On Resolving Efficient Dominating Set of Cycle and Comb Product Graph

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Abstract. The graph used in this paper is a connected, bounded, and undirected graph G, is used which contains a set of vertex $V(G)$ and a set of edge $E(G)$. It is called the efficient dominating set of a graph if every point V in D or is adjacent to one vertex in D. For a set of solutions of G in an ordered set, it is distinguished by the distance of its point representation. Suppose we take any vertex in G, then $S = s_1, s_2, ..., s_k$ is a subset of $V(G)$ and the ordered set W of vertex representation is $r(p|S) = (d(p, s_1), d(p, s_2), ..., d(p, s_k))$. The set S can be is called the completion set of G if $r(u|S) \neq r(p|S)\ \forall\ u, v \in G$. And for subset Z of $V(G)$ it can be called the efficient dominating set if $r(u|Z) \neq r(p|Z)\ \forall\ u, v \in G$, then the minimum cardinality of resolving efficient dominating set is symbolized by $\gamma_{re}(G)$. The axiomatic deductive technique and the pattern detection method used in this study apply the principles of deductive proof to mathematical logic by using existing axioms, lemmas, and theorems to solve questions about the topic under study. Some theorems or definitions will be obtained in this study as a result of further analysis of previously existing theorems or definitions. The pattern identification approach follows a research method for locating efficient set completion patterns in the graph under consideration and the problem. In this paper we obtain $\gamma_{re}(G)$ from several cycle graphs $G_m \triangleright C_n$, namely $P_m \triangleright C_n$, $F_m \triangleright C_n$, $K_{l,m} \triangleright C_n$, in this paper the proving of resolving efficient dominating set is only on $n \equiv 0(mod\ 3)$.

Keyword: Resolving Efficient Dominating Set, Cycle Graph, Comb Product Graph

1 Introduction

A graph is an arrangement of objects called vertices that are linked together by connectors called edges. This paper employs graphs with finite, connected, and undirected properties designated as G. Graph introduced by Chartrand and Lesniak [7] that G is represented as a collection of unsorted pairings various vertices

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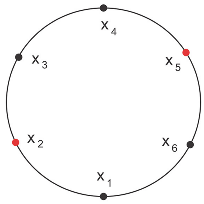

Fig. 1. Efficient dominating set of C_6.

of G called edges, which may or may not be empty, and a collection things that are both finite and nonempty known as vertices. Where $V(G)$ and $E(G)$ represent the vertices and edges of a graph G, severally. One of the numerous types of study in graph theory is the dominating set and graph metric dimensions.

The set of solvers of G is classified as a subset W of $V(G)$ if each G vertex is distinguished diverse by the representation of their distance to each vertex in the set sorted W Boutin [5]. In addition to set of solutions in graph theory there is also a dominating set. The subset of $V(G)$ and every vertex in G that does not exist in D is neighbors at the very least one the vertex in D is an explanation from Du and Wan [11] discusses the dominating set D of graph G, where $\gamma(G)$ indicates the dominating number, this is the lowest cardinality of the collection in G.

Lin M C et al. has investigated several types of dominating set [18,19]. According to Chartrand et al. [8], an independent set is one in which no two vertices inside the set are neighboring. Deng et al. [10] have defined the efficient dominating set as a dominating set. An independent subset of $V(G)$ is defined as the efficient dominating set of graph G. in which each vertex in $G - D$ is directly adjacent to a vertex in D. The efficient dominating number of graph G, represented by $\gamma_e(G)$, is the lowest cardinality of an efficient dominating set of graph G.

A side from the dominance of numbers, Research is also done on graphs metric dimensions. Slater explained the metric measurement theory [20], also Harary and Melters. [14] Metric dimensions according to Slater with a different terminology, which is finding numbers. x and y vertices in a graph G are separated by the symbol $d(x, y)$ for the ordered set $S = s_1, s_2, ..., s_k \subseteq V(G)$ and vertex $p \in V(G), r(p|S) = (d(p, s_1), d(p, s_2), ..., d(p, s_k))$ is a representation of a vertex p in relation to S. And the set W is termed a graph G resolving set if $r(p|W)6 = r(u|W)$ for each vertex u and v in G. The least The resolving set has a cardinality of dim (G), which stands for the metric dimension (Fig. 1).

