 - s_4$ path has the same color path, so $c(s_3 s_4) = 3$. Based on assume above, thus $rc(DW_n) \ge 3$. Next, we will prove $rc(DW_n) \le 3$, defined a function $c : E(DW_n) \to \{1, 2, 3\}$ as follows:

$$\begin{aligned} c(s_i s_{i+1}, t_i t_{i+1}) &= \begin{cases} 1, & if \quad i = 1, 4 \\ 2, & if \quad i = 2, 5 \\ 3, & if \quad i = 3 \end{cases} \\ c(r s_i) &= 1, \qquad if \ \ 1 \le i \le n \\ c(r t_i) &= 2, \qquad if \ \ 1 \le i \le n \\ c(s_n s_1, t_n t_1) &= 3 \end{aligned}$$

The maximum value of an edge function above is $c = t_n t_1$, so $rc(DW_n) \le 3$.

Based on the conditions above, we get $3 \ge rc(DW_n) \ge 3$. Thus, it can be concluded that $rc(DW_n) = 3$ for $n = 5, 6$.

The following step is to examine the rainbow path of a double wheel graph as presented in Table 2.

Case 3. If ≥ 7. According to Lemma 1, $rc(G) \ge diam(G)$, let $rc(DW_n) \ge 3$. Assume $rc(DW_n) = 3$, if $c(r t_i) = 3$ then $c(t_3 r) = c(r t_5)$ or $c(t_3 t_4) = c(t_4 t_5)$. It's a contradiction because the rainbow $t_3 - t_5$ path has the same color path, so

Table 2. The rainbow path that connects u to v in DW_n.

Case	If	x	y	Rainbow Path	Condition
1	$n = 5, 6$	r	s_a	r, s_a	$1 \leq a \leq n$
2	$n = 5, 6$	r	t_a	r, t_a	$1 \leq a \leq n$
3	$n = 5, 6$	s_a	t_a	s_a, r, t_a	$1 \leq a, a \leq n$
4	$n = 5, 6$	s_a	s_b	s_a, s_b	$a \neq b$ $1 \leq a, b \leq n$ $b = a + 1,$ $b = a - 1$
		t_a	t_b	t_a, t_b	
5	$n = 5, 6$	s_a	s_b	s_a, s_b	$a \neq b$ $a, b = 1, n$
		t_a	t_b	t_a, t_b	
6	$n = 5$	s_a	s_b	s_a, s_b	$a \neq b$ $2 \leq a, b \leq n$ $a, b = odd,$ $a, b = even$
		t_a	t_b	t_a, t_b	
7	$n = 5$	s_a	s_b	s_a, s_c, s_b	$a \neq b$ $a, b = 1, 3,\ c = 2$
		t_a	t_b	t_a, t_c, t_b	
8	$n = 5$	s_a	s_b	s_a, s_c, s_b	$a \neq b \neq c$ $1 \leq a, b \leq n$ $b = a + 3, c = 1, 5,$ $b = a - 3, c = 1, 5$
		t_a	t_b	t_a, t_c, t_b	
9	$n = 6$	s_a	s_b	s_a, s_c, s_b	$a \neq b \neq c$ $1 \leq a, b \leq n$ $b = a + 2,$ $a < c < b$ $b = a - 2,$ $a > c > b$
		t_a	t_b	t_a, t_c, t_b	
10	$n = 6$	s_a	s_b	s_a, s_c, s_b	$a \neq b \neq c$ $1 \leq a, b \leq n$ $b = a + 4, c = 1, 6,$ $b = a - 4, c = 1, 6$
		t_a	t_b	t_a, t_c, t_b	
11	$n = 6$	s_a	s_b	s_a, s_c, s_d, s_b	$a \neq b \neq c$ $1 \leq a, b \leq n$ $b = a + 3,$ $a < c < d < b$ $b = a - 3,$ $a > c > d > b$
		t_a	t_b	t_a, t_c, t_d, t_b	

Table 3. The rainbow path that connects u to v in DW_n.

