On the Resolving Efficient Domination Number of Path and Comb Product of Special Graph

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Abstract. Suppose G is a bounded, connected, and undirected graph. Let $V(G)$ and $E(G)$ be the vertex set and edge set, respectively. The subset D of $V(G)$ is a subset of the graph G that dominates it efficiently if every vertex in G is either in D or adjacent to a vertex in D. The subset M of $V(G)$ is the complete set of G if every vertex in G is differentiated differently based on its representation to each vertex in the ordered set M. Suppose $M = \{b_1, b_2, b_3, \ldots, b_k\}$ is a subset of $V(G)$. The representation of a vertex $n \in G$ on an ordered set M is $r(n|M) = \{(d(n, b_1), d(n, b_2), d(n, b_3) \ldots, d(n, b_k)\}$. A set M is called a set of solutions of G if $r(u|M) \neq r(n|M)\ \forall\ u, n \in G$. The subset Z of $V(G)$ is called the set of efficient dominating solutions of a graph G if is an efficient dominating set and $r(u|M) \neq r(n|M)\ \forall\ u, n \in G$. Let $\gamma_{re}(G)$ denote the minimum cardinality of an efficient domination number. We call it efficient dominance number resolution on graphs. The research methods used in this research are the axiomatic deductive method and the pattern detection method. The axiomatic deductive method is a research method that applies the principles of deductive proof (from general to specific) that apply to mathematical logic by using axioms, lemmas, and existing theorems to solve a problem on the topic under study. In this research, some theorems or definitions will be obtained from the results of further analysis of some previously existing theorems or definitions. Next is the pattern detection method, which is a research method to find efficient set completion patterns in the graph under study and the problem to be solved. In this article, the efficiency solution of the $\gamma_{re}G$ domination number of several will be obtained comb product graph is $BF_a \triangleright P_b$, $TS_a \triangleright P_b$, and $L_a \triangleright P_b$.

Keywords: Resolving Efficient Dominating Number · Path Graph · Comb Product of Special Graph

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1 Introduction

We use connected, undirected, and finite graphs denoted by G in this paper. W. Chartrand and Lesniak [4] explaining that a graph is a finite set of vertices and edges, defined as $G(V, E)$, where E is a non-empty set. Vertices and V is the set of edges. Each edge connects one node with another node, and each node can have many edges connecting it with other nodes. A graph $G = (V(G), E(G))$ consists of 2 finite sets, namely the first $V(G)$ is the set of points or vertices denoted by V, then the second $E(G)$ is the set of sides or edges denoted with E (Daniel Farida and Taneo Prida N.L.) [5]. In graph theory there are many kinds of studies, one of which is the set of differentiating dominance and graph metric dimensions.

FU Yumin [9] explains that suppose G is a directed graph with vertex set $V(G)$ and edge set $E(G)$. We say that is a finite graph if $V(G)$ and $E(G)$ are both finite. The notation $e(a, b)$ will be used to represent an edge directed from vertex a to vertex b and we say that a dominates b and b is dominated by a. Also, we refer to an as the start vertex of edge e and b as the end vertex of edge e. The subset D of V is the dominating set for G if every vertex that is not in D is the starting vertex of a leading edge node in D. The minimal dominant set is the dominant set from the undirected graph various riddle questions arise.

Bange et al. [1] explains that in a graph G with vertex set V and edge set E, the neighborhood of vertex n is the set of vertices neighboring n, $N(n) = x \in V : nx \in E$. $N[n] \cup n$ is the closed neighborhood of n. A vertex set D or a subset of V is a dominating set for G if every vertex of G is either a part of D or neighboring a component of D. That is, D dominates if $n \in V$ implies $N[N] \cap D \geq \phi$. Bange et al. [2] also explained that Dominating set D is an efficient dominating set if for each $n \in V$ we have $|N[n] \cap D| = 1$, or equivalently, if the distance between any two vertices in D is at least three. Such a set might represent a collection of transmitting stations, no two of which produce interference.

