On Rainbow Vertex Antimagic Coloring of Shell Related Graphs

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Abstract. A rainbow vertex antimagic coloring namely rainbow vertex coloring and antimagic labeling. For a bijective function $f\,(E(G)) \rightarrow \{1, 2, 3\cdots, |E\,(G)|\}$, the associate weight of a vertex $v \epsilon V\,(G)$ against f is $W_f\,(v) = \sum_{e \epsilon E(v)} f\,(e)$, where $E(v)$ is the set of vertices incident to v. The function f is known as vertex antimagic labeling if every vertex has a distinct weight. Any path P in a graph G labeled with edges called a rainbow path if any two interior vertices on the path $u - v$ have different weights. If there is a rainbow path $u - v$ between every pair of u and v vertices, then f is referred to as the rainbow vertex antimagic labeling of G. Graph G is called rainbow vertex antimagic coloring when each $u - v$ edge has a vertex weight color $wf(v)$. The deductive analytic method is used in this research to analyze the issues that have been discovered, define the issue, and demonstrate patterns and theorems. With n ranging, the steps required are to draw a graph, determine the cardinality, search for the pattern of the rainbow vertex connection number as the lower bound, construct the bijective function to obtain the upper bound, and if the upper bound meets the lower bound, than new theorems and their proofs will be generated. Using the results of the proven theorem analysis will form a conclusion. $BF(n, m)$ and shell flower graph $[C_{n,n-3}, k_2]^k$

Keywords: rainbow vertex coloring · rainbow vertex antimagic coloring · shell graph

1 Introduction

Graph theory provides many discoveries for researchers such as rainbow coloring, antimagic labeling, antimagic rainbow coloring, and many others. In this paper, we will focus on rainbow vertex antimagic coloring, which is formed from rainbow vertex coloring and antimagic labeling.

Rainbow vertex coloring means any path between u and v in graph G has different internal vertices color [12]. As introduced by Krivelevich and Yuster in 2010 as cited in Hasan *et al.* [3], the rainbow vertex connection of a connected

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graph G is also known as rvc(G), where rvc(G) is the smallest number of colors required to make G rainbow vertex-connected [10]. The rainbow coloring of graph G, where each vertex is connected by a path whose internal vertex has a different color, is the simplest explanation for the rainbow vertex-connection graph [6]. On the slinky graph (Sl_n, C_4), Akadji *et al.* [1] has determined the precise value of the rainbow vertex connection number and the strong rainbow vertex connection number. On a small-world Farey graph, Darayon *et al.* [2] has discovered the number of rainbow vertex connections. Heggernes *et al.* [5] created the rainbow vertex coloring for chordal and bipartite graphs.

Antimagic labeling was proposed by Hartsfield and Ringel [4]. For $G(V, E)$, a graph G is called antimagic if there is a bijection $f(E(G)) \rightarrow \{1, 2, 3 \cdots, |E(G)|\}$, so every vertex has distinct weight, where the node-weights of v vertices are the sum of the labels of all edges adjacent to v [7]. The associated weight of a vertex $v \epsilon V(G)$ against f is $W_f(v) = \sum_{e \epsilon E(v)} f(e)$ [8]. If $wf(u) = wf(v)$ for two different points $u, v \epsilon V(G)$, then f is called the antimagic labeling of G.

In this paper, we will combine rainbow vertex coloring and antimagic labeling to make rainbow vertex antimagic coloring as research conducted by marsidi [8,9]. For a bijective function $f(E(G)) \rightarrow \{1, 2, 3 \cdots, |E(G)|\}$, the associate weight of a vertex $v \epsilon V(G)$ against f is $W_f(v) = \sum_{e \epsilon E(v)} f(e)$, where $E(v)$ is the set of vertices incident to v. If for every two vertices of u and v, there is a rainbow path $v - v'$ then f is called the rainbow vertex antimagic labeling of G. Graph G called rainbow vertex antimagic coloring when each $u - v$ edge has a vertex weight color $wf(v)$. The least amount of colors selected from all rainbow coloring caused by rainbow vertex antimagic labeling of graph G is the rainbow vertex antimagic connection number of G, denoted by $rvac(G)$.

