On Rainbow Antimagic Coloring of Some Classes of Graphs

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Abstract. This study aims to develop rainbow antimagic coloring. A graph G is said to be a rainbow antimagic coloring if there is a bijection function $f : V(G) \rightarrow \{1, 2, ..., V(G)\}$, and the associated weight of an edge $uv \in E(G)$ under f is $w_f(uv) = f(u) + f(v)$. If there is $w_f(uv) \neq w_f(u'v')$ for every two edges $uv, u'v'$ in $E(P)$, a path P in a vertex-labeled graph G is said to be a rainbow path. The purpose of this paper is to study the rainbow antimagic coloring and determine the smallest number of rainbow antimagic connection number, denoted by $rac(G)$, where G is the vertex amalgamation of some graphs, namely fan graph F_p, dragon graph Dg_p, bow graph $B_{p,q}$, and wheel graph W_p. In this paper, we complete previous research of rainbow antimagic connection number of vertex amalgamation product of fan graph until reaching the lower bound.

Keywords: Edge coloring · rainbow antimagic coloring · vertex amalgamation

1 Introduction

Graph theory is one of the most recent breakthroughs in mathematics. In graph theory, an object is referred to as a vertex and a connection between things as an edge. A graph G comprises a set E of 2-element subsets of V called edges and a set V of finite nonempty objects called vertices. A pair of two sets, V and E, make up graph G. Because of this, some people type $G = (V, E)$ [4]. The degree of vertex v in a graph G is the number of vertices adjacent to v and denoted by $d(v)$ [12]. Graph theory has various topics, such as rainbow antimagic coloring. Rainbow antimagic coloring is the combination of rainbow connection and antimagic labeling.

The concept of rainbow connection was first introduced by Chartrand et al. [3]. There are adjacent edges that may have the same color in the rainbow

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connection notion, where the color employs the integers $\{1, 2, ..., k\}$. A path P can form a rainbow path if no adjacent edges in graph G have the same colors. Graph G is rainbow connected if a rainbow path connects any two of its vertices. The edge coloring that makes the graph G rainbow connected can be defined as a rainbow connection. Rainbow coloring research aims to determine the smallest number of colors, often known as the rainbow connection number, or $rc(G)$ [2]. The exact values of rainbow connection number for several special graph operations, such as cartesian product, the composition of two special graphs, tensor product, and amalgamation of special graphs, were discovered by Dafik et al. in [5]. In [15], Parmar et al. found the rainbow connection number of H graphs. In [6], Dafik et al. found the exact value of rainbow and strong rainbow connection numbers of comb product graphs.

Apart from rainbow connections, antimagic labeling is another exciting topic in graphs. This topic is described as labeling (given a number) the vertices of a graph so that no two distinct edges have the same weight. The weight of an edge is the sum of the labels of two vertices that incident it [9]. In [17], Simanjuntak et al. discussed two new $(a, d)-$ antimagic labeling. In [19], Wang and Zhang discussed the antimagic labeling of regular graphs.

This study investigates the idea of rainbow antimagic coloring. If every vertex of graph G is labeled with antimagic labels and then edge weights of antimagic labels are used to construct a rainbow path, the rainbow path is said to have rainbow antimagic coloring [18]. This study aims to find $rac(G)$, the antimagic rainbow connection number, which is the smallest possible number of colors obtained from antimagic rainbow coloring. The rainbow antimagic connection number of graphs is determined by Dafik et al., who also specify the lower and upper bounds of the $rac(G)$ of graphs [18]. Previous researchers have discovered the rainbow antimagic coloring of some graphs. Kusumawardani et al. [11] found the rainbow antimagic connection number of flower graphs and gear graph. Sulistiyono et al. [18] found the rainbow antimagic connection number of path graph, ladder graph, triangular ladder graph, and diamond graph. Al-Jabbar et al. [1] found the rainbow antimagic connection number of triangular book, book graph, and generated book graph. Dafik et al. [7] found the rainbow connection number of complete graph, cycle graph, fan graph, wheel graph, friendship graph, and tree graph. Septory et al. [16] found the rainbow antimagic connection number of jahangir graph, complete bipartite graph,lemon graph, double star graph, and firecracker graph. Budi et al. [2] found the rainbow antimagic connection number of dragonfly graph, lollipop graph, stacked book graph, dutch windmill graph, and flowerpot graph. Nisviasari et al. [14]. Joedo et al. [10] found the rainbow antimagic connection number of tadpole graph. Joedo et al. [10] found the rainbow antimagic connection number of $Amal(P_n, v, m)$, $Amal(S_n, v, m)$, $Amal(Br_{n,d}, v, m)$, $Amal(P_{3,n}, v, m)$, $Amal(F_n, v, m)$, $Amal(Tb_n, v, m)$.

In this paper, we discussed the $rac(G)$ where G is the vertex amalgamation of some graphs, namely, fan graph (F_m), dragon graph (Dg_m), bow graph ($B_{m,p}$), and wheel graph (W_m). For the $rac(G)$ of vertex amalgamation product of fan

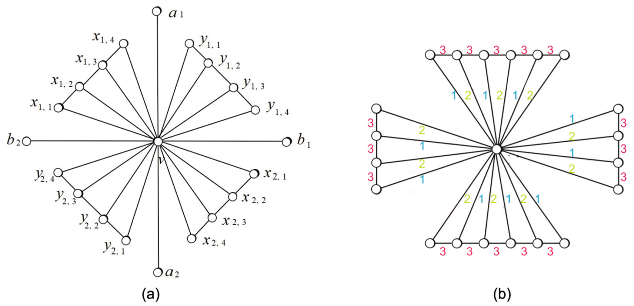

Fig. 1. (a) Vertex Amalgamation of Dragon, $Amal(Dg_3, v, 2)$, (b) Rainbow coloring of $Amal(B_{6,4}, v, 2)$ graph

graph, we improve previous research by [10] until we reach the lower bound. The previous research on the vertex amalgamation product of fan graph can be seen in Theorem 3. Vertex amalgamation was introduced by Lee [13]. Let G be a graph and $v_0 \in V(G)$ be a fixed vertex of G. For $t \geq 2$, vertex amalgamation denoted by $Amal(G, v_0, t)$ is the graph constructed by amalgamating t copies of G with the fixed vertex v_0 [13]. The illustration of the vertex amalgamation of the graph can be seen in Fig. 1(a).

All Lemma, Propositions, and Theorems that we used to prove the theorem in this paper are as follows.

Lemma 1. [18] Let G be any connected graph. Let $rc(G)$ and $\triangle(G)$ be the rainbow connection number of G and the maximum degree of G, respectively. Then, $rac(G) \geq max\{rc(G), \triangle(G)\}$.

Proposition 1. [14] If G is a nontrivial connected graph of size m, then $diam(G) \leq rc(G) \leq src(G) \leq m$.

Proposition 2. Let G be a nontrivial connected graph of size m. Then

(a) rc(G)=1 if and only if G is complete, src(G)=1 if only if G is complete;
(b) rc(G)=2 if and only if src(G)=2;
(c) rc(G)=m if and only if G is a tree, src(G)=m if and only if G is a tree.

Theorem 1. [8] Let $Amal(F_p, v, t)$ be a vertex amalgamation product of fan graph, then $rc(Amal(F_p, v, t)) = 3$.

Theorem 2. [8] Let $Amal(W_p, v, t)$ is a vertex amalgamation of wheel graph, then

$$rc(Amal(W_p, v, t)) = \begin{cases} 3, & \text{for } p \geq 5 \\ & t \geq 2 \text{ or } p = 4 \text{ and } t \geq 3. \\ 2, & \text{for } p = 4 \text{ and } t = 2 \end{cases}$$

Theorem 3. [10] Let $Amal(F_p, v, t)$ be a vertex amalgamation product of fan graph with $p \geq 3, t \geq 2$, therefore $rac(Amal(F_p, v, t)) = pt + 1$.

2 Result and Discussion

In this paper, we discuss some new results of the rainbow antimagic coloring of vertex amalgamation of some graphs.