The problem of metric dimensions was first introduced by Harary and Melter [7], but it was also investigated by Slater [14]. Let $G = (V, E)$ be a simple connected graph $u, n \in V$, the distance between two vertices u and n denoted as $d(u, n)$ is defined as the length of the shortest path connecting them. Let $M = \{b1, b2, b3, b4, \ldots bk\}$ ordered vertex set of subsets V, and $n \in V$, the representation of n with respect to W denoted as $r(n|M)$ is $k - tuples(d(n|b1), d(n|b2), d(n|b3) \ldots d(n|bk))$. M is called a resolving set if each vertex in G has a different representation of M. The resolving set with minimum cardinality is referred to as the minimum resolving set, while the cardinality of this resolving set is referred to as the metric dimension of graph G, denoted as $Dim(G)$ [16].

Birgham et al. [3] combines the concept of dominance sets and metric dimensions expressed as resolving domination sets. The domination number of resolving set is the minimum cardinality and is represented by $\gamma_{re}G$. The dominating set of a graph G is a subset Y of $V(G)$ with a representation of its distance to every vertex in Y that defines every edge of G but not just the dominating set of G. Some types of resolving for dominating numbers are resolving for strong dominating numbers, resolving perfect dominating numbers, and resolving efficient dominating numbers, see [9,11,13].

Faisal et, al. [8] explained that in graph theory, there are known operations between two or more graphs, for example corona, cartesian, combined, normal, add, and comb operations. There are S Dhanalakshmi and N Parvathi [15] a definition of comb product namely comb is a graph obtained by joining a single pendant edge to each vertex of a path.

Recently, Kusumawardani, et al. [12] created a new understanding by combining the theory of complete sets and the theory of efficient dominating numbers which is expressed as efficient dominating number completion. Z is called an efficient dominating set of some graph G if it only satisfies the characteristics of the efficient set and dominating set, but also satisfies the characteristics of the complete set or $r(x|Z) = r(y|Z)x, yG$. $\gamma_{re}G$ shows the dominating efficiency the domination amount of the solution which is the minimum cardinality efficient set of graph G solutions, see [10].

All graphs studied in this paper are comb products. We use two graphs G_1 and G_2 which are finite, connected, and undirected graphs. $G_1 \triangleright G_2$ denotes a new graph obtained by obtaining a copy of graph G_1 and as many duplicates as possible of as many vertices in graph G_1 and inserting the ith copy of graph G_2 to the ith vertex of graph G_1 expressed as the comb product of graph G_1 and graph G_2 [6]. In this article, the efficiency solution of the $\gamma_{re}G$ domination number of several will be obtained comb product graph is $BF_a \triangleright P_b$, $TS_a \triangleright P_b$, and $L_a \triangleright P_b$.

2 Research Method

This research is categorized into exploratory research, namely research that aims to explore things that researchers want to know and research results can be used as a basis for further research. In solving the problem, this study uses pattern detection and axiomatic deductive methods. The pattern detection method is to look for patterns to construct an efficient differentiating domination set of solutions to obtain a metric dimension value with an efficient differentiating domination set of solutions in such a way as to obtain a minimum cardinality value with different point coordinates. While the axiomatic deductive method is a research method that uses the principles of deductive proof that apply in mathematical logic by using existing axioms or theorems to solve a problem. The research design for metric dimensions with sets of efficient differentiating domination solutions on special graphs. The description of this research design is as follows: Determine the graph to be used for metric dimension analysis with domination solution sets, then determine the cardinality of the vertices and edges of the graph used, then proceed determines the distinguishing set M, after that calculates the vertex representation of M to get a different result for each point then calculates the minimum cardinality of the distinguishing set to determine the metric dimension value with the connected distinguishing set after that determines the functions $dim(G)$ and $\gamma_{re}(G)$ and ends with determine the theorem of the research results on a special graph and prove it by calculating the formulation of the point coordinates. [17]

3 Result and Discussion

We found an efficient solution to solve the domination of line comb products. Among them namely $BF_a \triangleright P_b$, $TS_a \triangleright P_b$, and $L_a \triangleright P_b$. We use the theorem as follows.