Rainbow vertex antimagic coloring has been developed by many researchers on various kinds of graphs. For instance, in Marsidi's [8] research on the rainbow vertex antimagic coloring of tree graphs. For the paths Pn, wheels Wn, friendships Fnm, and fans Fn in 2022, Marsidi [9] determined rainbow vertex antimagic coloring. We will calculate the value of the rainbow vertex antimagic connection number $rvac$ of two graphs related to shells, the shell-butterfly graph $BF(n, m)$ and the shell flower graph $[Cn, n - 3, k2]k$.

2 Method

The deductive analytic method used in this research. The deductive analytic method is a method that solves problems through simple concepts or previously known rules, then moves further to more definite conclusions. The methods taken in this research include formulating the issues raised and researching various sources on how to color rainbow vertices on graphs. To analyze the issues that have been discovered, define the issue, demonstrate patterns and theorems. With n ranging, we each draw a shell butterfly graph and shell flower graph. After that, determine the cardinality of graph G. Search for the pattern of the rainbow vertex connection number $rvc(G)$ of G as the lower bound of $ravc(G)$. Construct the bijective function to obtain the upper bound of $rvac(G)$. $rvac(G)$ will be obtained if the upper bound meets the lower bound. After that, new Theorems and their proofs will be generated. Using the results of the proven theorem analysis will form a conclusion.

3 Preliminaries

The definitions, observations, and theorems related to the idea of rainbow vertex antimagic coloring will be presented in this section to support the discovery of rainbow vertex antimagic coloring of the graph related to shells.

Remark 1. *[8] Let G be a connected graph, $rvac(G) \geq rvc(G)$.*

Definition 1. *[13] Taking $(k-3)$ concurrent chords from a Ck results in a graph known as a shell graph $C(k, k-3)$. Recognise that the shell graph and the fan graph are identical $F_{n-1} = P_{n-1} + K_1$.*

Definition 2. *[11] A double shell graph, which consists of two disjoint shells with a common apex and each shell being of any order, can be made into a butterfly graph by adding two pendent edges at the apex.*

Definition 3. *[13] By joining the end vertex of K_2 to the apex of the shell, $'k'$ copies of the union of the shells $C(n-3)$ and K_2 result in a shell-flower graph. This graph denoted as $[C(n, n-3) \cup K_2]^k$.*

Theorem 1. *[3] Let G be a connected graph with $diam(G)$, $rvc(G) \geq diam(G) - 1$.*

Theorem 2. *[10] Let S_n be a star graph. For every positive integer $n \geq 3$, $rvac(S_n) = n + 1$.*

4 Result and Discussion

This research establishes new theorems for rainbow vertex antimagic coloring on graphs that relate to shells, such as the shell-butterfly graph $BF(n, m)$ and shell flower graph $[C_{n,n-3}, k_2]^k$.

Theorem 3. *Let $BF_(m, n)$ be a shell-butterfly graph. For every positive integer $m \geq 3$ and $n \geq 3$, $rvc(BF_(m, n)) = 1$.*

Proof. if J is $BF(m, n)$ graph with a vertex set $V(J) = \{A\} \cup \{A_i; 1 \leq i \leq 2\} \cup \{u_i; 1 \leq i \leq m\} \cup \{v_i; 1 \leq i \leq n\}$ and edge set $E(J) = \{AA_i; 1 \leq i \leq 2\} \cup \{Au_i; 1 \leq i \leq n\} \cup \{Av_i; 1 \leq i \leq m\} \cup \{u_iu_{i+1}; 1 \leq i \leq n-1\} \cup \{v_iv_{i+1}; 1 \leq i \leq m-1\}$. The cardinality of vertex set J is $|V(J)| = m + n + 3$ and $|E(J)| = 2m + 2n$. Since $diam(J)$ is 2, based on Theorem 1 $rvc(J) \geq diam(J) - 1 = 1$. It is clear that $rvc(J) \geq 1$. Define the vertex coloring of $e : V(J) \rightarrow 1$:

Table 1. The Rainbow Vertex of $u-v$ Path of $BF(m,n)$.