Case	x	y	Rainbow Path	Condition
1	r	s_a	r, s_a	$1 \leq a \leq n$
2	r	t_a	r, t_a	$1 \leq a \leq n$
3	s_a	t_a	s_a, r, t_a	$1 \leq a, a \leq n$
4	s_a	s_b	s_a, r, s_b	$a \neq b \neq c,\ 1 \leq a, b \leq n$ $b = even,\ a = odd$ $a = even,\ b = odd$
	t_a	t_b	t_a, r, t_b	
5	s_a	s_b	s_a, s_c, r, s_b	$b \neq c \neq a,\ 1 \leq a, b \leq n$ $b, a = even \iff a, b = odd$ $c = a + 1,\ 1 \leq a \leq n - 1$ $c = a - 1,\ a = n$
	t_a	t_b	t_a, t_c, r, t_b	

$c(rt_i) = 4$ if i even. Based on assume above, thus $rc(DW_n) \geq 4$. Next, we will prove $rc(DW_n) \leq 4$, defined a function $c : E(DW_n) \rightarrow \{1, 2, 3, 4\}$ as follows:

$$c(rs_i) = \begin{cases} 1, if \ \ i = odd \\ 2, if \ \ i = even \end{cases}$$
$$c(rt_i) = \begin{cases} 3, if \ \ i = odd \\ 4, if \ \ i = even \end{cases}$$
$$c(s_i s_{i+1}, s_n s_1) = 3$$
$$c(t_i t_{i+1}, t_n t_1) = 1$$

The maximum value of the edge function above is $c = rt_i$ if i even, so $rc(DW_n) \leq 4$.

Based on the conditions above, we get $4 \geq rc(DW_n) \geq 4$. Thus, it can be concluded that $rc(DW_n) = 4$ for $n \geq 7$.

The following step is to examine the rainbow path of a double wheel graph as presented in Table 3.

Theorem 2. *It is established that a double wheel graph with $n \geq 3$ has rainbow connection numbers, $rac(DW_n) = 2n$.*

Proof. A double wheel graph has a vertex set $V(DW_n) = \{r\} \cup \{s_i; 1 \leq i \leq n\} \cup \{t_i; 1 \leq i \leq n\}$ and an edge set $E(DW_n) = \{rs_i; 1 \leq i \leq n\} \cup \{rt_i; 1 \leq i \leq n\} \cup \{s_i s_{i+1}; 1 \leq i \leq n-1\} \cup \{s_n s_1\} \cup \{t_i t_{i+1}; 1 \leq i \leq n-1\} \cup \{t_n t_1\}$. Therefore, the set of vertices $|V(DW_n)|$ and the set of edges $|E(DW_n)|$ have cardinality of $2n + 1$ and $4n$, respectively.

We aim to demonstrate that $rac(DW_n) = 2n$ for $n \geq 3$ by utilizing the lower and upper bounds. Initially, we demonstrate the validity of the lower bound $rac(DW_n) \geq 2n$ with the aid of Lemma 2, as follows:

$$\begin{aligned} rac(DW_n) &\geq max\{rc(DW_n), \Delta(DW_n)\} \\ 2n &= max\{4, 2n\} = 2n \end{aligned}$$

So the lower bound of a double wheel graph is $rac(DW_n) \geq 2n$.

Next, we will prove the upper bound $rac(DW_n) \leq 2n$. A function $f : V(DW_n) \to \{1, 2, 3, ..., |V(DW_n)|\}$ in the graph is defined as follows:

$$\begin{aligned} f(r) &= n+1 \\ f(s_i) &= \begin{cases} \frac{i+1}{2}, & if\ \ i\ odd,\ 1 \leq i \leq n-1 \\ \frac{3+3i}{2}, & if\ \ i\ odd,\ i = n \\ \frac{2n+i+2}{2}, & if\ \ i\ even,\ 1 \leq i \leq n \end{cases} \\ f(t_i) &= \begin{cases} 2i+1, & if\ \ i = n \\ i+1, & if\ \ i = n-1 \\ \frac{n+i}{2}, & if\ \ i\ odd,\ n\ odd \\ \frac{n+i+1}{2}, & if\ \ i\ odd,\ n\ even \\ \frac{3n+i+3}{2}, & if\ \ i\ even,\ n\ odd \\ \frac{3n+i+2}{2}, & if\ \ i\ even,\ n\ even \end{cases} \end{aligned}$$