Theorem 4. Let $Amal(F_p, v, t)$ be a vertex amalgamation product of fan graph with $p \geq 3, t \geq 2$, furthermore $rac(Amal(F_p, v, t)) = pt$.

Proof. Let $Amal(F_p, v, t)$ be a vertex amalgamation product of a fan graph with vertex set $V(Amal(F_p, v, t)) = \{v\} \cup \{x_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p\}$ and $E(Amal(F_p, v, t)) = \{vx_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p\} \cup \{x_{i,j}x_{i,j+1}, 1 \leq i \leq t, 1 \leq j \leq p-1\}$. So, we get $|V(Amal(F_p, v, t))| = pt+1$ and $|E(Amal(F_p, v, t))| = (2p-1)t$. Then, we proved the correctness of the lower bound and the upper bound, namely $rac(Amal(F_p, v, t)) \geq pt$ and $rac(Amal(F_p, v, t)) \leq pt$.

First of all, we proof the lower bound of $rac(Amal(F_p, v, t))$ is $rac(Amal(F_p, v, t)) \geq pt$. Based on Lemma 1, we know that $rac(G) \geq max\{\triangle(G), rc(G)\}$, so that:

$$\begin{aligned} rac(Amal(F_p, v, t)) &\geq max\{\triangle(Amal(F_p, v, t)), rc \\ &\quad (Amal(F_p, v, t))\} \\ &= max\{pt, 3\} = pt \end{aligned}$$

So, we get $rac(Amal(F_p, v, t)) \geq pt$ with $p \geq 3, t \geq 2$.

After that, we proof the upper bound of $rac(Amal(F_p, v, t))$ is $rac(Amal(F_p, v, t)) \leq pt$. To find the upper bound of $rac(Amal(F_p, v, t))$, we devide it into two cases, namely $t \equiv 0(mod2)$ and $t \equiv 1(mod2)$.

Case 1. For $t \equiv 0(mod\ 2), p \geq 3, t \geq 2$.

We define the vertex function of the $Amal(F_p, v, t)$ graph with $l : V(Amal(F_p, v, t)) \to \{1, 2, ..., |V(Amal(F_p, v, t))|\}$ for $x_{i,j} \in V(Amal(F_p, v, t))$ are as follows

$$l(v) = \begin{cases} \frac{(p+1)t}{2} + 1, & \text{if } p \equiv 1(mod\ 2) \\ \frac{pt}{2} + 1, & \text{if } p \equiv 0(mod\ 2) \end{cases}$$

$$l(x_{i,j}) = \begin{cases} \frac{(j-1)t}{2} + i, & \text{if } 1 \leq i \leq t, j \equiv 1(mod\ 2) \\ \frac{(p+j-1)t}{2} + i + 1, & \text{if } p \equiv 1(mod\ 2), j \equiv 0(mod\ 2) \\ \frac{(p+j)t}{2} - t + i + 1, & \text{if } p \equiv 0(mod\ 2), j \equiv 0(mod\ 2) \end{cases}$$

We get the edge weights from a predetermined vertex function, which will be used to color the edges. The edge weights of the graph $Amal(F_p, v, t)$ are as follows

$$w(vx_{i,j}) = \begin{cases} \frac{(p+j)t}{2} + i + 1, & \text{if } p \equiv 1(mod\ 2), 1 \leq j \leq p, \\ & j \equiv 1(mod\ 2) \\ \frac{jt}{2} + pt + i + 2, & \text{if } p \equiv 1(mod\ 2), j \equiv 0(mod\ 2) \\ \frac{(p+j-1)t}{2} + i + 1, & \text{if } p \equiv 0(mod\ 2), j \equiv 1(mod\ 2) \\ \frac{jt}{2} + (p-1)t + i + 2, & \text{if } p \equiv 0(mod\ 2), j \equiv 0(mod\ 2) \end{cases}$$

$$w(x_{i,j}x_{i,j+1}) = \begin{cases} \frac{(p-1)t}{2} + jt + 2i + 1, & \text{if } p \equiv 1(mod\ 2), 1 \leq j \leq p \\ \frac{(p+2j-2)t}{2} + 2i + 1, & \text{if } p \equiv 0(mod\ 2), 1 \leq j \leq p \end{cases}$$

The next step is to see that the edge weights induce rainbow antimagic coloring on the graph $Amal(F_p, v, t)$. Let $A_{1,1}$ be the set of the edge weights of graph $Amal(F_p, v, t)$ for $t \equiv 0(mod\ 2), p \geq 3, t \geq 2$. Then, $A_{1,1} = \{\frac{(p+1)t}{2} + 2, \frac{(p+1)t}{2} + 3, ..., \frac{(p-1)t}{2} + pt + t + 2\} - \{(p+1)t + 2\}$, thus the number of distinct colors of $A_{1,1}$ is pt. To determine the upper bound, we should find the cardinality of the set $A_{1,1}$ In this step, we use an arithmetic sequence to know the cardinality of the set $\{\frac{(p+1)t}{2} + 2, \frac{(p+1)t}{2} + 3, ..., \frac{(p-1)t}{2} + pt + t + 2\}$.

$$\begin{aligned} U_s &= a + (s-1)d \\ \frac{(p-1)t}{2} + pt + t + 2 &= \frac{(p+1)t}{2} + 2 + (s-1)1 \\ pt &= s - 1 \\ s &= pt + 1 \end{aligned}$$

Then, we determine the cardinality of the set $A_{1,1}$:

$$\begin{aligned} |A_{1,1}| &= |\{\tfrac{(p+1)t}{2} + 2, \tfrac{(p+1)t}{2} + 3, ..., \tfrac{(p-1)t}{2} + pt + t + \\ &\quad 2\}| - |\{(p+1)t + 2\}| \\ |A_{1,1}| &= pt + 1 - 1 \\ |A_{1,1}| &= pt \end{aligned}$$

It shows the edge weight induces a rainbow antimagic coloring of pt colors. So, we get the upper bound of $rac(Amal(F_p, v, t))$ is $rac(Amal(F_p, v, t)) \leq pt$ for $t \equiv 0(mod\ 2), p \geq 3, t \geq 2$.

Case 2. For $t \equiv 1(mod\ 2), p \geq 3, t \geq 2$.

We define the vertex function of the $Amal(F_p, v, t)$ graph with $l : V(Amal(F_p, v, t)) \to \{1, 2, ..., |V(Amal(F_p, v, t))|\}$ for $x_{i,j} \in V(Amal(F_p, v, t))$ are as follows

$$l(v) = \begin{cases} \frac{(p+1)t}{2} + t - 1, & \text{if } p \equiv 1(mod\ 2) \\ \frac{pt}{2} + t - 1, & \text{if } p \equiv 0(mod\ 2) \end{cases}$$

$$l(x_{i,j}) = \begin{cases} \frac{(j-1)t}{2} + i, & \text{if } p \geq 3, j \equiv 1(mod\ 2) \\ \frac{(p+j)t}{2} + 2 - i, & \text{if } p \equiv 0(mod\ 2), j \neq 2, \\ & j \equiv 0(mod\ 2), 3 \leq i \leq t \\ & \text{if } 1 \leq i \leq 2, j \equiv 0(mod\ 2), \\ & 2 \leq j \leq p \\ \frac{pt}{2} + t + 1 - i, & \text{if } p \equiv 0(mod\ 2), 3 \leq i \leq t, \\ & j = 2 \\ \frac{(p+j+1)t}{2} + 2 - i, & \text{if } p \equiv 1(mod\ 2), j \equiv 0(mod \\ & 2), j \neq 2, 3 \leq i \leq t \\ & \text{if } p \equiv 1(mod\ 2), j \equiv 0(mod \\ & 2), 2 \leq j \leq p \\ \frac{(p+1)t}{2} + t + 1 - i, & \text{if } p \equiv 1(mod\ 2), j = 2, \\ & 3 \leq i \leq n. \end{cases}$$