Theorem 1. For every positive integer $a \geq 5$ and $b \geq 2$,

$$\gamma_{re}(BF_a \triangleright P_b) = \begin{cases} a\lceil \frac{b}{3} \rceil, & \text{if } b \equiv 0, 2 \pmod 3 \\ a\lceil \frac{b}{3} \rceil + 1, & \text{if } b \equiv 1 \pmod 3 \end{cases}$$

Proof. $BF_a \triangleright P_b$ is the comb product graph of a butterfly graph and a path graph which is a copy of BF_a graph where vertices x_i, y_i and A are nodes attached to BF_a graph. The set of vertices is $V(BF_a \triangleright P_b) = \{A, A_j, x_i, y_i, x_{ij}, y_{ij}; 1 \leq i \leq a, 1 \leq j \leq b-1\}$. And the edge set is $E(BF_a \triangleright P_b) = \{AA_1\} \cup \{Ax_i, Ay_i; 1 \leq i \leq a\} \cup \{x_i x_{i+1}, y_i y_{i+1}; 1 \leq i \leq a-1\} \cup \{x_i x_{i1}, y_i y_{i1}; 1 \leq i \leq a\} \cup \{A_j A_{j+1}, x_{ij} x_{ij+1}, y_{ij} y_{ij+1}; 1 \leq i \leq a, 1 \leq j \leq b-2\}$.
The cardinality of its vertex and edge are respectively $|V(BF_a \triangleright P_b)| = 2ab + b$ and $|E(BF_a \triangleright P_b)| = 2ab + 2a - 3$. We select subset D as follows

$$D = \begin{cases} \{A_j, x_{ij}, y_{ij}; A, 1 \leq i \leq a, \leq j \leq b-1; \\ j \equiv 1 \pmod 3\} \text{for } b \equiv 0, 2 \pmod 3)\} \\ \{x_{ij}, y_{ij}; 1 \leq i \leq a, 1 \leq j \leq b-1; j \equiv 2 \pmod 3)\} \\ \text{for } b \equiv 1 \pmod 3)\} \\ \{A_j; \frac{1 \leq j \leq b-1}{j}, \frac{A}{A}; j \equiv 3 \pmod 3)\} \\ \text{for } b \equiv 1 \pmod 3)\} \end{cases}$$

We Have $|D| = a\lceil \frac{b}{3} \rceil$, for $b \equiv 0, 2 (mod\ 3)$ and $|D| = a\lceil \frac{b}{3} \rceil + 1$, for $b \equiv 1 (mod\ 3)$. The steps that D complete the set of efficient domination with the minimum cardinality we show as follows.

First, we will prove that D satisfies the properties from the set of efficient domination. For any $A_j, x_{ij}, y_{ij} \in D, d(A_j, x_{ij}, y_{ij}) \geq 3$, so that means $|N(A_j, A, x_{ij}, x_i, y_{ij}, y_i \in V(BF_a \triangleright P_b) - D) \cap D| = 1$ and each vertex $A_j, A, x_{ij}, x_i, y_{ij}, y_i \in V(BF_a \triangleright P_b) - D$ is dominated by exactly one vertex in D. So we say a subset of D is an efficiently dominating set (Figs. 1 and 2).