Case	x	y	**Rainbow Path** $u-v$	**Condition**
1	u_i	A_j	u_i, A, A_j	$1 \le i \le n$ $1 \le j \le 2$
2	u_i	u_j	$u_i,\ A,\ u_j$	$1 \le i \le n$ $1 \le j \le n$
3	v_i	v_j	$v_i,\ A,\ v_j$	$1 \le i \le m$ $1 \le j \le m$
4	v_i	A_j	$u_i,\ A,\ A_i$	$1 \le i \le m$ $1 \le j \le 2$
5	u_i	v_j	$u_i,\ A,\ v_j$	$1 \le i \le n$ $1 \le j \le m$
6	A_i	A_i	$A_i,\ A,\ A_i$	$1 \le i \le 2$

$$f(A) = 1$$
$$f(A_i) = 1$$
$$f(x_i, y_j) = 1; 1 \le i \le n, 1 \le j \le m$$

We will show the following rainbow path of graph that is formed can be seen in the Table 1:

Based on Table 1, we know that every path has a different interior vertex on the path of $u-v$. Thus, based on the lower bound and upper bound proven that $rvc(J) = 1$.

Theorem 4. *If G is $BF(m,n)$ graph with $m,n \ge 3$ and $m = n$, $rvac(G) = 4$.*

Proof. Let $BF(m,n)$ be shell butterfly graph with a vertex set $V(G) = \{A\} \cup \{A_i; 1 \le i \le 2\} \cup \{v_i; 1 \le i \le m\} \cup \{u_i; 1 \le i \le m\}$ and edge set $E(G) = \{AA_i; 1 \le i \le 2\} \cup \{Av_i; 1 \le i \le m\} \cup \{Au_i; 1 \le i \le m\} \cup \big\{v_iv_{i+1}; 1 \le i \le m-1\big\} \cup \{u_iu_{i+1}; 1 \le i \le m-1\}$. The cardinality of vertex set G is $|V(G)| = 2m+3$ and $|E(G)| = 4m$.

To prove the rainbow vertex antimagic coloring of G, first, we have to show the lower bound of $rvac(G)$. Since the vertex A has a degree much greater than the others, it must have a different vertex weight. Besides, The vertex A_i is a pendant of 2, so it has a different weight than the other. Therefore, the vertex A would have a different weight with vertex u_n and v_m because vertex A has quite a lot of difference of degree with the vertex u_n and v_m. So we got the lower bound in total $rvac(G) = 4$.

The next step is to determined the upper bound of the graph. There are 3 case that will be explained, where the cases have the same $rvac$ value but with different function patterns and weights. The three cases include when $m = 3, m = 4$ and $m \ge 5$ which will be explained below.

Case 1. For $m = 3$

To indicate the upper bound of rainbow vertex antimagic coloring of the shell butterfly graph, it is necessary to know the function and weight of each vertex. In case 1, because the value of m is 3 (fixed), the function and its weight as follows

$$\begin{aligned}
f(AA_i) &= 3m + i + 1 \\
f(Av_i) &= \begin{cases} \frac{3m+i}{i} & ; \imath = \ odd \\ im - 5 & ; \imath \ even \end{cases} \\
f(Au_i) &= \begin{cases} 2m + \lceil \frac{i}{2} \rceil & ; \ i = \ odd \\ im - 4 & ; \imath = \ even \end{cases} \\
f(v_i v_{i+1}) &= 2mi - m \qquad ; 1 \le i \le m - 1 \\
f(u_i u_{i+1}) &= n + i + 1 \qquad ; 1 \le i \le n - 1
\end{aligned}$$

The edge label function can determine the vertex weight, such that we obtain:

$$\begin{aligned}
w_f(A) &= 46 \\
w_f(A_i) &= 3m + i + 1 \quad ; \ 1 \le i \le 2 \\
w_f(u_i) = w_f(v_i) &= 4m + 1
\end{aligned}$$