The concepts of dominating sets and metric dimensions are combined by Birgham et al. [6] this is indicated by $\gamma_r(G)$ as the settlement dominating set (its minimum cardinality). The subset Y of $V(G)$, that is not only the nominated set of G but also defines each vertex of G by indicating distance from each vertex in Y, is the most important collection of graph G solutions. Some of the previous results that discuss the types of solutions for resolving perfect domi-

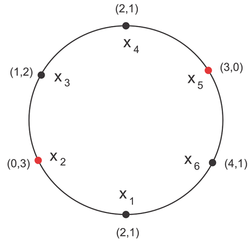

Fig. 2. Resolving Efficient dominating set of C_6.

nating number, resolving dominating number are resolving strong dominating number and resolving efficient dominating set [2,15,16,18,19,21], which are our references in writing this article, see [1,12,13] (Fig. 2).

Recently Kusumawardani, et al. [17] citing the opinion of Bange, et al. [4] if graph G has an efficient dominating set, then comes the cardinality of every efficient dominating set equals the dominating number of G. As a result, and each efficient dominant set has the same cardinality. The smallest cardinality of a graph efficient dominating set G is labeled with $\gamma_e(G$ and designated with efficient dominating numbers. There are some graphs with efficient dominating set, such as the order 4 cycle graph [3].

All graphs examined in this study are combed product graphs, with the notation $G1 \triangleright G2$ consisting of two finite, linked, and undirected graphs which are new graphs, obtained from a copy of graph $G1$ and a duplicate of graph $G2$ as many nodes from $G1$ then grafts $G2$ on $G1$. [9]. In the cycle graph resolving efficient dominating set we find only on $n \equiv 0(mod\ 3)$. and in this article will be discussed $\gamma_{re}(G)$ from several cycle graphs $G_m \triangleright C_n$, namely $P_m \triangleright C_n$, $F_m \triangleright C_n$, $K_{l,m} \triangleright C_n$.

2 Research Methods

This study used the axiomatic deductive technique and the pattern identification method. The axiomatic deductive methodology and research method applies the concepts of deductive proof (from general to specific) to mathematical logic by solving problems about the issue under study using axioms, lemmas, and existing theorems. Some theorems or definitions will be obtained from further analysis of previously existing theorems or explanations in this study. The pattern identification methodology is based on a research strategy for discovering efficient set completion patterns in the graph and problem under discussion. Let G represent the efficient domination number with the lowest cardinality. On a graph, this is known as efficient dominance number resolution.

We present give various suggestions and theorems in this section as per earlier results. They apply the same study methodology to different graphs. They are,

1) Agustin, et al. [1]

$$\gamma_{rst}(C_n) = \begin{cases} 2, & if\ n \equiv 3 \\ \left\lceil \frac{n}{3} \right\rceil, & otherwise \end{cases}$$

2) Hakim, et al. [13] have identified the precise value of a given graph's comb product, i.e. $\gamma_{re}(K_n \triangleright C_3) = \gamma_{re}(K_n \triangleright P_3) = n$, and $\gamma_{re}(W_n \triangleright P_3) = \gamma_{re}(W_n \triangleright C_3) = \gamma_{re}(S_n \triangleright P_2) = n + 1$

3 Results and Discussion

We found the resolution to be efficient dominating set of cycle comb products on $n \equiv 0(mod\ 3)$. Those are $P_m \triangleright C_n$, $F_m \triangleright C_n$, $K_{l,m} \triangleright C_n$. The theorems that will be used are listed below.

Theorem 1. For each positive integer, $m \geq 2, n \geq 3$

$$\gamma_{re}(P_m \triangleright C_n) = m(\tfrac{n}{3})$$

Proof. $P_m \triangleright C_n$ is a comb product graph of a graph P_m, and m is a copy of the graph P_n, with the vertices x_i representing the sticking spots on the graph P_m. The vertex set is $V(P_m \triangleright C_n) = \{x_i; 1 \leq i \leq m\} \cup \{x_{i,j}; 1 \leq i \leq m; 1 \leq j \leq n-1\}$, and the edge set is $E(P_m \triangleright C_n) = \{x_i, x_{i+1}; 1 \leq i \leq m-1\} \cup \{x_i, x_{i,j}; 1 \leq i \leq m; 1 \leq j \leq n\}$. The order and size respectively are $|V(P_m \triangleright C_n)| = m + mn - m$ and $|E(P_m \triangleright C_n)| = m - 1 + mn$. We choose subset D as below with $j \equiv 0(mod\ 3)$.

$$D = \{x_{i,j}; 1 \leq i \leq m\} \begin{cases} \{1 \leq j \leq n-1\} \\ \{2 \leq j \leq n-1\} \end{cases}$$

We can start putting the dominating vertex from vertex 1 and 2, we have $m(\frac{n}{3})$ for $n \equiv 0(mod\ 3)$. The next stages demonstrate that D is resolving an efficient dominant set with the lowest cardinality.