We get the edge weights from a predetermined vertex function, which will be used to color the edges. The edge weights of the graph $Amal(F_p, v, t)$ are as follows

$$w(vx_{i,j})=\begin{cases} \frac{(p+j+1)t}{2}+i-1, & \text{if } p \equiv 0(mod\ 2), j \equiv 1 \\ & (mod\ 2), 1 \leq i \leq t, \\ & 1 \leq j \leq p \\ (p+2)t-i, & \text{if } p \equiv 0(mod\ 2), j=2 \\ \frac{jt}{2}+pt+t+1-i, & \text{if } p \equiv 0(mod\ 2), j \neq 2, \\ & j \equiv 0(mod\ 2) \\ \frac{(p+j)t}{2}+t+i-1, & \text{if } p \equiv 1(mod\ 2), j \equiv 1 \\ & (mod\ 2) \\ (p+3)t-i+1, & \text{if } p \equiv 1(mod\ 2), j=2 \\ \frac{jt}{2}+(p+2)t+1-i, & \text{if } p \equiv 1(mod\ 2), j \neq 2, \\ & j \equiv 0(mod\ 2) \end{cases}$$

$$w(x_{i,j}x_{i,j+1})=\begin{cases} \frac{(p+j+2)t}{2}+2, & \text{if } p \equiv 1(mod\ 2), 1 \leq i \\ & \leq 2, j=1 \text{ or } j=3 \\ \frac{(p+j)t}{2}+t+1, & \text{if } p \equiv 1(mod\ 2), 3 \leq i \\ & \leq t, j=1 \text{ or } j=3 \\ \frac{(2j+p+1)t}{2}+2, & \text{if } p \equiv 1(mod\ 2), 1 \leq i \\ & \leq t, 3 \leq j \leq p \\ \frac{(p+2j-1)p}{2}+2, & \text{if } p \equiv 0(mod\ 2), 1 \leq i \\ & \leq 2, j=1 \text{ or } j=3 \\ \frac{(p+j-1)t}{2}+t+1, & \text{if } p \equiv 0(mod\ 2), 3 \leq i \\ & \leq t, j=1 \text{ or } j=3 \\ \frac{(2j+p)t}{2}+2, & \text{if } p \equiv 0(mod\ 2), 1 \leq i \\ & \leq t, 3 \leq j \leq p \end{cases}$$

The next step is to see that the edge weights induce rainbow antimagic coloring on the graph $Amal(F_p, v, t)$. Let $A_{1,2}$ be the set of the edge weights of graph $Amal(F_p, v, t)$ for $t \equiv 1(mod\ 2), p \geq 3, t \geq 2$. Then, $A_{1,2} = \{\frac{(p+1)t}{2}+t, \frac{(p+1)t}{2}+t+1, ..., \frac{(p-1)t}{2}+(p+2)t\} - \{(p+1)t+2t-2\}$, thus the number of distinct colors of $A_{1,2}$ is pt. To determine the upper bound, we should find the cardinality of $A_{1,2}$. In this step, we use an arithmetic sequence to know the cardinality of the set $\{\frac{(p+1)t}{2}+t, \frac{(p+1)t}{2}+t+1, ..., \frac{(p-1)t}{2}+(p+2)t\}$.

$$\begin{aligned} U_s &= a+(s-1)d \\ \frac{(p-1)t}{2}+(p+2)t &= \frac{(p+1)t}{2}+t+(s-1)1 \\ pt &= s-1 \\ s &= pt+1 \end{aligned}$$

Then, we determine the cardinality of the set $A_{1,2}$:

Table 1. The rainbow path from x to y on $Amal(F_p, v, t)$ graph.

case	x	y	rainbow path
1	$x_{i,j}$	v	$x_{i,j}, v$
2	$x_{i,j}$	$x_{k,l}$	$x_{i,j}, v, x_{k,l}$

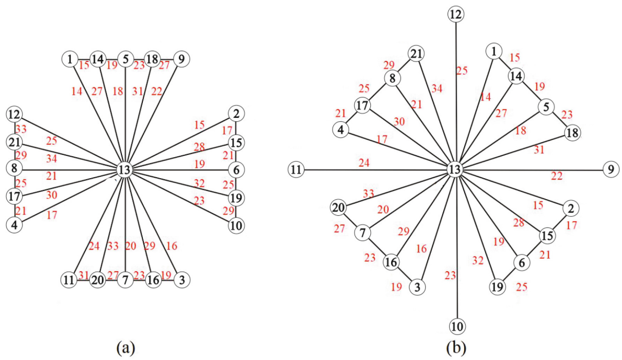

Fig. 2. The RAC of (a) $Amal(F_5, v, 4)$, (b) $Amal(Dg_2, v, 2)$.

$$\begin{aligned}|A_{1,2}| &= |\{\tfrac{(p+1)t}{2} + t, \tfrac{(p+1)t}{2} + t + 1, ..., \\ &\quad \tfrac{(p-1)t}{2} + (p+2)t\}| - |\{(p+1)t + 2t - 2\}| \\ &= pt + 1 - 1 = pt\end{aligned}$$

It shows the edge weight induces a rainbow antimagic coloring of pt colors. So, we get the upper bound of $rac(Amal(F_p, v, t))$ is $rac(Amal(F_p, v, t)) \leq pt$ for $t \equiv 1(mod\ 2), p \geq 3, t \geq 2$.

If there are as many colors as pt, a rainbow path can be constructed based on Cases 1 and 2. Table 1 shows the rainbow path. So, for $p \geq 3, t \geq 2.$, we get $rac(Amal(F_p, v, t)) \leq pn$.

From the discussion above, we have $pt \leq rac(Amal(F_p, v, t)) \leq pt$. So, for $p \geq 3, t \geq 2$, the proof of the formula $rac(Amal(F_p, v, t)) = pt$ has been completed. □

Figure 2 (a) illustrated $rac(Amal(F_p, v, t))$, with the vertex labels on the graph being black numbers and the edge weight on the graph being red numbers.

Theorem 5. Let $Amal(Dg_p, v, t)$ be a vertex amalgamation product of dragon graph with $p \geq 2, t \geq 2$, then $rac(Amal(Dg_p, v, t)) = (2p + 6)t$.

Proof. Let $Amal(Dg_p, v, t)$ be a vertex amalgamation product of dragon graph with vertex set $V(Amal(Dg_p, v, t)) = \{v\} \cup \{x_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p + 2\} \cup \{y_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p + 2\} \cup \{a_i, 1 \leq i \leq t\} \cup \{b_i, 1 \leq i \leq t\}$ and edge set $E(Amal(Dg_p, v, t)) = \{va_i, 1 \leq i \leq t\} \cup \{vb_i, 1 \leq i \leq t\} \cup \{vx_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p + 2\} \cup \{vy_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p + 2\} \cup \{x_{i,j}x_{i,j+1}, 1 \leq$

$i \leq t, 1 \leq j \leq p+1\} \cup \{y_{i,j}y_{i,j+1}, 1 \leq i \leq t, 1 \leq j \leq p+1\}$. So, we get $|V(Amal(Dg_p, v, t))| = (2p+7)t$ and $|E(Amal(Dg_p, v, t))| = 2pt + 6t + 2p + 2$.

First of all, we proof the lower bound of $rac(Amal(Dg_p, v, t))$ is $rac(Amal(Dg_p, v, t)) \geq (2p+6)t$. We know that the $Amal(Dg_p, v, t)$ graph contain the star graph S_t with size $2t$. Based on Preposition 2, we have $rc(Amal(Dg_p, v, t)) = 2t$. Then, we have $rac(G) \geq max\{rc(G), \triangle(G)\}$, so that:

$$\begin{aligned} rac(Amal(Dg_p, v, t)) &\geq max\{rc(Amal(Dg_p, v, t)), \\ &\quad \triangle(Amal(Dg_p, v, t))\} \\ &= max\{2t, (2p+6)t\} \\ &= (2p+6)t \end{aligned}$$

So, we get $rac(Amal(Dg_p, v, t)) \geq (2p+6)t$ with $p \geq 2, t \geq 2$.