Based on a vertex function, the edge weight of a double wheel graph is as follows:

$$\begin{aligned} w_f(rs_i) &= \begin{cases} \frac{2n+i+3}{2}, & if\ \ i\ odd,\ 1 \leq i \leq n-1 \\ \frac{5n+5}{2}, & if\ \ i\ odd,\ i = n \\ \frac{4n+i+4}{2}, & if\ \ i\ even,\ 1 \leq i \leq n \end{cases} \\ w_f(rt_i) &= \begin{cases} 3i+2, & if\ \ i = n \\ 2i+3, & if\ \ i = n-1 \\ \frac{3n+i+2}{2}, & if\ \ i\ odd,\ n\ odd \\ \frac{3n+i+3}{2}, & if\ \ i\ odd,\ n\ even \\ \frac{5n+i+5}{2}, & if\ \ i\ even,\ n\ odd \\ \frac{5n+i+4}{2}, & if\ \ i\ even,\ n\ even \end{cases} \\ w_f(s_i s_{i+1}) &= \begin{cases} 3i+5, & if\ \ i\ = n-1,\ n\ odd \\ n+i+2, & if\ \ i = odd,\ 1 \leq i \leq n-1 \\ n+i+2, & if\ \ i = even,\ 1 \leq i \leq n-2 \end{cases} \\ w_f(s_n s_1) &= \begin{cases} \frac{3n+5}{2}, & if\ \ n = odd \\ \frac{3n+4}{2}, & if\ \ n = even \end{cases} \\ w_f(t_i t_{i+1}) &= \begin{cases} 2i+3, & if\ \ i = n-2,\ n\ odd \\ 3i+4, & if\ \ i = n-1,\ n\ even \\ 2n+2+i, & if\ \ i\ even,\ 1 \leq i \leq n-1 \\ 2n+2+i, & if\ \ i\ odd,\ 1 \leq i \leq n-3 \end{cases} \\ w_f(t_n t_1) &= \begin{cases} \frac{5n+3}{2}, & if\ \ n = odd \\ \frac{5n+4}{2}, & if\ \ n = even \end{cases} \end{aligned}$$

Table 4. The rainbow path that connects u to v in DW_n.

Case	x	y	Rainbow Path	Condition
1	r	s_a	r, s_a	$1 \leq a \leq n$
2	r	t_a	r, t_a	$1 \leq a \leq n$
3	s_a	t_a	s_a, r, t_a	$1 \leq a, a \leq n$
4	s_a	s_b	s_a, r, s_b	$a \neq b,\ 1 \leq a, b \leq n$
5	t_a	t_b	t_a, r, t_b	$b \neq a,\ 1 \leq a, b \leq n$

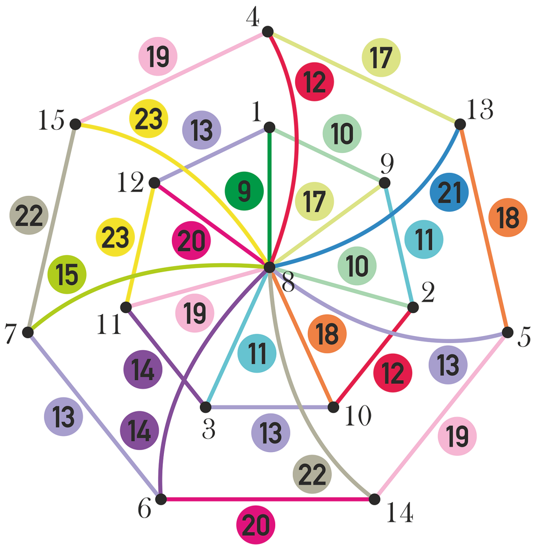

Fig. 1. Rainbow antimagic coloring of DW_7.