Second, we need to prove that our chosen subset D satisfies the characteristics of the resolving set. Here is a way to find out the distance representing every vertex connected to elements in D that are dissimilar to each other, we can see the distance function of two vertices in the graph $BF_a \triangleright P_b$.

$$
\begin{aligned}
&d(AA_{jRD}) &&= j_{RD} \\
&d(A_jA_{jRD}) &&= |j_{RD} - j| \\
&d(Ax_{ijRD}), d(Ay_{ijRD}) &&= \begin{cases} j_{RD} + 1 \\ j + i_{RD} + j_{RD} \end{cases} \\
&d(x_iA_{jRD}), d(y_iA_{jRD}) &&= j_{RD} + 1 \\
&d(x_{ij}A_{jRD}), d(y_{ij}A_{jRD}) &&= i + j + j_{RD} \\
&d(x_ix_{ijRD}), d(y_iy_{ijRD}) &&= \begin{cases} j_{RD}, \\ \text{ if } i_{RD} = i \\ |i_{RD} - i| + j_{RD}, \\ \text{ if } i_{RD} \neq i \end{cases} \\
&d(x_{ij}x_{ijRD}), d(y_{ij}y_{ijRD}) &&= \begin{cases} |j_{RD} - j|, \\ \text{ if } i_{RD} = i \\ |i_{RD} - i| + j + j_{RD}, \\ \text{ if } i_{RD} \neq i \end{cases} \\
&d(x_iy_{ijRD}), d(y_ix_{ijRD}) &&= i + j_{RD} + 1 \\
&d(x_{ij}y_{ijRD}), d(y_{ij}x_{ijRD}) &&= j + j_{RD} + 2
\end{aligned}
$$

We know that every representation of the elements in D must be different from each other considering the subset of D that we have and that we have selected. So, D shows the characteristics of the resolving set.

Third, we need to prove that D is an efficient dominating set with minimum cardinality. If $|D_0| < a\lceil \frac{b}{3} \rceil$, so we have $|D_0| = a\lceil \frac{b}{3} \rceil\ - 1$ for $b \equiv 0, 2 \pmod 3$ and $|D_0| < a\lceil \frac{b}{3} \rceil\ + 1$, so we have $|D_0| = a\lceil \frac{b}{3} \rceil$ for $b \equiv 1 \pmod 3$. Below are possible circumstances that will occur

1) For $b \equiv 0 (mod\ 3)$
   - If $\{x_{i,1}\} \notin D_0 \rightarrow \exists\ x_i, x_{i,1}, x_{i,2}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If $\{x_{i,j} : j \equiv 1 (mod\ 3); 5 \leq j \leq b-2\} \notin D_0 \rightarrow \exists\ x_{i,j+1}, x_{i,j}, x_{i,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
2) For $b \equiv 1 (mod\ 3)$
   - If vertex $x_1 \notin D_0 \rightarrow \exists\ x_1, x_i$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If vertex $x_{1,b-1} \notin D_0 \rightarrow \exists\ x_{1,b-1}, x_{1,b-2}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If any vertex $\{x_{i,j} : 2 \leq i \leq a; 2 \leq i \leq b-2; j \equiv 2 (mod\ 3)\} \notin D_0 \rightarrow \exists\ x_{i,j+1}, x_{i,j}, x_{i,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.

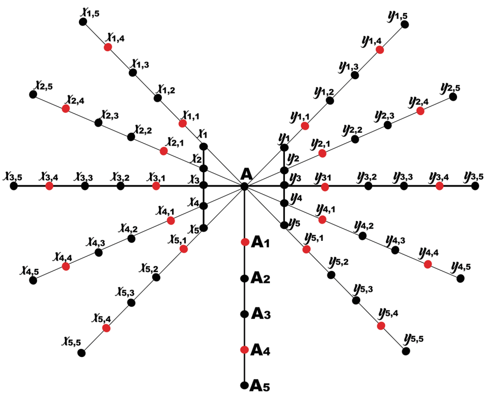

Fig. 1. The resolving efficient dominating numbers of $BF_5 \triangleright P_6$

- If any vertex $\{x_{1,j} : 3 \le j \le b-1; j \equiv 0(mod\ 3)\} \notin D_0 \rightarrow \exists\ x_{1,j+1}, x_{1,j}, x_{1,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.