After that, we proof the upper bound of $rac(Amal(Dg_p, v, t))$ is $rac \leq (2p+6)t$. We define the vertex function of the $Amal(Dg_p, v, t)$ graph with $l : V(Amal(Dg_p, v, t)) \rightarrow \{1, 2, ..., |V(Amal(Dg_p, v, t))|\}$ for $x_{i,j} \in V(Amal(Dg_p, v, t))$ as follows

$$l(v) = \begin{cases} 6t + p - 2, & \text{if } p \equiv 1(mod\ 2) \\ 6t + p - 1, & \text{if } p \equiv 0(mod\ 2) \end{cases}$$

$$l(x_{i,j}) = \begin{cases} (j-1)t + 2i - 1, & \text{if } j \equiv 1(mod\ 2), 1 \leq j \leq \\ & p+2, p \geq 2 \\ (j+4)t + p + 2i - 2, & \text{if } j \equiv 0(mod\ 2), 1 \leq j \leq \\ & p+2, p \equiv 0(mod\ 2) \\ (j+4)t + p + 2i - 3, & \text{if } j \equiv 0(mod\ 2), 1 \leq j \leq \\ & p+2, p \equiv 1(mod\ 2) \end{cases}$$

$$l(y_{i,j}) = \begin{cases} (j-1)t + 2i, & \text{if } j \equiv 1(mod\ 2), 1 \leq j \leq \\ & p+2, p \geq 2 \\ (j+4)t + p + 2i - 1, & \text{if } j \equiv 0(mod\ 2), 1 \leq j \leq \\ & p+2, p \equiv 0(mod\ 2) \\ (j+4)t + p + 2i - 2, & \text{if } j \equiv 0(mod\ 2), 1 \leq j \leq \\ & p+2, p \equiv 1(mod\ 2) \end{cases}$$

$$l(a_i) = \begin{cases} 2p(t-1) + 2i + 3, & \text{if } p \equiv 0(mod\ 2) \\ (2p+4)t + 2i, & \text{if } p \equiv 1(mod\ 2) \end{cases}$$

$$l(b_i) = \begin{cases} 2(pt - p + i + 2), & \text{if } p \equiv 0(mod\ 2), 1 \leq i \leq t \\ 2(pt + 2t + i) + 1, & \text{if } p \equiv 1(mod\ 2), 1 \leq i \leq t \end{cases}$$

We get the edge weights from a predetermined vertex function, which will be used to color the edges. The edge weights of the graph $Amal(Dg_p, v, t)$ are as follows

$$w(vx_{i,j})=\begin{cases}(j+5)t+p+2i-3, & \text{if } p\equiv 1(mod\ 2),\\ & j\equiv 1(mod\ 2),\\ & 1\le j\le p+2\\ (j+10)t+2(p+i)-5, & \text{if } p\equiv 1(mod\ 2),\\ & j\equiv 0(mod\ 2),\\ & 1\le j\le p+2\\ (j+5)t+p+2i-2, & \text{if } p\equiv 0(mod\ 2),\\ & j\equiv 1(mod\ 2),\\ & 1\le j\le p+2\\ (j+10)t+2(p+i)-3, & \text{if } p\equiv 0(mod\ 2),\\ & j\equiv 0(mod\ 2),\\ & 1\le j\le p+2\end{cases}$$

$$w(va_i)=\begin{cases}(2p+10)t+p+2i-2, & \text{if } p\equiv 1(mod\ 2),\\ & 1\le i\le t\\ (2p+6)t-p+2i+2, & \text{if } p\equiv 0(mod\ 2),\\ & 1\le i\le t\end{cases}$$

$$w(vb_i)=\begin{cases}(2p+10)t+p+2i-1, & \text{if } p\equiv 1(mod\ 2),\\ & 1\le i\le t\\ (2p+6)t-p+2i+3, & \text{if } p\equiv 0(mod\ 2),\\ & 1\le i\le t\end{cases}$$

$$w(vy_{i,j})=\begin{cases}(j+5)t+p+2i-2, & \text{if } p\equiv 1(mod\ 2),\\ & j\equiv 1(mod\ 2),\\ & 1\le j\le p+2\\ (j+10)t+2p+2i-4, & \text{if } p\equiv 1(mod\ 2),\\ & j\equiv 0(mod\ 2),\\ & 1\le j\le p+2\\ (j+5)t+p+2i-1, & \text{if } p\equiv 0(mod\ 2),\\ & j\equiv 1(mod\ 2),\\ & 1\le j\le p+2\\ (j+10)t+2p+2i-2, & \text{if } p\equiv 0(mod\ 2),\\ & j\equiv 0(mod\ 2),\\ & 1\le j\le p+2\end{cases}$$

$$w(x_{i,j}x_{i,j+1})=\begin{cases}(2j+4)t+4i+p-4, & \text{if } p\equiv 1(mod\ 2),\\ & p\ge 2\\ (2j+4)t+4i+p-3, & \text{if } p\equiv 0(mod\ 2),\\ & p\ge 2\end{cases}$$

Table 2. The rainbow path from x to y on $Amal(Dg_p, v, t)$ graph.

case	x	y	rainbow path	case	x	y	rainbow path
1	$x_{i,j}$	v	$x_{i,j}, v$	8	$y_{i,j}$	a_i	$y_{i,j}, v, a_i$
2	$x_{i,j}$	$x_{k,l}$	$x_{i,j}, v, x_{k,l}$	9	$y_{i,j}$	b_i	$y_{i,j}, v, b_i$
3	$x_{i,j}$	$y_{i,j}$	$x_{i,j}, v, y_{i,j}$	10	a_i	v	a_i, v
4	$x_{i,j}$	a_i	$x_{i,j}, v, a_i$	11	a_i	a_j	a_i, v, a_j
5	$x_{i,j}$	b_i	$x_{i,j}, v, b_i$	12	a_i	b_i	a_i, v, b_i
6	$y_{i,j}$	v	$y_{i,j}, v$	13	b_i	v	b_i, v
7	$y_{i,j}$	$y_{k,l}$	$y_{i,j}, v, y_{k,l}$	14	b_i	b_j	b_i, v, b_j

$$w(y_{i,j}y_{i,j+1}) = \begin{cases} (2j+4)t + 4i + p - 2, & \text{if } p \equiv 1(mod\ 2), \\ & p \geq 2 \\ (2j+4)t + 4i + p - 1, & \text{if } p \equiv 0(mod\ 2), \\ & p \geq 2 \end{cases}$$

The next step is to see that the edge weights induce rainbow antimagic coloring on the graph $Amal(Dg_p, v, t)$. Let A_2 be the set of the edge weights of graph $Amal(Dg_p, v, t)$. Then, $A_2 = \{p + 6t - 1, p + 6t, ..., (2p + 10)t + p + 2t - 1\} - \{2p + 12t - 4\}$, thus the number of distinct colors of the edge weights is $(2p + 6)t$. To determine the upper bound, we should find the cardinality of set A_2. In this step, we use an arithmetic sequence to know the cardinality of the set $\{p + 6t - 1, p + 6t, ..., (2p + 10)t + p + 2t - 1\}$.

$$\begin{aligned} U_s &= a + (s-1)d \\ (2p+10)t + p + 2t - 1 &= p + 6t - 1 + (s-1)1 \\ 2pt + 6t &= s - 1 \\ s &= (2p+6)t + 1 \end{aligned}$$

Then, we determine the cardinality of the edge weight set A_2:

$$\begin{aligned} |A_2| &= |\{p + 6t - 1, p + 6t, ..., (2p+10)t + p + 2t - \\ &\quad 1\}| - |\{2p + 12t - 4\}| = (2p+6)t + 1 - 1 \\ |A_2| &= (2p+6)t \end{aligned}$$

It shows the edge weight induced a rainbow antimagic coloring of $(2p + 6)t$ colors. If there are $(2p + 6)t$ colors, a rainbow path can be built based on the edge weight function that has been discovered. Table 2 shows the rainbow path. So, for $p \geq 2, t \geq 2$, we get $rac(Amal(Dg_p, v, t)) \leq (2p + 6)t$.