According to the vertex weights above, the graph $BF(m, n)$ with $m = 3$ has the following four different weights:

$$\begin{aligned}
w(BF(m, n)) &= \{4m + 1, 3m + 2, 3m + 3, 46\} \\
|w(BF(m, n))| &= 4.
\end{aligned}$$

It shows the vertex weight induces a rainbow vertex antimagic coloring of 4 colors that have a rainbow point trajectory from $u - v$. Every vertex $u, u \epsilon V(BF(m, n))$ is given the color w_f, so the internal vertices for each vertex have different weights. Therefore, it is demonstrated that $rvac(BF(m, n)) = 4$ for $m = 3$ based on the lower and upper bounds.

Case 2. For $m = 4$

Same as the first case, in this case, we can see the value of m is 4 (fixed). so to find the function and its weight as follows

$$\begin{aligned}
f(AA_i) &= 3m + i + 1 \\
f(Av_i) &= \begin{cases} \frac{m+i+5}{i} & ; \imath = \ odd \\ 2m - i - 2 & ; \imath \ even \end{cases} \\
f(Au_i) &= \begin{cases} \lfloor \frac{4m=4i+2}{i} \rfloor & ; \ i = \ odd \\ i^2 - \frac{2i+m}{m} & ; \imath = \ even \end{cases} \\
f(v_i v_{i+1}) &= \begin{cases} 3m - \lceil \frac{4i-1}{i} \rceil & ; \ i = \ odd \\ 2m - 1 & ; \imath = \ even \end{cases} \\
f(u_i u_{i+1}) &= \begin{cases} 2m - \lceil \frac{m-2+i}{i} \rceil & ; \ i = \ odd \\ 3m & ; \imath = \ even \end{cases}
\end{aligned}$$

The edge label function can determine the vertex weight, such that we obtain:

$$\begin{aligned}
w_f(A) &= 89 \\
w_f(A_i) &= 3m + i + 2 \quad ; \ 1 \le i \le 2 \\
w_f(u_i) = w_f(v_i) &= 5m - 1
\end{aligned}$$

According to the vertex weights above, the graph $BF(m,n)$ with $m=3$ has the following four different weights:

$$\begin{aligned} wBF(m,n) &= \{5m-1, 3m+3, 3m+4, 89\} \\ |wBF(m,n)| &= 4. \end{aligned}$$

It shows the vertex weight induces a rainbow vertex antimagic coloring of 4 colors that have a rainbow point trajectory from $u-v$. Every vertex $u, u\epsilon V(BF(m,n))$ is given the color w_f, so the internal vertices for each vertex have different weights. Therefore, it is demonstrated that $rvac(BF(m,n)) = 4$ for $m=4$ based on the lower and upper bounds.

Case 3. For $m \geq 5$

$$\begin{aligned} f(AA_i) &= 4m+i-2 \\ f(Av_i) &= \begin{cases} (1,3) & ; \imath = (v_{m-4}, v_{m-1}) \\ 2i-1 & ;\ 3 \leq i \leq m-3 \\ 3m-1 & ; \imath = v_{m-2}, m\ odd \\ m & ; \imath = v_{m-2}, m\ even \end{cases} \\ f(Au_i) &= \begin{cases} 4n-2i+3 & ;\ 1 \leq i \leq n-2 \\ n-1 & ; \imath = u_{n-1}, m\ odd \\ 3n-2 & ; \imath = u_{n-1}, m\ even \end{cases} \\ f(v_iv_{i+1}) &= \begin{cases} 4m-2i & ;\ 1 \leq i \leq m-3,\ i\ odd \\ m^2-3 & ; \imath = m-1 \\ 2m-(i+2) & ;\ 2 \leq i \leq m-3,\ i\ even \\ m+i-3 & ; \imath = m-2 \end{cases} \\ f(u_iu_{i+1}) &= \begin{cases} 2n+i-1 & ;\ i\ odd \\ i & ;\ 2 \leq i \leq n-1,\ i\ even \\ 4n-4 & ; \imath = n \end{cases} \end{aligned}$$