3) For $b \equiv 2(mod\ 3)$
- If vertex $x_{i,1} \notin D_0 \rightarrow \exists\ x_i, x_{i,1}, x_{i,2}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
- If vertex $x_{i,b-1} \notin D_0 \rightarrow \exists\ x_{i,b-1}, x_{i,b-2}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
- If any vertex $\{x_{i,j}; 1 < j < b-1; j \equiv 1(mod\ 3)\} \notin D_0 \rightarrow \exists\ x_{i,j+1}, x_{i,j}, x_{i,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.

The subset D_0 does not satisfy the efficient dominating set, it contradicts the efficient dominating completion set. Therefore, $|D| = a\lceil \frac{b}{3} \rceil$ must be the minimum cardinality of the resolved efficient domination set of $BF_a \triangleright P_b$ when $b \equiv 0, 2(mod\ 3)$ and $|D| = a\lceil \frac{b}{3} \rceil + 1$ must be the minimum cardinality of the resolved efficient domination set of $BF_a \triangleright P_b$ when $b \equiv 1(mod\ 3)$. It completed the proof. □

Theorem 2. For every positive integer $a \ge 3$ *and* $b \ge 2$,

$$\gamma_{re}(TS_a \triangleright P_b) = \begin{cases} a\lceil \frac{b}{3} \rceil, & \text{if } b \equiv 2(mod 3) \\ a\lfloor \frac{b}{3} \rfloor + 1, & \text{if } b \equiv 1(mod 3) \end{cases}$$

Proof. $TS_a \triangleright P_b$ is the comb product graph of a triangular snake graph and a path graph which is a copy of TS_a graph where vertices x_i is node attached to TS_a graph. The set of vertices is $V(TS_a \triangleright P_b) = \{x_i, x_{ij}; 1 \le i \le a, 1 \le j \le b-1\}$. And the edge set is $E(TS_a \triangleright P_b) = \{x_i x_i + \lfloor \frac{a}{2} \rfloor; 1 \le i \le \lfloor \frac{a}{2} \rfloor\} \cup \{x_i x_i + \lfloor \frac{a}{2} \rfloor; 1 \le i \le \lfloor \frac{a}{2} \rfloor\} \cup \{x_i x_{i+1}; \lfloor \frac{a}{2} \rfloor \le i \le a-1\} \cup \{x_i x_{ij}; 1 \le i \le a; j = 1\} \cup \{x_i x_{ij+1}; 1 \le i \le a; 1 \le j \le b-2\}$

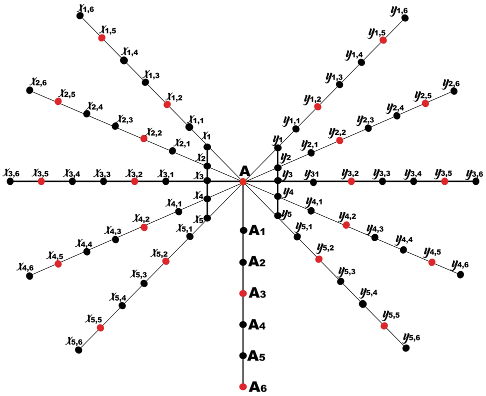

Fig. 2. The resolving efficient dominating numbers of $BF_5 \triangleright P_7$

The cardinality of its vertex and edge are respectively $|V(TS_a \triangleright P_b)| = ab$ and $|E(TS_a \triangleright P_b)| = 3\lfloor \frac{a}{3} \rfloor + ab - a$. We select subset D as follows

$$D = \begin{cases} \{x_{ij}; 1 \leq i \leq a, 1 \leq j \leq b-1; j \equiv 1(mod\ 3)\}, \\ \text{ for } b \equiv 2(mod\ 3) \\ \{x_i, x_{ij}; 1 \leq i \leq a, 1 \leq j \leq b-1; j \equiv 0(mod\ 3)\} \\ \text{ and } j \equiv 2(mod\ 3), \text{ for } b \equiv 1(mod\ 3) \end{cases}$$

We Have $|D| = a\lceil \frac{b}{3} \rceil$, for $b \equiv 2 \pmod 3$ and $|D| = a\lfloor \frac{b}{3} \rfloor + 1$, for $b \equiv 1 \pmod 3$. The steps that D complete the set of efficient domination with the minimum cardinality we show as follows.