From the discussion above, we have $(2p+6)t \leq rac(Amal(Dg_p, v, t)) \leq (2p+6)t$. So, for $p \geq 2, t \geq 2$, the proof of the formula $rac(Amal(Dg_p, v, t)) = (2p+6)t$ is complete. □

Figure 2 (b) illustrates $rac(Amal(Dg_p, v, t))$, with the vertex labels on the graph as black numbers and the edge weight on the graph as red numbers.

Theorem 6. Let $Amal(B_{p,q}, v, t)$ be a vertex amalgamation of bow graph with $p \geq 3, q \geq 3, t \geq 2$, then $rc(Amal(B_{p,q}, v, t)) = 3$.

Proof. Let $Amal(B_{p,q}, v, t)$ be a vertex amalgamation product of bow graph with vertex set $V(Amal(B_{p,q}, v, t)) = \{v\} \cup \{x_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p\} \cup \{y_{i,j}, 1 \leq i \leq p, 1 \leq j \leq q\}$ and edge set $E(Amal(B_{p,q}, v, t)) = \{vx_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p\} \cup \{vy_{i,j}, 1 \leq i \leq t, 1 \leq j \leq q\} \cup \{x_{i,j}x_{i,j+1}, 1 \leq i \leq t, 1 \leq j \leq p-1\} \cup \{y_{i,j}y_{i,j+1}, 1 \leq i \leq t, 1 \leq j \leq q-1\}$. So, we get $|V(Amal(B_{p,q}, v, t))| = (p+q)t+1$ and $|E(Amal(B_{p,q}, v, t))| = (2p + 2q - 2)t$.

The first step to prove this theorem is we find the lower bound of $rc(Amal(B_{p,q}, v, t))$. Based on Preposition 1, we have the lower bound of $rc(Amal(B_{p,q}, v, t)) \geq diam(Amal(B_{p,q}, v, t))$, then $rc(Amal(B_{p,q}, v, t)) \geq 2$. Assume that $rc(Amal(B_{p,q}, v, t)) = 2$. Let c' be a rainbow 2-coloring of $Amal(B_{p,q}, v, t)$. We analyze three vertices, namely $x_{i,j}, x_{i+1,k}$, and $y_{i,l}$. Because every path with a length of 2-connecting them is unique, the edges of this path should be colored differently. Let $c'(x_{i,j}v) = 1$ and $c'(x_{i+1,k}v) = 2$. Based on this condition, we have a rainbow path between $x_{i,j}$ and $x_{i+1,k}$. After that, we determine a rainbow path between $x_{i,j}$ and $y_{i,l}$. Then, $c'(y_{i,l}v) = 2$. Based on this condition, we do not have a rainbow path between $x_{i+1,k}$ and $y_{i,l}$. We find the contradiction. So, we have $rc(Amal(B_{p,q}, v, t)) \geq 3$.

The next step is we find the upper bound of $rc(Amal(B_{p,q}, v, t)) \leq 3$ by labeling each edges use this formula:

$$c(vx_{i,j}) = \begin{cases} 1 & \text{if } 1 \leq i \leq t, j \equiv 1(mod\ 2), 1 \leq j \leq p \\ 2 & \text{if } 1 \leq i \leq t, j \equiv 0(mod\ 2), 1 \leq j \leq p \end{cases}$$
$$c(vy_{i,j}) = \begin{cases} 1 & \text{if } 1 \leq i \leq t, j \equiv 1(mod\ 2), 1 \leq j \leq q \\ 2 & \text{if } 1 \leq i \leq t, j \equiv 0(mod\ 2), 1 \leq j \leq q \end{cases}$$
$$c(x_{i,j}x_{j+1}) = 3 \text{ if } 1 \leq i \leq t, 1 \leq j \leq p-1$$
$$c(y_{i,j}y_{j+1}) = 3 \text{ if } 1 \leq i \leq t, 1 \leq j \leq q-1$$

Based on the explanation above, we have $3 \leq rc(Amal(B_{p,q}, v, t)) \leq 3$. So, $rc(Amal(B_{p,q}, v, t)) = 3$ for $p \geq 3, q \geq 3$, and $t \geq 2$. It completes the proof. □

The illustration of the rainbow coloring of $Amal(B_{p,q}, v, t)$ graph can be seen in Fig. 1(b).

Theorem 7. Let $Amal(B_{p,q}, v, t)$ be a vertex amalgamation product of bow graph with $p \geq 3, q \geq 3, t \geq 2$, then $rac(Amal(B_{p,q}, v, t)) = (p + q)t$.

Proof. Let $Amal(B_{p,q}, v, t)$ be a vertex amalgamation product of bow graph with vertex set $V(Amal(B_{p,q}, v, t)) = \{v\} \cup \{x_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p\} \cup \{y_{i,j}, 1 \leq i \leq t, 1 \leq j \leq q\}$ and $E(Amal(B_{p,q}, v, t)) = \{vx_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p\} \cup \{vy_{i,j}, 1 \leq i \leq t, 1 \leq j \leq q\} \cup \{x_{i,j}x_{i,j+1}, 1 \leq i \leq t, 1 \leq j \leq p-1\} \cup \{y_{i,j}y_{i,j+1}, 1 \leq i \leq t, 1 \leq j \leq q-1\}$. So, we get $|V(Amal(B_{p,q}, v, t))| = (p+q)t+1$ and $|E(Amal(B_{p,q}, v, t))| = (2p + 2q - 2)t$.

First of all, we proof the lower bound of $rac(Amal(B_{p,q}, v, t))$ was $rac(Amal(B_{p,q}, v, t)) \geq (p + q)t$. Based on Lemma 1 and Theorem 5, we have

$$\begin{aligned} rac(Amal(B_{p,q},v,t)) &\geq max\{rc(Amal(B_{p,q},v,t)),\\ &\quad \triangle(Amal(B_{p,q},v,t))\}\\ &= max\{3,(p+q)t\} = (p+q)t \end{aligned}$$

So, we get $rac(Amal(B_{p,q},v,t)) \geq (p+q)t$ with $p \geq 3, q \geq 3, t \geq 2$.

After that, we proof the upper bound of $rac(Amal(B_{p,q},v,t))$ is $rac(Amal(B_{p,q},v,t)) \leq (p+q)n$. We define the vertex function of the $Amal(B_{p,q},v,t)$ graph with $l : V(Amal(B_{p,q},v,t)) \rightarrow \{1,2,...,|V(Amal(B_{p,q},v,t))|\}$ for $x_{i,j} \in V(Amal(B_{p,q},v,t))$ as follows

$$l(v) = \lceil \tfrac{p}{2} \rceil t + \lceil \tfrac{q}{2} \rceil (t-1) + q + 1, \text{ if } p \geq 3, q \geq 3, t \geq 2$$

$$l(x_{i,j}) = \begin{cases} \frac{j+1}{2} + (i-1)\lceil \frac{p}{2} \rceil, & \text{if } 1 \leq i \leq t, \\ & j \equiv 1 (mod\ 2), \\ & 1 \leq j \leq p \\ \lceil \frac{p}{2} \rceil t + \lfloor \frac{p}{2} \rfloor (i-1) + \frac{j}{2} & \text{if } 1 \leq i \leq t, \\ +qt+1, & j \equiv 0 (mod\ 2), \\ & 1 \leq j \leq p \end{cases}$$

$$l(y_{i,j}) = \begin{cases} \lceil \frac{p}{2} \rceil t + \frac{j+1}{2} + (i-1)\lceil \frac{q}{2} \rceil, & \text{if } 1 \leq i \leq t, \\ & j \equiv 1 (mod\ 2), \\ & 1 \leq j \leq q \\ \lceil \frac{p}{2} \rceil t + \lceil \frac{q}{2} \rceil t + \frac{j}{2}, & \text{if } i = 1, \\ & j \equiv 0 (mod\ 2), \\ & 1 \leq j \leq q \\ \lceil \frac{p}{2} \rceil t + \lceil \frac{q}{2} \rceil (t+1) + & \text{if } 2 \leq i \leq t, \\ \lfloor \frac{q}{2} \rfloor (i-2) + \frac{j}{2}, & j \equiv 0 (mod\ 2), \\ & 1 \leq j \leq q \end{cases}$$