First, we shall demonstrate that D satisfies the efficient dominating set characteristic. For any $x_{i,j}, x_{R,D} \in D$, $d(x_{i,j}, x_{R,D}) \geq 3$, thus $|N(x_{i,j}, x_{R,D} \in V(P_m \triangleright C_n) - D) \cap D| = 1$ and every vertex in $V(P_m \triangleright C_n) - D$ is dominated by precisely one vertex in D. As a result, we concluded that subset D is an efficient dominating set.

Second, we shall demonstrate that the subset D we have chosen likewise meets the resolving set property. To determine whether each vertex's distance representation is correct. respecting the items in D differs, consider any two vertices' distance function in $P_m \triangleright C_n$ as shown below.

$$d(x_{i,jRD})\ for\ i_{RD} = i, 1 \leq j_{RD} \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$\begin{aligned} d(x_i) &= j_{RD} \\ d(x_{i,j}) &= |j - j_{RD}|, \\ &\quad if\ 1 \leq j \leq \left\lceil \tfrac{n}{2} \right\rceil + j_{RD} - 1 \\ d(x_{i,j}) &= \left|\left\lceil \tfrac{n}{2} \right\rceil - j\right| + j_{RD} + \left\lfloor \tfrac{n}{2} \right\rfloor, \\ &\quad \text{if } \left\lceil \tfrac{n}{2} \right\rceil + j_{RD} \leq j \leq n-1 \end{aligned}$$

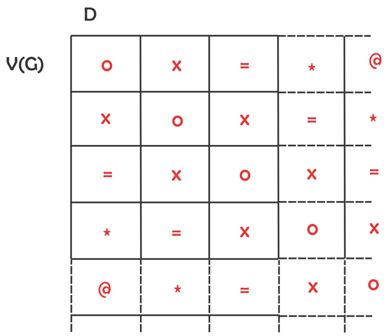

Fig. 3. The distance pattern of the set of points to the set of dominating vertex of $P_m \triangleright C_n$.

$$d(x_{i,jRD})\ for\ i_{RD} = i, \left\lceil \frac{n}{2} \right\rceil + 2 \leq j_{RD} \leq n-2$$

$$\begin{aligned} d(x_i) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lfloor \tfrac{n}{2} \right\rfloor - j_{RD} \\ d(x_{i,j}) &= \left| j - j_{RD} \right|, \\ & if\ 3 \leq j \leq n-1 \\ d(x_{i,j}) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lfloor \tfrac{n}{2} \right\rfloor - j_{RD} + j, \\ & if\ 1 \leq j \leq j_{RD} - \left\lceil \tfrac{n}{2} \right\rceil - 3 \end{aligned}$$

$$d(x_{i,jRD})\ for\ i_{RD} \neq i, 1 \leq j_{RD} \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$\begin{aligned} d(x_i) &= j_{RD} + i - 1 \\ d(x_{i,j}) &= j + j_{RD} + 1, \\ & if\ 1 \leq j \leq \left\lfloor \tfrac{n}{2} \right\rfloor \\ d(x_{,ij}) &= \left| \left\lceil \tfrac{n}{2} \right\rceil - j \right| + j_{RD} + \left\lceil \tfrac{n}{2} \right\rceil, \\ & if\ \left\lceil \tfrac{n}{2} \right\rceil + j_{RD} \leq j \geq n-1 \end{aligned}$$

$$d(x_{i,jRD})\ for\ i_{RD} \neq i, \left\lceil \frac{n}{2} \right\rceil + 2 \leq j_{RD} \leq n-2$$

$$\begin{aligned} d(x_i) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lfloor \tfrac{n}{2} \right\rfloor - j_{RD} + i - 1 \\ d(x_{i,j}) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lfloor \tfrac{n}{2} \right\rfloor - j_{RD} + j + i - 1 \\ d(x_{i,j}) &= \left\lceil \tfrac{n}{2} \right\rceil \text{ the numbers repeating with a countdown.} \end{aligned}$$

The vertex $V(G)$ representation pattern for the dominating set (D) is shown below. Figure 3 depicts a block for the distance of a collection of points to a collection of dominant points, making it easier for readers to see and look for patterns in cycle graphs, particularly cycle graphs $n \equiv 0 (mod\ 3)$.