First, we will prove that D satisfies the properties from the set of efficient domination. For any $x_{ij} \in D, d(x_{ij}) \geq 3$, so that means $|N(x_{ij}, x_i \in V(TS_a \triangleright P_b) - D) \cap D| = 1$ and each vertex $x_{ij}, x_i \in V(TS_a \triangleright P_b) - D$ is dominated by exactly one vertex in D. So we say a subset of D is an efficiently dominating set (Fig. 3).

Second, we need to prove that our chosen subset D satisfies the characteristics of the resolving set. Here is a way to find out the distance representing every vertex connected to elements in D that are dissimilar to each other, we can see the distance function of two vertices in the graph $TS_a \triangleright P_b$ as follows.

$$d(x_i x_{ijRD}) = \begin{cases} j_{RD}, & \text{if } i_{RD} = i \\ j_{RD} + 1, & \text{if } i - 1 \leq i_{RD} \leq i + 1, \\ & \text{ for } i = \text{ positive integer}, \\ & \text{if } i + 1 \leq i_{RD} \leq i + 2, \\ & \text{and } i - 2 \leq i_{RD} \leq i - 1, \\ & \text{for } i = \text{negative integer} \end{cases}$$

$$d(x_i x_{ijRD}) = \begin{cases} j_{RD}+2, & \text{if } i+3 \le i_{RD} \le i+4, \\ & \text{and } i-4 \le i_{RD} \le i-3 \\ & \text{for } i = \text{negative integer}, \\ & \text{if } i+2 \le i_{RD} \le i+3, \\ & \text{and } i-3 \le i_{RD} \le i-2, \\ & \text{for } i = \text{positive integer} \\ j_{RD}+3, & \text{if } i+5 \le i_{RD} \le i+6, \\ & \text{and } i-6 \le i_{RD} \le i-5 \\ & \text{for } i = \text{negative integer}, \\ & \text{if } i+4 \le i_{RD} \le i+5, \\ & \text{and } i-5 \le i_{RD} \le i-4, \\ & \text{for } i = \text{positive integer}, \\ & \text{and apply multiples} \end{cases}$$

$$d(x_{ij} x_{ijRD}) = \begin{cases} |j_{RD}-j|, & \text{if } i_{RD} = i \\ j_{RD}+j+1, & \text{if } i-1 \le i_{RD} \le i+1, \\ & \text{for } i = \text{positive integer}, \\ & \text{if } i+1 \le i_{RD} \le i+2 \\ & \text{and } i-2 \le i_{RD} \le i-1, \\ & \text{for } i = \text{negative integer} \\ j_{RD}+j+2, & \text{if } i+3 \le i_{RD} \le i+4 \\ & \text{and } i-4 \le i_{RD} \le i-3 \\ & \text{for } i = \text{negative integer}, \\ & \text{if } i+2 \le i_{RD} \le i+3 \\ & \text{and } i-3 \le i_{RD} \le i-2, \\ & \text{for } i = \text{positive integer} \\ j_{RD}+j+3, & \text{if } i+5 \le i_{RD} \le i+6 \\ & \text{and } i-6 \le i_{RD} \le i-5 \\ & \text{for } i = \text{negative integer}, \\ & \text{if } i+4 \le i_{RD} \le i+5 \\ & \text{and } i-5 \le i_{RD} \le i-4, \\ & \text{for } i = \text{positive integer}, \\ & \text{and apply multiples} \end{cases}$$

We know that every representation of the elements in D must be different from each other considering the subset of D that we have and that we have selected. So, D shows the characteristics of the resolving set.