We get the edge weights from a predetermined vertex function, which will be used to color the edges. The edge weights of the graph $Amal(B_{p,q},v,t)$ are as follows

$$w(vx_{i,j}) = \begin{cases} \lceil \frac{p}{2} \rceil 2t + \lceil \frac{q}{2} \rceil (t-1) + \lfloor \frac{p}{2} & \text{if } 1 \leq i \leq t, \\ \rfloor (i-1) + \frac{j}{2} + (t+1)q + 2 & j \equiv 0 (mod\ 2), \\ & 1 \leq j \leq p \\ \lceil \frac{p}{2} \rceil (t+i-1) + \lceil \frac{q}{2} \rceil (t-1) + & \text{if } 1 \leq i \leq t, \\ \frac{j+1}{2} + q + 1, & j \equiv 1 (mod\ 2), \\ & 1 \leq j \leq p \end{cases}$$

$$w(vy_{i,j})=\begin{cases} \lceil \frac{p}{2} \rceil 2t + \lceil \frac{q}{2} \rceil (t+i-2)+ & \text{if } 1 \leq i \leq t, \\ \frac{j+1}{2} + q + 1, & j \equiv 1 (mod\ 2), \\ & 1 \leq j \leq q \\ \lceil \frac{p}{2} \rceil 2t + \lceil \frac{q}{2} \rceil (2t-1)+ & \text{if } i = 1, \\ \frac{j}{2} + q + 1, & j \equiv 0 (mod\ 2), \\ & 1 \leq j \leq q \\ \lceil \frac{p}{2} \rceil 2t + \lceil \frac{q}{2} \rceil (2t) + \lfloor \frac{q}{2} \rfloor & \text{if } 2 \leq i \leq t, \\ (i-2) + \frac{j}{2} + q + 1, & j \equiv 0 (mod\ 2), \\ & 1 \leq j \leq q \end{cases}$$

$$w(x_{i,j}x_{i,j+1}) = \lceil \frac{p}{2} \rceil (t+i-1) + \lfloor \frac{p}{2} \rfloor (i-1) + qt + j + 2, \text{ if } 1 \leq i \leq t, 1 \leq j \leq p-1$$

$$w(y_{i,j}y_{i,j+1})=\begin{cases} \lceil \frac{p}{2} \rceil 2t + \lceil \frac{q}{2} \rceil (t+i-1) & \text{if } i = 1, 1 \leq j \leq q \\ +j+1, & \\ \lceil \frac{p}{2} \rceil 2t + \lceil \frac{q}{2} \rceil (t+i)+ & \text{if } 2 \leq i \leq t, \\ \lfloor \frac{q}{2} \rfloor (i-2) + j + 1, & 1 \leq j \leq q \end{cases}$$

The next step is to see that the edge weights induce rainbow antimagic coloring on the graph $Amal(B_{p,q}, v, t)$. Let A_3 be the set of the edge weights of graph $Amal(B_{p,q}, v, t)$. Then, $A_3 = \{\frac{pt}{2} + \frac{(t-1)q}{2} + q + 2, \frac{pt}{2} + \frac{(t-1)q}{2} + q + 3, ..., pt + \frac{(t-1)q}{2} + \frac{pt}{2} + (t+1)q + 2\} - \{pt + (t+1)q + 2\}$, thus the number of distinct colors of the edge weights is $(p+q)t$. To determine the upper bound, we should find the cardinality of the set of edge weights. In this step, we use an arithmetic sequence to know the cardinality of the set $\{\frac{pt}{2} + \frac{(t-1)q}{2} + q + 2, \frac{pt}{2} + \frac{(t-1)q}{2} + q + 3, ..., pt + \frac{(t-1)q}{2} + \frac{pt}{2} + (t+1)q + 2\}$.

$$\begin{aligned} U_s &= a + (s-1)d \\ pt + \frac{(t-1)q}{2} + \frac{pt}{2} + &= \frac{pt}{2} + \frac{(t-1)q}{2} + q + 2 + \\ (t+1)q + 2 \quad & (s-1)1 \\ (p+q)t &= s - 1 \\ s &= (p+q)t + 1 \end{aligned}$$

Then, we determine the cardinality of edge weight set A_3:

$$\begin{aligned} |A_3| &= |\{\frac{pt}{2} + \frac{(t-1)q}{2} + q + 2, \frac{pt}{2} + \frac{(t-1)q}{2} + q + 3, \dots, \\ & pt + \frac{(t-1)q}{2} + \frac{pt}{2} + (t+1)q + 2\}| - \\ & |\{pt + (t+1)q + 2\}| \\ |A_3| &= (p+q)t + 1 - 1 \\ |A_3| &= (p+q)t \end{aligned}$$

It shows the edge weight induces a rainbow antimagic coloring of $(p+q)t$ colors. If there are $(p+q)t$ colors, a rainbow path can be built based on the edge weight function that has been discovered. Table 3 shows the rainbow path. So, for $p \geq 3, q \geq 3, t \geq 2$, we get $rac(Amal(B_{p,q}, v, t))$ is $rac(Amal(B_{p,q}, v, t)) \leq (p+q)t$.

Table 3. The rainbow path from x to y on $Amal(B_{p,q}, v, t)$ graph.

case	x	y	rainbow path
1	$x_{i,j}$	v	$x_{i,j}, v$
2	$x_{i,j}$	$x_{k,l}$	$x_{i,j}, v, x_{k,l}$
3	$x_{i,j}$	$y_{i,j}$	$x_{i,j}, v, y_{i,j}$
4	$y_{i,j}$	v	$y_{i,j}, v$
5	$y_{i,j}$	$y_{k,l}$	$y_{i,j}, v, y_{k,l}$

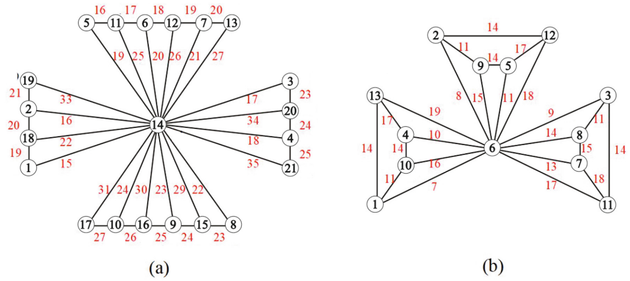

Fig. 3. The RAC of (a) $Amal(B_{4,6}, v, 2)$, (b) $Amal(W_4, v, 3)$.

From the discussion above, we have $(p+q)t \leq rac(Amal(B_{p,q}, v, t)) \leq (p+q)t$. So, for $p \geq 3, q \geq 3, t \geq 2$, the proof of the formula $rac(Amal(B_{p,q}, v, t)) = (p+q)t$ is complete. □

Figure 3 (a) illustrated $rac(Amal(B_{p,q}, v, t))$, with the vertex labels on the graph as black numbers and the edge weight on the graph as red numbers.

Theorem 8. Let $Amal(W_p, v, t)$ be a vertex amalgamation product of wheel graph with $p \geq 3, t \geq 2$, then $rac(Amal(W_p, v, t)) = pt$.

Proof. Let $Amal(W_p, v, t)$ be a vertex amalgamation product of wheel graph with vertex set $V(Amal(W_p, v, t)) = \{v\} \cup \{x_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p\}$ and $E(Amal(W_p, v, t)) = \{vx_{i,j}, 1 \leq i \leq t, 1 \leq j \leq p\} \cup \{x_{i,j}x_{i,j+1}, 1 \leq i \leq t, 1 \leq j \leq p-1\} \cup \{x_{i,1}x_{i,p}, 1 \leq i \leq t\}$. So, we get $|V(Amal(W_p, v, t))| = pt + 1$ and $|E(Amal(W_p, v, t))| = 2pt$.