We know that each vertex's representation respecting the elements in D must be dissimilar from one another because we chose any two vertices' distance

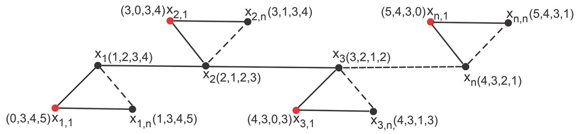

Fig. 4. The Resolving Efficient dominating set of $P_m \triangleright C_n$.

function based on the vertex D. As a result, D quench the complete set property (Fig. 4).

Third, we demonstrate that D is an efficient dominating set with the smallest cardinality. Supposing $|D_1| < m(\frac{n}{3})$ so we have $|D_1| =< m(\frac{n}{3})$ for $n \equiv 0(mod 3)$. Here are the conditions For $n \equiv 0(mod\ 3)$

- any vertex $\{x_{i,1}\} \notin D_1 \rightarrow \exists\ x_i, x_{i,1}, x_{i,2}$ that are not dominated by D_1. As a result, D_1 is not the most efficient dominating set.
- any vertex $\{x_{i,j}; b \equiv 1(mod\ 3); 4 \leq j \leq n-2\} \notin D_1 \rightarrow \exists\ x_{i,j+1}, x_{i,j}, x_{i,j-1}$ which aren't dominated by D_1. As a result, there are more efficient dominant sets than D_1.

Theorem 2. For each positive integer, $m \geq 3, n \geq 3$

$$\gamma_{re}(F_m \triangleright C_n) = (m+1)(\tfrac{n}{3})$$

Proof. $F_m \triangleright C_n$ is a comb product graph of a graph C_m and m is a copy of the graph P_n, with the vertices x_i representing the sticking spots on the graph F_m. The vertex set is $V(F_m \triangleright C_n) = \{y_i; 1 \leq i \leq n\} \cup \{x_{j,i}; 1 \leq j \leq m; 1 \leq i \leq n\}$ and the set edge of $F_m \triangleright C_n$ is $V(F_m \triangleright C_n) = \{y_i, y_n\} \cup \{y_1, x_{j,i}; 1 \leq j \leq m\} \cup \{x_{j,1}, x_{j+1,1}; 1 \leq j \leq m-1\} \cup \{y_i, y_{i+1}; 1 \leq i \leq n-1\} \cup \{x_{j,i}, x_{j,i+1}; 1 \leq j \leq m; 1 \leq i \leq n-1\} \cup \{x_{j,1}, x_{j,n}; 1 \leq j \leq m\}$. The order and size respectively are $F_m \triangleright C_n$ is $|V(F_m \triangleright C_n)| = n.m + n$ and edge cardinality of $F_m \triangleright C_n$ is $|E(F_m \triangleright C_n)| = 2m + m.n + n - 1$. We select subset D as shown below with $j \equiv 0(mod\ 3)$.

$$D = \begin{cases} \{y_i; 2 \leq i \leq n\} \\ \{x_{i,j}; 1 \leq i \leq m; 2, 3 \leq j \leq n\} \end{cases}$$

We have $(m+1)(\frac{n}{3})$ for $n \equiv 0(mod\ 3)$. The next stages demonstrate that D is resolving an efficient dominant set with the lowest cardinality.

First, we shall demonstrate that D satisfies the efficient dominating set characteristic. For any $x_{i,j}, x_{R,D} \in D$, $d(x_{i,j}, x_{R,D}) \geq 3$, thus $|N(x_{i,j}, x_{R,D} \in V(F_m \triangleright C_n) - D) \cap D| = 1$ and every vertex in $V(F_m \triangleright C_n) - D$ is dominated by precisely one vertex in D. As a result, we concluded that subset D is an efficient dominating set.