From the edge labeling functions above, we can determine the vertex weights which induce a rainbow vertex antimagic coloring as follow.

$$\begin{aligned} w_f(A) &= 4m^2+6m+1 \\ w_f(A_i) &= 4m+i-2 \quad ;\ 1 \leq i \leq 2 \\ w_f(u_i) = W_f(v_i) &= (m-5)6+25 \end{aligned}$$

To determine the number of vertex weight based on the set of vertex weight which is

$$w = \left\{4m^2+6m+1, 4m+i-2, 4m, 6m-5\right\}$$

$$|w| = 4$$

It shows the vertex weight induces a rainbow vertex antimagic coloring of 4 colors that have a rainbow point trajectory from $u-v$. Every vertex $u, u\epsilon V(BF(m,n))$ is given the color w_f, so the internal vertices for each vertex have different weights. Therefore, it is demonstrated that $rvac(BF(m,n)) = 4$

Table 2. The Rainbow Vertex of $u-v$ Path of $BF(m,n)$.

Case	x	y	**Rainbow Path** $u-v$	**Condition**
1	A_i	A_i	$A_i,\ A,\ A_i$	$1 \le i \le 2$
2	A_i	u_j	$A_i,\ A,\ u_j$	$1 \le i \le 2$ $1 \le j \le m$
3	A_i	v_j	$A_i,\ A,\ v_j$	$1 \le i \le n$ $1 \le j \le m$
4	u_j	u_k	$u_j,\ A,\ u_k$	$1 \le j \le m$ $1 \le k \le m$
5	v_j	v_k	$v_j,\ A,\ v_k$	$1 \le j \le m$ $1 \le k \le m$
6	u_j	v_k	$y_j,\ A,\ y_k$	$1 \le j \le m$ $1 \le k \le m$

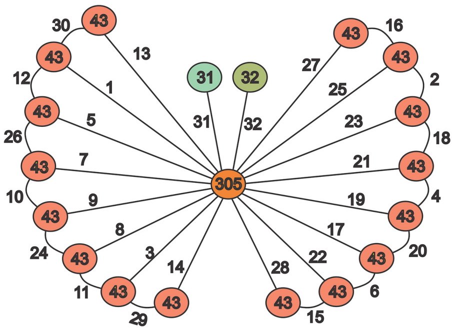

Fig. 1. rvac of $BF(m,n)$ for $m = 8$.

for $m \ge 5$ based on the lower and upper bounds. Table 2 provides four cases of rainbow paths in $BF(m,n)$, as shown below, to provide more information.

We can conclude from Table 2 that Graph $BF(m,n)$ with $m = n$ has rainbow vertex antimagic coloring. As a result, we get $rvac(BF(m,n)) = 4$. For the illustration of $rvac(BF(m,n))$ is provided in Fig. 1. The vertex weights have in own distinct colors adjusted to make it easier to see the rainbow antimagic path.

Theorem 5. *Let $[C(n,n-3) \cup K_2]^k$ be a shell flower graph. For every positive integer $k \ge 3$, $rvc([C(n,n-3) \cup K_2]^k) = 1$.*

Proof. If K is $[C(n,n-3) \cup K_2]^k$ graph with a vertex set $V(K) = \{A\} \cup \{x_i; 1 \le i \le k\} \cup \{y_{i,j}; 1 \le i \le k, 1 \le j \le n-1\}$ and edge set $E(K) = \{Ax_i; 1 \le i \le k\} \cup \{Ay_{i,j}; 1 \le i \le k\ , 1 \le j \le n-1\} \cup \big\{y_{ij)=}y_{ij+1}; 1 \ \le \ i \ \le k\, 1 \le j \le n-2\big\}$. The cardinality of vertex set K is $|V(K)| = nk+1$ and $|E(K)| = 2kn - 2k$. Since $diam(K)$ is 2, based on Theorem 1 $rvc(K) \ge diam(K) - 1 = 1$. It is clear that $rvc(K) \ge 1$.