First of all, we proof the lower bound of $rac(Amal(W_p, v, t))$ is $rac(Amal(W_p, v, t)) \geq mn$. We know that $rac(G) \geq max\{\triangle(G), rc(G)\}$, so that:

$$\begin{aligned} rac(Amal(W_p, v, t)) &\geq max\{\triangle(Amal(W_p, v, t)), \\ &\quad rc(Amal(W_p, v, t))\} \\ &= max\{3, pt\} = pt \end{aligned}$$

So, we get $rac(Amal(W_p, v, t)) \geq pt$ with $p \geq 3, t \geq 2$.

After that, we proof the upper bound of $rac(Amal(W_p, v, t))$ is $rac(Amal(W_p, v, t)) \leq pt$. To find the upper bound of $rac(Amal(W_p, v, t))$, we divide it into four cases.

Case 1. For $p \equiv 0(mod\ 2), t = 2, p \geq 3$.

We define the vertex function of the $Amal(W_p, v, t)$ graph with $l : V(Amal(W_p, v, t)) \to \{1, 2, ..., |V(Amal(W_p, v, t))|\}$ for $x_{i,j} \in V(Amal(W_p, v, t))$ as follows

$$l(v) = p + 1$$
$$l(x_{i,j}) = \begin{cases} j + i - 1, & \text{if } j \equiv 1(mod\ 2), 1 \leq j \leq p \\ p + j + i - 1, & \text{if } j \equiv 0(mod\ 2), 1 \leq j \leq p \end{cases}$$

We get the edge weights from a predetermined vertex function, which will be used to color the edges. The edge weights of the graph $Amal(W_p, v, t)$ are as follows

$$w(vx_{i,j}) = \begin{cases} p + j + i, & \text{if } 1 \leq i \leq 2, j \equiv 1(mod\ 2), \\ & 1 \leq j \leq p \\ 2p + i + j, & \text{if } 1 \leq i \leq 2, j \equiv 0(mod\ 2), \\ & 1 \leq j \leq p \end{cases}$$
$$w(x_{i,j}x_{i,j+1}) = 2j + p + 2i - 2, \text{ if } 1 \leq i \leq 2, 1 \leq j \leq p - 1$$
$$w(x_{i,1}x_{i,p}) = 2p + 2i - 1, \text{ if } 1 \leq i \leq 2$$

The next step is to see that the edge weights induce rainbow antimagic coloring on the graph $Amal(W_p, v, t)$. Let $A_{4,1}$ be the set of the edge weights of graph $Amal(W_p, v, t)$ for $p \equiv 0(mod\ 2), t = 2, p \geq 3$. Then, $A_{4,1} = \{p + 2, p + 1, ..., pt + p + 2\} - \{2p + 2\}$, thus the number of distinct colors of the edge weights is pn. To determine the upper bound, we should find the cardinality of the set $A_{4,1}$. In this step, we use an arithmetic sequence to know the cardinality of the set $\{p + 2, p + 1, ..., pt + p + 2\}$.

$$U_s = a + (s - 1)d$$
$$pt + p + 2 = p + 2 + (s - 1)1$$
$$pt = s - 1$$
$$s = pt + 1$$

Then, we determine the cardinality of the set $A_{4,1}$:

$$|A_{4,1}| = |\{p + 2, p + 1, ..., pt + p + 2\}| - |\{2p + 2\}|$$
$$|A_{4,1}| = pt + 1 - 1$$
$$|A_{4,1}| = pt$$

It shows the edge weight induced a rainbow antimagic coloring of pt colors. So, we get $rac(Amal(W_p, v, t)) \leq pt$ for $p \equiv 0(mod\ 2), t = 2, p \geq 3$.

Case 2. For $p \equiv 0(mod\ 2), p \geq 3, t \geq 3$.

We define the vertex function of the $Amal(W_p, v, t)$ graph with $l : V(Amal(W_p, v, t)) \to \{1, 2, ..., |V(Amal(W_p, v, t))|\}$ for $x_{i,j} \in V(Amal(W_p, v, t))$ as follows

$$l(v)=\frac{pt}{2},\text{ if } p\equiv 0(mod\ 2), p\geq 3, t\geq 3$$

$$l(x_{i,j})=\begin{cases}\frac{(j-1)t}{2}+i, & \text{if } 1\leq i\leq t-1, j\equiv 1(mod\ 2),\\ & 1\leq j\leq p\\ \frac{(j-1)t}{2}+i, & \text{if } i=t, j\equiv 1(mod\ 2), 1\leq j\leq\\ & p-3\\ \frac{pt}{2}+1, & \text{if } i=t, j=p-1\\ \frac{(p+j)t}{2}+2-i, & \text{if } 1\leq i\leq t, j\equiv 0(mod\ 2), 1\leq\\ & j\leq p\end{cases}$$

We get the edge weights from a predetermined vertex function, which will be used to color the edges. The edge weights of the graph $Amal(W_p, v, t)$ are as follows

$$w(vx_{i,j})=\begin{cases}\frac{(p+j-1)t}{2}+i, & \text{if } 1\leq i\leq t-1, j\equiv 1\\ & (mod\ 2), 1\leq j\leq p\\ \frac{(p+j+1)t}{2}, & \text{if } i=t, j\equiv 1(mod\ 2),\\ & 1\leq j\leq p-3\\ pt+1, & \text{if } i=t, j=p-1\\ \frac{(2p+j)t}{2}+2-i, & \text{if } 1\leq i\leq t, j\equiv 0(mod\ 2),\\ & 1\leq j\leq p\end{cases}$$

$$w(x_{i,j}x_{i,j+1})=\begin{cases}\frac{(p+2j)t}{2}+2, & \text{if } 1\leq i\leq t-1,\\ & j\equiv 1(mod\ 2),\\ & 1\leq j\leq p-1\\ \frac{(3p+2j+2)t}{2}+2-i, & \text{if } i=t, j\equiv 1(mod\ 2),\\ & 1\leq j\leq p-3\\ \frac{(2p+j)t}{2}+3-i, & \text{if } i=t, p-2\leq j\leq\\ & p-1\end{cases}$$

$$w(x_{i,1}x_{i,p}) = pt+2, \text{if } 1\leq i\leq t$$

The next step is to see that the edge weights induce rainbow antimagic coloring on the graph $Amal(W_p, v, t)$. Let $A_{4,2}$ be the set of the edge weights of graph $Amal(W_p, v, t)$. Then, $A_{4,2} = \{\frac{pt}{2}+1, \frac{pt}{2}+2, ..., \frac{3pt}{2}+1\} - \{pt\}$, thus the number of distinct colors of the edge weights is pt. To determine the upper bound, we should find the cardinality of the set $A_{4,2}$. In this step, we use an arithmetic sequence to know the cardinality of the set $\{\frac{pt}{2}+1, \frac{pt}{2}+2, ..., \frac{3pt}{2}+1\}$.

$$\begin{aligned} U_s &= a+(s-1)d\\ \frac{3pt}{2}+1 &= \frac{pt}{2}+1+(s-1)1\\ pt &= s-1\\ s &= pt+1 \end{aligned}$$

Then, we determine the cardinality of the set $A_{4,2}$:

$$\begin{aligned} |A_{4,2}| &= |\{\frac{pt}{2}+1, \frac{pt}{2}+2, ..., \frac{3pt}{2}+1\}| - |\{pt\}|\\ &= pt+1-1 = pt \end{aligned}$$

It shows the edge weight induces a rainbow antimagic coloring of pt colors. So, we get $rac(Amal(W_p, v, t)) \leq pt$ for $p \equiv 0(mod\ 2), p \geq 3, t \geq 3$.

Case 3. For $p \equiv 1(mod\ 2), p \geq 3, t = 2$.