Second, we shall demonstrate that the subset D we have chosen likewise meets the resolving set property. To determine whether each vertex's distance representation is correct. respecting the items in D differs, consider any two vertices' distance function in $F_m \triangleright C_n$ as shown below.

$$d(y_{iRD})\ for\ 1 \leq i_{RD} \leq \left\lceil \frac{n}{2} \right\rceil + i_{RD} - 1$$

$$\begin{aligned} d(y_i) &= |i_{RD} - i| \\ d(y_i) &= \left\lceil \tfrac{n}{2} \right\rceil - i + j_{RD} + \left\lfloor \tfrac{n}{2} \right\rfloor, \\ &\quad if\ \left\lceil \tfrac{n}{2} \right\rceil + j_{RD} \leq i \leq n \end{aligned}$$

$$d(y_{iRD})\ for\ \left\lceil \frac{n}{2} \right\rceil + 3 \leq i_{RD} \leq n - 1$$

$$\begin{aligned} d(y_i) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lfloor \tfrac{n}{2} \right\rfloor - j_{RD} + i, \\ &\quad if\ 1 \leq i \leq i_{RD} - \left\lceil \tfrac{n}{2} \right\rceil \\ d(y_i) &= |i_{RD} - i|, \\ &\quad if\ i_{RD} - \left\lfloor \tfrac{n}{2} \right\rfloor \leq i \leq n \end{aligned}$$

$$d(y_{iRD})\ for\ 2 \leq i \leq n, 1 \leq i_{RD} \leq \left\lceil \frac{n}{2} \right\rceil + i_{RD} - 1$$

$$\begin{aligned} d(x_{i,j}) &= j + j_{RD} + i - 3, \\ &\quad if\ 1 \leq j \leq \left\lceil \tfrac{n}{2} \right\rceil \\ d(x_{i,j}) &= \left\lceil \tfrac{n}{2} \right\rceil - j + j_{RD} + \left\lfloor \tfrac{n}{2} \right\rfloor + i - 1, \\ &\quad if\ \left\lceil \tfrac{n}{2} \right\rceil + 1 \leq j \leq n \end{aligned}$$

$$d(y_{iRD})\ for\ 2 \leq i \leq n, \left\lceil \frac{n}{2} \right\rceil + 3 \leq i_{RD} \leq n - 1$$

$$\begin{aligned} d(x_{i,j}) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lfloor \tfrac{n}{2} \right\rfloor - j_{RD} + j + i + 1 - 2, \\ &\quad if\ 1 \leq j \leq \left\lceil \tfrac{n}{2} \right\rceil \\ d(x_{i,j}) &= \text{Starting from } \left\lceil \tfrac{n}{2} \right\rceil \\ &\quad \text{the numbers repeating with a countdown.} \end{aligned}$$

$$d(x_{i,jRD})\ for\ 1 \leq i_{RD} \leq \left\lceil \frac{n}{2} \right\rceil + i_{RD} - 1$$

$$\begin{aligned} d(y_i) &= j_{RD} + i - 1, \\ &\quad if\ 1 \leq i \leq \left\lceil \tfrac{n}{2} \right\rceil \\ d(y_i) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lceil \tfrac{n}{2} \right\rceil + j_{RD} - i, \\ &\quad if\ \left\lceil \tfrac{n}{2} \right\rceil + 1 \leq i \leq n \end{aligned}$$

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$$\begin{aligned} d(y_i) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lfloor \tfrac{n}{2} \right\rfloor - j_{RD} + i + 1, \\ &\quad if\ 1 \leq i \leq \left\lceil \tfrac{n}{2} \right\rceil \\ d(y_i) &= \text{Starting from } \left\lceil \tfrac{n}{2} \right\rceil \\ &\quad \text{the numbers repeating with a countdown.} \end{aligned}$$

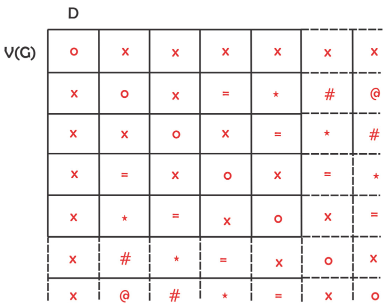

Fig. 5. The distance pattern of the set of points to the set of dominating vertex of $F_m \triangleright C_n$.