The edge weight function above produces the sets edge weight as follows:

$$\begin{aligned} W(DW_n) &= \left\{\tfrac{2n+4}{2}, \ldots, \tfrac{3n+4}{2}, \ldots, \tfrac{4n+3}{2}\right\}, \\ &\quad \left\{\tfrac{4n+6}{2}, \ldots, \tfrac{5n+7}{2}, \ldots, \tfrac{6n+5}{2}\right\} \\ &= n + n \\ |W(DW_n)| &= 2n \end{aligned}$$

The number of edge weights is $2n$ based on the sets of edge weights, so $rac(DW_n) \leq 2n$.

Based on the lower bound and upper bound, we get $2n \geq rac(DW_n) \geq 2n$. Thus, it can be concluded that $rac(DW_n) = 2n$ for $n \geq 3$.

The following step is to examine the rainbow path of a double wheel graph as presented in Table 4.

An illustration of $rac(DW_n)$ for $n = 7$ can be observed in Fig. 1.

Theorem 3. *It is established that the value $rc(PC_n) = 3$ where $n \geq 2$.*

Table 5. The rainbow path that connects u to v in PC_n.

Case	x	y	**Rainbow Path**	**Condition**
1	r	s	r, s	
2	r	t_a	r, s, t_a	$1 \leq a \leq n$
3	r	u	r, s, t_a, u	$1 \leq a \leq n$
4	s	t_a	s, t_a	$1 \leq a \leq n$
5	s	u	s, t_a, u	$1 \leq a \leq n$
6	t_a	u	t_a, u	$1 \leq a \leq n$
7	t_a	t_b	t_a, s, t_b	$b \neq a,\ 1 \leq a, b \leq n$ $b\ even,\ a\ odd$ $a\ even,\ b\ odd$
8	t_a	t_b	t_a, s, t_c, t_b	$b \neq a,\ 1 \leq a, b \leq n$ $a, b = odd \iff b, a = even$ $c = b - 1,\ b > a$ $c = b + 1,\ b < a$

Proof. A parachute graph has a vertex set $V(PC_n) = \{r\} \cup \{s\} \cup \{u\} \cup \{t_i;\ 1 \leq i \leq n\}$ and an edge set $E(PC_n) = \{rs\} \cup \{st_i;\ 1 \leq i \leq n\} \cup \{ut_i;\ 1 \leq i \leq n\} \cup \{t_i t_{i+1};\ 1 \leq i \leq n-1\}$. Therefore, the set of vertices $|V(PC_n)|$ and the set of edges $|E(PC_n)|$ have cardinality of $n + 3$ and $3n$.

According to Lemma 1, $rc(G) \geq diam(G)$, where $max\{d(u, v)\}$ is calculated for $v = u$ and $u = r$ then $diam(PC_n) = 3$, so $rc(PC_n) \geq 3$. Next, we will prove $rc(PC_n) \leq 3$, defined the function $c : E(PC_n) \rightarrow \{1, 2, 3\}$ as follows:

$$
\begin{aligned}
c(rs) &= 1 \\
c(st_i) &= \begin{cases} 2, & if\ \ i\ odd \\ 3, & if\ \ i\ even \end{cases} \\
c(ut_i) &= \begin{cases} 2, & if\ \ i\ even \\ 3, & if\ \ i\ odd \end{cases} \\
c(t_i t_{i+1}) &= 2, \qquad if\ \ 1 \leq i \leq n-1
\end{aligned}
$$

The highest value that can be obtained from the edge function $c = st_i$ if i odd, so $rc(PC_n) \leq 3$.

Based on the conditions above, we get $3 \geq rc(PC_n) \geq 3$. Thus, it can be concluded that $rc(PC_n) = 3$ for $n \geq 2$.

The following step is to examine the rainbow path of a parachute graph as presented in Table 5.