Table 3. The Rainbow Vertex of $u - v$ Path of $[C(n, n-3) \cup K_2]^k$.

Case	x	y	Rainbow Path $u - v$	Condition
1	x_i	x_j	x_i, A, x_j	$1 \le i \le k$ $1 \le j \le k$
2	x_i	y_{ij}	x_i, A, x_{ij}	$1 \le i \le k$ $1 \le j \le k$
3	y_{ij}	y_{tq}	y_{ij}, A, y_{tk}	$1 \le i \le k$ $1 \le j \le n-1$ $1 \le l \le k$ $1 \le q \le n-1$

Define the vertex coloring of $e : V(K) \to 1$:

$$f(A) = 1$$
$$f(x_i) = 1$$
$$f(y_{ij}) = 1; 1 \le i \le k, 1 \le j \le n-1$$

We will show the following rainbow path of graph that is formed can be seen in the Table 3:

Based on Table 3, we know that every path has a different interior vertex on the path $u-v$. Based on the lower and upper bound, it is proved that $rvc(K) = 1$.

Theorem 6. *If B is $[C(4,1) \cup K_2]^k$ graph with $k \ge 3$, $rvac(B) = k+1$.*

Proof. Let B be shell flower graph with a vertex set $V(B) = \{A\} \cup \{x_i; 1 \le i \le k\} \cup \{y_{i,j}; 1 \le i \le k, 1 \le j \le 3\}$ and edge set $E(B) = \{Ax_i; 1 \le i \le k\} \cup \{Ay_{i,j}; 1 \le i \le k\ , 1 \le j \le 3\} \cup \{y_{ij)=}y_{ij+1}; 1 \le i \le k\, 1 \le j \le 2\}$. The cardinality of vertex set B is $|V(B)| = 4k+1$ and $|E(B)| = 6k$.

To prove the rainbow vertex antimagic coloring of B, first, we have to show the lower bound of $rvac(B)$. Since the vertex A has a degree much greater than the others, it must have a different vertex weight. Besides, The vertex x_i is a pendant of k, which is the same as the star graph, so it has a different weight than the other. Based on from Theorem 2. $rvac(S_n) = n+1$, we can conclude the $rvac$ of graph B. So we got the lower bound in total $rvac(B) = k+1$.

To indicate the upper bound of rainbow vertex antimagic coloring of the shell butterfly graph, we construct the bijective function of edge labels that we divided into four cases,where the cases have the same $rvac$ value but with different function patterns and weights. The three cases include when $k = 3, k = 4, k = 5$ and $k \ge 6$ which will be explained below.

Case 1. For $k = 3$

The illustration in Fig. 2 (b) shows how to find the function of graph (B) in case 2, where the value of k is fixed at 4. From that example, it is clear that