$$d(x_{i,jRD})\ for\ i_{RD} = i, 1 \leq j_{RD} \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$\begin{aligned} d(x_{i,j}) &= |j_{RD} - j| \\ &\quad if\ 1 \leq j \leq \lceil \tfrac{n}{2} \rceil + 1 \\ d(x_{i,j}) &= \lceil \tfrac{n}{2} \rceil - j + j_{RD} + \lfloor \tfrac{n}{2} \rfloor, \\ &\quad if\ \lceil \tfrac{n}{2} \rceil + 2 \leq j \leq n \end{aligned}$$

$$d(x_{i,jRD})\ for\ i_{RD} = i,\ \left\lceil \frac{n}{2} \right\rceil + 3 \leq i_{RD} \leq n - 1$$

$$\begin{aligned} d(x_{i,j}) &= \lceil \tfrac{n}{2} \rceil + \lfloor \tfrac{n}{2} \rfloor - j_{RD} + j, \\ &\quad if\ 1 \leq j \leq i_{RD} - \lceil \tfrac{n}{2} \rceil \\ d(x_{i,j}) &= |j_{RD} - j|, \\ &\quad if\ j_{RD} - \lfloor \tfrac{n}{2} \rfloor \leq j \leq n \end{aligned}$$

The vertex $V(G)$ representation pattern for the dominating set (D) is shown below. Figure 5 depicts a block for the distance of a collection of points to a collection of dominant points, making it easier for readers to see and look for patterns in cycle graphs, particularly cycle graphs $n \equiv 0(mod\ 3)$.

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- any vertex $\{x_{i,j}; b \equiv 1(mod\ 3); 4 \leq j \leq n-2\} \notin D_1 \rightarrow \exists\ x_{i,j+1}, x_{i,j}, x_{i,j-1}$ which aren't dominated by D_1. As a result, there are more efficient dominant sets than D_1.

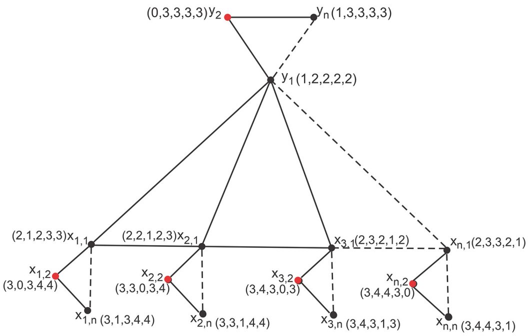

Fig. 6. The Resolving Efficient dominating set of $F_m \triangleright C_n$.

Theorem 3. For each positive integer, $l, m \geq 2, n \geq 3$

$$\gamma_{re}(K_{lm} \triangleright C_n) = (l+m)(\tfrac{n}{3})$$

Proof. Graph $K_{lm} \triangleright C_n$ is a comb product graph of a graph K_{lm} and l and m copies graph C_n with the vertices x_i representing the sticking point on the graph K_{lm}. The vertex set is $V(K_{lm} \triangleright C_n) = \{x_i; 1 \leq i \leq l\} \cup \{y_j; 1 \leq j \leq m\} \cup \{x_{i,a}; 1 \leq i \leq l; 1 \leq a \leq n-1\} \cup \{y_{j,a}; 1 \leq i \leq m; 1 \leq a \leq n-1\}$, and the edge set is $E(K_{Lm} \triangleright C_n) = \{x_i, y_i; 1 \leq i \leq l; 1 \leq j \leq m\} \cup \big\{x_1, x_{i,a}; 1 \leq i \leq l; a = 1, n-1\big\} \cup \big\{x_{i,a}, x_{i,a+1} = 1 \leq i \leq l; 1 \leq a \leq n-1\big\} \cup \big\{y_j, y_{j,a}; 1 \leq k \leq m, 1 \leq a \leq n-1\big\} \cup \{y_{j,a}, y_{j,a+1}; 1 \ \leq j \leq m; 1 \leq a \leq n-1\}$. The order and size respectively are $|V(K_{lm} \triangleright C_n)| = l+m+l(n-1)+m(n-1)$ and $|E(K_l m \triangleright C_n)| = lm+ln+mn$. We select subset D shown below with $j \equiv 0(mod\ 3)$.

$$D = \begin{cases} \{x_{i,j}; 1 \leq i \leq l; 1, 2 \leq j \leq n\} \\ \{x_{i,j}; 1 \leq i \leq m; 1, 2 \leq j \leq n\} \\ \{y_{i,j}; 1 \leq i \leq l; 1, 2 \leq j \leq n\} \\ \{y_{i,j}; 1 \leq i \leq m; 1, 2 \leq j \leq n\} \end{cases}$$

For $n \equiv 0(mod\ 3)$, we have $(l+m)(\frac{n}{3})$. The next stages demonstrate that D is resolving an efficient dominant set with the lowest cardinality.