Theorem 4. *It is established that the value* $rac(PC_n) = n + 2$ *where* $n \geq 2$.

Proof. A parachute graph has a vertex set $V(PC_n) = \{r\} \cup \{s\} \cup \{u\} \cup \{t_i;\ 1 \leq i \leq n\}$ and an edge set $E(PC_n) = \{rs\} \cup \{st_i;\ 1 \leq i \leq n\} \cup \{ut_i;\ 1 \leq i \leq n\} \cup \{t_i t_{i+1};\ 1 \leq i \leq n-1\}$. Therefore, the set of vertices $|V(PC_n)|$ and the set of edges $|E(PC_n)|$ have cardinality of $n + 3$ and $3n$.

We aim to demonstrate that $rac(PC_n) = n+2$ for $n \geq 2$ by utilizing the lower and upper bounds. Initially, we demonstrate the validity of the lower bound $rac(PC_n) \geq n+2$ with the aid of Lemma 2, as follows:

$$\begin{aligned} rac(PC_n) &\geq max\{rc(PC_n), \Delta(PC_n)\} \\ n+2 &= max\{3, n+1\} \\ n+2 &= (n+1)+1 \end{aligned}$$

Assume $rac(PC_n) = n+1$, if there are possible permutations of vertex labels that result in $n+1$ edge weights, then there are the same vertex labels. This contradicts the definition of antimagic labeling, where all dot labels must be distinct. Furthermore, edge weight numbers produce $n+n$ weights, and these edge weight numbers are not on the degree edge. So the lower bound of a parachute graph is $rac(PC_n) \geq n+2$.

Subsequently, we will demonstrate the upper bound of $rac(PC_n) \leq n+2$. A function $f : V(PC_n) \rightarrow \{1, 2, 3, ..., |V(PC_n)|\}$ in the graph is defined as follows:

$$\begin{aligned} f(r) &= n+3 \\ f(s) &= n \\ f(u) &= n+2 \\ f(t_i) &= \begin{cases} \frac{i+1}{2}, & if \;\; i = odd \\ n+1, & if \;\; i = 2 \\ \frac{2n-i+2}{2}, & if \;\; i = even, \; i \geq 4 \end{cases} \end{aligned}$$

Based on a vertex function, the edge weight of a parachute graph is as follows:

$$\begin{aligned} w_f(rs) &= 2n+3 \\ w_f(st_i) &= \begin{cases} \frac{2n+i+1}{2}, & if \;\; i = odd \\ 2n+1, & if \;\; i = 2 \\ \frac{4n-i+2}{2}, & if \;\; i = even, \; i \geq 4 \end{cases} \\ w_f(ut_i) &= \begin{cases} \frac{2n+i+5}{2}, & if \;\; i = odd \\ 2n+3, & if \;\; i = 2 \\ \frac{4n-i+6}{2} & if \;\; i = even, \; i \geq 4 \end{cases} \\ w_f(t_i t_{i+1}) &= \begin{cases} n+2, & i = 1 \\ n+3, & i = 2 \\ n+1, & i = odd, \; i \geq 3 \\ n+2, & i = even, \; i \geq 4 \end{cases} \end{aligned}$$

The edge weight function above produces the sets edge weight as follows:

$$\begin{aligned} W(PC_n) = &\{\tfrac{2n+2}{2}, ..., \tfrac{3n+2}{2}, ..., \tfrac{4n-2}{2}\}, \\ &\{2n, 2n+1, 2n+3\} \\ = &\; n-1+3 \\ |W(PC_n)| = &\; n+2 \end{aligned}$$

The number of edge weights is $2n$ based on the sets of edge weights, so $rac(PC_n) \leq n+2$.

Table 6. The rainbow path that connects u to v in PC_n.