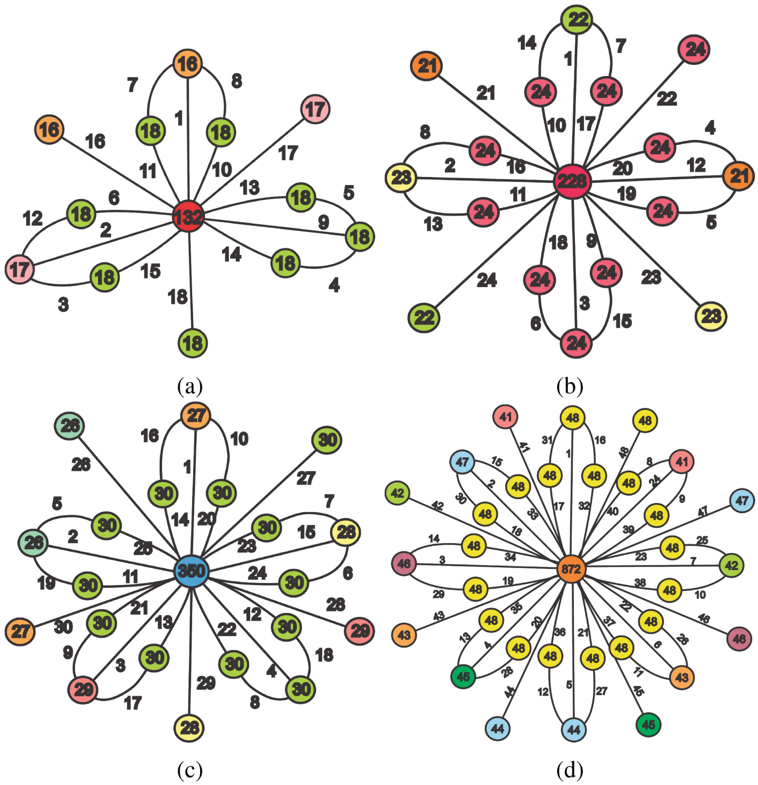

Fig. 2. rvac of $[C(4,1) \cup K_2]^k$ with (a) $k=3$, (b)$k=4$,(c) $k=5$, and (d)$k \geq 6$.

graph (B) requires $k+1$ colors to create a rainbow point trajectory from $u-v$. The following are the vertex weights that result in a rainbow-colored antimagic vertex coloring.

$$\begin{aligned} w_f(A) &= 132 \\ w_f(y_{ij}) &= \begin{cases} 18 & ;\ 1 \leq i \leq k, j=(1,3) \\ 5k+i & ;\ 1 \leq i \leq k, j=2 \end{cases} \\ w_f(x_i) &= 5k+i \qquad ;\ 1 \leq i \leq k \end{aligned}$$

To determine the number of vertex weight based on the set of vertex weight which is

$$w = \underbrace{\{5k+i, ..., 18\}}_{k} + \{132\}$$
$$|w| = k+1.$$

It shows the vertex weight induces a rainbow vertex antimagic coloring of $k+1$ colors, which means it concludes the proof of $rvac(G)=k+1$. Figure 2(a), showed that every path of $u-v$ has different interior colors of vertices. Based on the lower and upper bound, it is proved that $rvac(B) = k + 1$.

Case 2. For $k = 4$

For case 2, the value of k is 4 (fixed), then to find the function and weight, We can see it from the illustration in Fig. 2 (b). From Fig. 2 (b), it can be seen that need $k + 1$ colors to make rainbow point trajectory from $u - v$. It showed weight of it vertices as follows

$$\begin{aligned} w_f(A) &= 228 \\ w_f(y_{ij}) &= \begin{cases} 24 & ;\ 1 \le i \le 4, j = (1,3) \\ 5k + i + 1 & ;\ 1 \le i \le 3, j = 2 \\ 21 & ;\ i = 4, j = 2 \end{cases} \\ w_f(x_i) &= 5k + i \qquad ;\ 1 \le i \le k \end{aligned}$$

Based on the weight vertex, which is

$$w = \underbrace{\{5k + i, 5k + i + 1, ..., 24\}}_{k} + \{225\}$$
$$|w| = k + 1.$$

It demonstrates how the vertex weight causes $k + 1$ colors with rainbow point trajectories from u to v to exhibit rainbow vertex antimagic coloring. Each internal vertex for each vertex has a different weight because each vertex $u, u' \epsilon V(BF(m, n))$ is given the color wf. Thus, based on the lower and upper bound, it is proved that $rvac(BF(m, n)) = k + 1$ for $m \ge 5$.