First, we shall demonstrate that D satisfies the efficient dominating set characteristic. For any $x_{i,j}, x_{R,D} \in D$, $d(x_{i,j}, x_{R,D}) \geq 3$, thus $|N(x_{i,j}, x_{R,D} \in V(K_{l,m} \triangleright C_n) - D) \cap D| = 1$ and every vertex in $V(K_{l,m} \triangleright C_n) - D$ is dominated by precisely one vertex in D. As a result, we concluded that subset D is an efficient dominating set.

Second, we shall demonstrate that the subset D we have chosen likewise meets the resolving set property. To determine whether each vertex's distance representation is correct. respecting the items in D differs, consider any two vertices' distance function in $K_{l,m} \triangleright C_n$ as shown below.

$$d(x_{i,jRD}, y_{i,jRD})\ for\ i_{RD} = i, 1 \leq j_{RD} \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$\begin{aligned} d(x_i, y_i) &= j_{RD} \\ d(x_{i,j}, y_{i,j}) &= |j - j_{RD}|, \\ &\quad if\ 1 \leq j \leq \left\lceil \tfrac{n}{2} \right\rceil + j_{RD} - 1 \\ d(x_{i,j}, y_{i,j}) &= \left| \left\lceil \tfrac{n}{2} \right\rceil - j \right| + j_{RD} + \left\lfloor \tfrac{n}{2} \right\rfloor, \\ &\quad if\ \ \left\lceil \tfrac{n}{2} \right\rceil + j_{RD} \leq j \leq n - 1 \end{aligned}$$

$$d(x_{i,jRD}, y_{i,jRD})\ for\ i_{RD} = i, \left\lceil \frac{n}{2} \right\rceil + 2 \leq j_{RD} \leq n - 2$$

$$\begin{aligned} d(x_i, y_i) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lfloor \tfrac{n}{2} \right\rfloor - j_{RD} \\ d(x_{i,j}, y_{i,j}) &= |j - j_{RD}|, \\ &\quad if\ 3 \leq j \leq n - 1 \\ d(x_{i,j}, y_{i,j}) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lfloor \tfrac{n}{2} \right\rfloor - j_{RD} + j, \\ &\quad if\ 1 \leq j \leq j_{RD} - \left\lceil \tfrac{n}{2} \right\rceil - 3 \end{aligned}$$

$$d(x_{i,jRD,})\ for\ i_{RD} \neq i, 1 \leq j_{RD} \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$\begin{aligned} d(y_i, y_{i,j}) &= j_{RD} + i + 1, \\ &\quad if\ 0 \leq j \leq \left\lfloor \tfrac{n}{2} \right\rfloor \\ d(y_{i,j}) &= \left\lceil \tfrac{n}{2} \right\rceil - j + j_{RD} + \left\lfloor \tfrac{n}{2} \right\rfloor + 1, \\ &\quad if\ \ \left\lceil \tfrac{n}{2} \right\rceil + j_{RD} \leq j \geq n - 1 \end{aligned}$$

$$d(x_{i,jRD,}, y_{i,jRD})\ for\ i_{RD} \neq i, \left\lceil \frac{n}{2} \right\rceil + 2 \leq j_{RD} \leq n - 2$$

$$\begin{aligned} d(y_{i}., y_{ij}) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lfloor \tfrac{n}{2} \right\rfloor - j_{RD} + j + 1, \\ &\quad if\ 1 \leq j \leq \left\lfloor \tfrac{n}{2} \right\rfloor \\ d(y_{i,j}) &= \text{Starting from } \left\lceil \tfrac{n}{2} \right\rceil \\ &\quad \text{the numbers repeating with a countdown.} \end{aligned}$$

$$d(y_{i,jRD})\ for\ i_{RD} \neq i, 1 \leq j_{RD} \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$\begin{aligned} d(x_i, x_{i,j}) &= j_{RD} + i + 1, \\ &\quad if\ 0 \leq j \leq \left\lfloor \tfrac{n}{2} \right\rfloor \\ d(x_{i,j}) &= \left\lceil \tfrac{n}{2} \right\rceil - j + j_{RD} + \left\lfloor \tfrac{n}{2} \right\rfloor + 1, \\ &\quad if\ \ \left\lceil \tfrac{n}{2} \right\rceil + j_{RD} \leq j \geq n - 1 \end{aligned}$$

$$d(x_{i,jRD,}, y_{i,jRD})\ for\ i_{RD} \neq i, \left\lceil \frac{n}{2} \right\rceil + 2 \leq j_{RD} \leq n - 2$$