Case	x	y	Rainbow Path	Condition
1	r	s	r, s	
2	r	t_a	r, s, t_a	$1 \leq a \leq n$
3	r	u	r, s, t_1, u	
4	s	t_a	s, t_a	$1 \leq a \leq n$
5	s	u	s, t_1, u	
6	t_a	u	t_a, u	$1 \leq a \leq n$
7	t_a	t_b	t_a, s, t_b	$a \neq b,\ 1 \leq a, b \leq n$

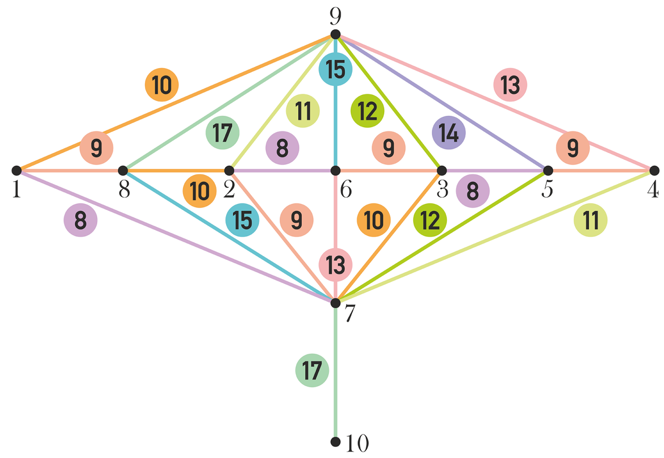

Fig. 2. Rainbow antimagic coloring of PC_7.

Based on the upper and lower bound , we get $n + 2 \leq rac(PC_n) \leq n + 2$. Thus, it can be concluded that $rac(PC_n) = n + 2$ for $n \geq 2$.

The following step is to examine the rainbow path of a parachute graph as presented in Table 6.

An illustration of $rac(PC_n)$ for $n = 7$ can be observed in Fig. 2.

Previous researchers have found results related to rainbow antimagic coloring, such as Sulistiyono *et al.* [6], who obtained the findings of rainbow antimagic connection numbers for path graph, ladder graph, triangular ladder graph, and diamond graph. Additionally, Jabbar *et al.* [7] found the rainbow antimagic connection numbers for book graph, triangular book graph, and generalized book graph. Septory *et al.* [8] also found the rainbow antimagic connection numbers for complete bipartite graph, jahangir graph, lemon graph, firecracker graph, and double star graph. Budi *et al.* [9] found the rainbow antimagic connection numbers for lollipop graph, dutch windmill graph, dragonfly graph, flowerpot graph, and stacked book graph. The rainbow antimagic connection numbers for the cycle graph, complete graph, fan graph, friendship graph, wheel graph, and tree graph were discovered by Dafik *et al.* [10]. For vertex amalgamations of the path graph, broom graph, star graph, fan graph, paw graph, and triangular book graph, Joedo *et al.* [11] discovered the rainbow antimagic connection numbers. Nisviasari *et al.* [12] found the rainbow antimagic connection numbers for tadpole graph. Septory *et al.* [13] found $rac(G)$ for the comb product of friendship graph with path, broom, star, and double star graphs.

Based on previous research, we have found the rainbow antimagic connection numbers for double wheel graph and parachute graph, which are $rac(DW_n) = 2n$ if $n \geq 3$ and $rac(PC_n) = n + 2$ if $n \geq 2$.

3 Conclusion

Based on our own research findings, four new theorems were developed for rainbow coloring and antimagic rainbow coloring on double wheel and parachute graph. Our research has been established that the value of $rac(DW_n) = 2n$ with $n \geq 3$, and established that the value of $rac(PC_n) = n + 2$ with $n \geq 2$. Additionally, it has been established that the value of $rc(DW_n) = 2$ if $n = 3, 4$, $rc(DW_n) = 3$ if $n = 5, 6$, and $rc(DW_n) = 4$ if $n \geq 7$. Furthermore, it has been established that the value of $rc(PC_n) = 3$ when $n \geq 2$.

Open Problem 1. Investigate the rainbow antimagic connection number of a different special graph or a graph formed by some operation.

Acknowledgment. This work was supported by the PUI-PT Combinatorics and Graph, CGANT, University of Jember of year 2023.

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