We define the vertex function of the $Amal(W_p, v, t)$ graph with $l : V(Amal(W_p, v, t)) \rightarrow \{1, 2, ..., |V(Amal(W_p, v, t))|\}$ for $x_{i,j} \in V(Amal(W_p, v, t))$ as follows

$$l(v) = p + 1, \text{ if } p \equiv 1(mod\ 2), t = 2$$

$$l(x_{i,j}) = \begin{cases} j + i - 1, & \text{if } 1 \leq i \leq 2, j \equiv 1(mod\ 2), 1 \leq j \leq p - 1 \\ p + t, & \text{if } i = 1, j = p \\ p, & \text{if } i = 2, j = p \\ p + j + i, & \text{if } 1 \leq i \leq 2, j \equiv 0(mod\ 2), 1 \leq j \leq p - 1 \end{cases}$$

We get the edge weights from a predetermined vertex function, which will be used to color the edges. The edge weights of the graph $Amal(W_p, v, t)$ are as follows

$$w(vx_{i,j}) = \begin{cases} p + j + i, & \text{if } 1 \leq i \leq 2, j \equiv 1(mod\ 2), 1 \leq j \leq p - 1 \\ pt + 3, & \text{if } i = 1, j = p \\ pt + 1, & \text{if } i = 2, j = p \\ pt + j + i + 1, & \text{if } 1 \leq i \leq 2, j \equiv 0(mod\ 2), 1 \leq j \leq p - 1 \end{cases}$$

$$w(x_{i,j}x_{i,j+1}) = \begin{cases} p + 2j + 2i, & \text{if } 1 \leq i \leq 2, 1 \leq j \leq p - 2 \\ pt + j + i + 2, & \text{if } i = 1, j = p - 1 \\ pt + j + i, & \text{if } i = 2, j = p - 1 \end{cases}$$

$$w(x_{i,1}x_{i,p}) = \begin{cases} j + p + i + 1, & \text{if } i = 1 \\ j + p + i - 1, & \text{if } i = 2 \end{cases}$$

The next step is to see that the edge weights induce rainbow antimagic coloring on the graph $Amal(W_p, v, t)$. Let $A_{4,3}$ be the set of the edge weights of graph $Amal(W_p, v, t)$. Then, $A_{4,3} = \{p + 2, p + 3, ..., pt + p + 2\} - \{2p + 2\}$, thus the number of distinct colors of the edge weights is pt. To determine the upper bound, we should find the cardinality of the set $A_{4,3}$. In this step, we use an arithmetic sequence to know the cardinality of the set $\{p+2, p+3, ..., pt+p+2\}$.

$$U_s = a + (s - 1)d$$
$$pt + p + 2 = p + 2 + (s - 1)1$$
$$pt = s - 1$$
$$s = pt + 1$$

Then, we determine the cardinality of the set $A_{4,3}$:

$$\begin{aligned}|A_{4,3}| &= |\{p+2, p+3, ..., pt+p+2\}| - |\{2p+2\}|\\ &= pt+1-1 = pt\end{aligned}$$

It shows the edge weight induces a rainbow antimagic coloring of pt colors. So, we get $rac(Amal(W_p, v, t)) \leq pt$ for $p \equiv 1(mod\ 2), p \geq 3, t = 2$.

Case 4. For $p \equiv 1(mod\ 2), p \geq 3, t \geq 3$.

We define the vertex function of the $Amal(W_p, v, t)$ graph with $l : V(Amal(W_p, v, t)) \rightarrow \{1, 2, ..., |V(Amal(W_p, v, t))|\}$ for $x_{i,j} \in V(Amal(W_p, v, t))$ as follows

$$l(v) = \frac{(p-1)t}{2} + 2$$

$$l(x_{i,j}) = \begin{cases} \frac{(j-1)t}{2} + i, & \text{if } 1 \leq i \leq t, j \equiv 1(mod\ 2),\\ & 1 \leq j \leq p-2\\ pt - (\frac{j}{2} - 1)t - i + 2, & \text{if } 1 \leq i \leq t, j \equiv 0(mod\ 2),\\ & 1 \leq j \leq p-1\\ \frac{(p-1)t}{2} + 1, & \text{if } i = t, j = p\\ \frac{(p-1)t}{2} + t + 2 - i, & \text{if } 1 \leq i \leq t-1, j = p \end{cases}$$

We get the edge weights from a predetermined vertex function, which will be used to color the edges. The edge weights of the graph $Amal(W_p, v, t)$ are as follows

$$w(vx_{i,j}) = \begin{cases} \frac{(p+j-2)t}{2} + i + 2, & \text{if } 1 \leq i \leq t,\\ & j \equiv 1(mod\ 2),\\ & 1 \leq j \leq p-2\\ \frac{(p-j-1)t}{2} + (p+1)t - i + 4, & \text{if } 1 \leq i \leq t,\\ & j \equiv 0(mod\ 2),\\ & 1 \leq j \leq p-1\\ (p-1)t + 3, & \text{if } i = t, j = p\\ pt - i + 4, & \text{if } 1 \leq i \leq t,\\ & j = p \end{cases}$$

$$w(x_{i,j}x_{i,j+1}) = \begin{cases} pt + 2, & \text{if } 1 \leq i \leq t, j \equiv 1(mod\\ & 2), 1 \leq j \leq p-2\\ pt + t + 2, & \text{if } 1 \leq i \leq t, j \equiv 0(mod\\ & 2), 1 \leq j \leq p-3\\ pt + 3, & \text{if } i = t, j = p-1\\ pt + 2t + 4 - 2i, & \text{if } 1 \leq i \leq t-1,\\ & j = p-1\\ \frac{(p+1)t}{2} + 2, & \text{if } 1 \leq i \leq t-1, j = p \end{cases}$$

$$w(x_{i,1}x_{i,p}) = \frac{(p+1)t}{2} + 1, \text{if } i = t$$

The next step is to see that the edge weights induce rainbow antimagic coloring on the graph $Amal(W_p, v, t)$. Let $A_{4,4}$ be the set of the edge weights of

Table 4. The rainbow path from x to y on $Amal(W_p, v, t)$ graph.

case	x	y	rainbow path
1	$x_{i,j}$	v	$x_{i,j}, v$
2	$x_{i,j}$	$x_{k,l}$	$x_{i,j}, v, x_{k,l}$

graph $Amal(W_p, v, t)$. Then, $A_{4,4} = \{\frac{(p-1)t}{2} + 3, \frac{(p-1)t}{2} + 4, ..., \frac{(p-3)t}{2} + (p+1)t + 3\} - \{(p-1)t + 4\}$, thus the number of distinct colors of the edge weights is pt. To determine the upper bound, we should find the cardinality of the set $A_{4,4}$. In this step, we use an arithmetic sequence to know the cardinality of the set $\{\frac{(p-1)t}{2} + 3, \frac{(p-1)t}{2} + 4, ..., \frac{(p-3)t}{2} + (p+1)t + 3\}$.

$$\begin{aligned} U_s &= a + (s-1)d \\ \frac{(p-3)t}{2} + (p+1)t + 3 &= \frac{(p-1)t}{2} + 3 + (s-1)1 \\ pt &= s - 1 \\ s &= pt + 1 \end{aligned}$$

Then, we determine the cardinality of the set $A_{4,3}$:

$$\begin{aligned} |A_{4,4}| &= |\{\frac{(p-1)t}{2} + 3, \frac{(p-1)t}{2} + 4, ..., \frac{(p-3)t}{2} + (p+1)t + \\ &\quad 3\}| - |\{(p-1)t + 4\}| \\ |A_{4,4}| &= pt + 1 - 1 = pt \end{aligned}$$

It shows the edge weight induces a rainbow antimagic coloring of pt colors. So, we get the upper bound of $rac(Amal(W_p, v, t)) \leq pt$ for $p \equiv 1(mod\ 2), p \geq 3, t \geq 3$.

Based on Case 1, Case 2, Case 3, and Case 4, if there are as many colors as pt, a rainbow path could be constructed. Table 4 shows the rainbow path. So, for $p \geq 3, t \geq 2$, we get $rac(Amal(W_p, v, n)) \leq pt$.

From the discussion above, we have $pt \leq rac(Amal(W_p, v, t)) \leq pt$. So, for $p \geq 3, t \geq 2$, the proof of the formula $rac(Amal(W_p, v, t)) = pt$ is complete. □

The illustration of $rac(Amal(W_p, v, t))$ in Fig. 3 (b) with the vertex labels on the graph being black numbers and the edge weight on the graph being red numbers.

3 Conclusions

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Open Problem

Determine the lower and upper bound of the rainbow antimagic coloring of edge amalgamation, cartesian product, and others.

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