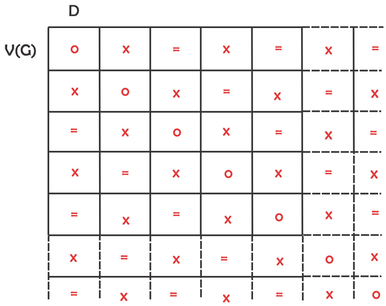

Fig. 7. The distance pattern of the set of points to the set of dominating points of $K_{l,m} \triangleright C_n$.

$$\begin{aligned} d(x_{i\cdot}, x_{ij}) &= \left\lceil \tfrac{n}{2} \right\rceil + \left\lfloor \tfrac{n}{2} \right\rfloor - j_{RD} + j + 1, \\ &\quad if\ 1 \le j \le \left\lfloor \tfrac{n}{2} \right\rfloor \\ d(x_{i,j}) &= \text{Starting from } \left\lceil \tfrac{n}{2} \right\rceil \\ &\quad \text{the numbers repeating with a countdown.} \end{aligned}$$

The vertex $V(G)$ representation pattern for the dominating set (D) is shown below. Figure 7 depicts a block for the distance of a collection of points to a collection of dominant points, making it easier for readers to see and look for patterns in cycle graphs, particularly cycle graphs $n \equiv 0(mod\ 3)$.

We know that each vertex's representation respecting the elements in D must be dissimilar from one another because we chose any two vertices' distance function based on the vertex D. As a result, D quench the complete set property (Fig. 8).

Third, we demonstrate that D is an efficient dominant set with the smallest cardinality. Supposing $|D_1| < (l+m)(\frac{n}{3})$ so we have $|D_1| =< (l+m)(\frac{n}{3})$ for $n \equiv 0(mod 3)$. Here are the conditions For $n \equiv 0(mod\ 3)$

- any vertex $\{x_{i,1}\} \notin D_1 \rightarrow \exists\ x_i, x_{i,1}, x_{i,2}$ that are not dominated by D_1. As a result, D_1 is not the most efficient dominating set.
- any vertex $\{x_{i,j}; b \equiv 1(mod\ 3); 4 \le j \le n-2\} \notin D_1 \rightarrow \exists\ x_{i,j+1}, x_{i,j}, x_{i,j-1}$ which aren't dominated by D_1. As a result, there are more efficient dominant sets than D_1.

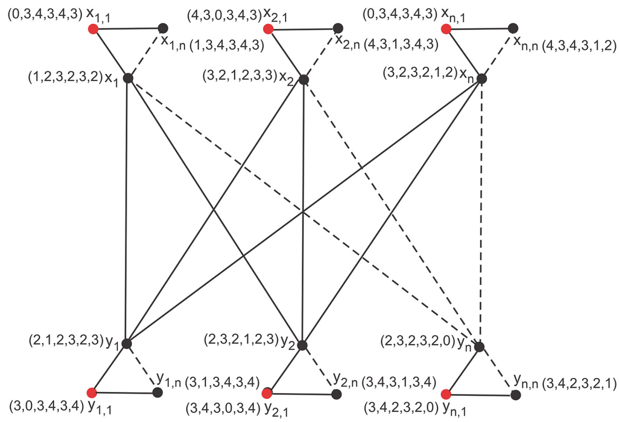

Fig. 8. The Resolving Efficient dominating set of $K_{l,m} \triangleright C_n$.

4 Conclusion

We examined and discover the precise values of resolving efficient dominating set on cycle and specific comb product in this study, namely $P_m \triangleright C_n$, $F_m \triangleright C_n$, $K_{l,m} \triangleright C_n$. In this work, the demonstration of resolving efficient dominating set is only on $n \equiv 0(mod\ 3)$. Because this is a novel research area that combines the investigation of metric dimension and efficient dominating set, many difficulties relating to this topic have yet to be uncovered. As a result, we present the following open issues.

Open Issues 1. Specify and verify the exact value of the dominating efficient differential number if G is an arbitrarily connected and infinite graph.

Open Issues 2. Specify the most efficient dominating number of various graph operations, such as joint, cartesian, and corona.

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