Case 3. For $k = 5$

Similar to the first and second cases, we can see that the value of k in this instance is 5 (fixed). Figure 2 (c) representation of rvac(G) illustrates how to find the function and its weight. According to Fig. 2 (c), a rainbow point trajectory from $u - v$ requires $k + 1$ colors. The weight of its vertices was displayed as follows.

$$\begin{aligned} w_f(A) &= 350 \\ w_f(y_{ij}) &= \begin{cases} 30 & ;\ 1 \le i \le 4, j = (1,3) \\ 5k + i + 1 & ;\ i\ odd, j = 2 \\ 5k + i & ;\ i\ even, j = 2 \\ 28 & ;\ i = 5, j = 2 \end{cases} \\ w_f(x_i) &= 5k + i \qquad ;\ 1 \le i \le k \end{aligned}$$

Based on the weight vertex, which is

$$w = \underbrace{\{5k + i, 5k + i + 1, ..., 28, ..., 30\}}_{k} + \{350\}$$
$$|w| = k + 1.$$

It shows the vertex weight induces a rainbow vertex antimagic coloring of $k + 1$ colors, which means it concludes the proof of $rvac(G) = k + 1$. Figure (c), showed that every path of $u - v$ has different interior colors of vertices. Based on the lower and upper bound, it is proved that $rvac(B) = k + 1$ (Table 4).

Table 4. The Rainbow Vertex of $u - v$ Path of $[C(n, n-3) \cup K_2]^k$.

Case	**x**	**y**	**rainbow path** $u - v$	**Condition**
1	x_i	x_j	x_i, A, x_j	$1 \leq i \leq k$ $1 \leq j \leq k$
2	x_i	y_{ij}	x_i, A, x_{ij}	$1 \leq i \leq k$ $1 \leq j \leq k$
3	y_{ij}	y_{tq}	y_{ij}, A, y_{tk}	$1 \leq i \leq k$ $1 \leq j \leq 3$ $1 \leq l \leq k$ $1 \leq q \leq 3$

Case 4. For $k \geq 6$

From the edge labeling functions above, we can determine the vertex weights which induces a rainbow vertex antimagic coloring as follow.

$$w_f(A) = 13(k-2)^2 + 57(k-2) + 62$$

$$\begin{aligned} w_f(y_{ij}) &= \begin{cases} 6k & ;\ 1 \leq i \leq k, j = (1,3) \\ 6k - i + 1 & ;\ 1 \leq i \leq k, j = 2 \end{cases} \\ w_f(x_i) &= 5k + i \qquad\quad ;\ 1 \leq i \leq k \end{aligned}$$

To determine the number of vertex weight based on the set of vertex weight which is

$$w = \underbrace{\{6k, ..., 6k - i + 1, ..., 5k + i, ...\}}_{k} + \{13(k-2)^2 + 57(k-2) + 62\}$$

$$|w| = k + 1.$$

For the illustration of $rvac(G)$ is provided in Fig. 2 (d) The vertex weights have their own distinct colors adjusted to make it easier to see the rainbow antimagic path. It shows the vertex weight induces a rainbow vertex antimagic coloring of $k+1$ colors, which means it concludes the proof of $rvac(G) = k+1$. We will show the following rainbow path of a graph that is formed can be seen in Table 4:

5 Conclusion

These results lead us to the discovery of four new theorems regarding the rainbow vertex connection number and rainbow vertex antimagic coloring for graphs connected to shells. The rainbow vertex connection number of shell-butterfly graph ($BF(m,n)$) with $m \geq 3$ and $n \geq 3$ is $rvc(BF(m,n)) = 1$. The rainbow vertex antimagic connection number of shell-butterfly graph ($BF(m,n)$) with $m, n \geq 3$ and $m = n$ is $ravc(BF(m,n)) = 4$. The rainbow vertex connection number shell flower graph $[C(n, n-3) \cup K_2]^k$ with $k \geq 3$ is $rvc([C(n, n-3) \cup K_2]^k) = 1$. The rainbow vertex antimagic connection number of shell flower graph $[C(4,1) \cup K_2]^k$ with $k \geq 3$ is $ravc([C(4,1) \cup K_2]^k) = k + 1$.Therefore, we have the following open problems.

Open Problem 1. Identify the rainbow vertex antimagic connection number $rvac(G)$ of a different graph related to shells.

Open Problem 2. Calculate the rainbow vertex antimagic connection number $ravc([C(\frac{4}{n}, n-3) \cup K_2]^k)$ with $n \geq 5$ and $k \geq 3$.

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