Third, we need to prove that D is an efficient dominating set with minimum cardinality. If $|D_0| < a\lceil \frac{b}{3} \rceil$, so we have $|D_0| = a\lceil \frac{b}{3} \rceil \; -1$ for $b \equiv 2 \pmod 3$ and

$|D_0| < a\lceil\frac{b}{3}\rceil + 1$, so we have $|D_0| = a\lceil\frac{b}{3}\rceil$ for $b \equiv 1 \pmod 3$. Below are possible circumstances that will occur

1) For $b \equiv 2 (mod\ 3)$
   - If $\{x_{i,1}\} \notin D_0 \rightarrow \exists\ x_i, x_{i,1}, x_{i,2}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If any vertex $\{x_{i,j} : j \equiv 1 (mod\ 3); 3 \leq j \leq b-2\} \notin D_0 \rightarrow \exists\ x_{i,j+1}, x_{i,j}, x_{i,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
2) For $b \equiv 1 (mod\ 3)$
   - If vertex $x_1 \notin D_0 \rightarrow \exists\ x_1, x_i$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - Vertex $x_{1,b-1} \notin D_0 \rightarrow \exists\ x_{1,b-1}, x_{1,b-2}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If any vertex $\{x_{i,j} : 3 \leq i \leq a; 2 \leq i \leq b-2; j \equiv 2 (mod\ 3)\} \notin D_0 \rightarrow \exists\ x_{i,j+1}, x_{i,j}, x_{i,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If any vertex $\{x_{1,j} : 3 \leq j \leq b-1; j \equiv 0 (mod\ 3)\} \notin D_0 \rightarrow \exists\ x_{1,j+1}, x_{1,j}, x_{1,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If any vertex $\{x_{1,j} : 3 \leq j \leq b-1; j \equiv 2 (mod\ 3)\} \notin D_0 \rightarrow \exists\ x_{1,j+1}, x_{1,j}, x_{1,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.

The subset D_0 does not satisfy the efficient dominating set, it contradicts the efficient dominating completion set. Therefore, $|D| = a\lceil\frac{b}{3}\rceil$ must be the minimum cardinality of the resolved efficient domination set of $TS_a \triangleright P_b$ when $b \equiv 2 (mod\ 3)$ and $|D| = a\lceil\frac{b}{3}\rceil + 1$ must be the minimum cardinality of the resolved efficient domination set of $TS_a \triangleright P_b$ when $b \equiv 1 (mod\ 3)$. It completed the proof. □

Theorem 3. For every positive integer $a \geq 2$ and $b \geq 2$,

$$\gamma_{re}(L_a \triangleright P_b) = \begin{cases} a\lceil\frac{b}{3}\rceil, \text{ if } a \equiv 0 (mod\ 4), and \text{ if } b \equiv 2 (mod\ 3), \\ a\lfloor\frac{b}{3}\rfloor + 1, \text{ if } a \equiv 0, 2 (mod\ 4), and \text{ if } b \equiv 1, 3 (mod\ 3) \end{cases}$$

Proof. $L_a \triangleright P_b$ is the comb product graph of a ladder graph and a path graph which is a copy of L_a graph where vertices x_i and y_i are nodes attached to L_a graph. The set of vertices is $V(L_a \triangleright P_b) = \{x_i, y_i, x_{ij}, y_{ij}; 1 \leq i \leq a, 1 \leq j \leq b-1\}$. And the edge set is $E(L_a \triangleright P_b) = \{x_i y_i; 1 \leq i \leq a\} \cup \{x_i x_{i+1}, y_i y_{i+1}; 1 \leq i \leq a-1\} \cup \{x_i x_{i1}, y_i y_{i1}; 1 \leq i \leq a\} \cup \{x_{ij} x_{ij+1}, y_{ij} y_{ij+1}; 1 \leq i \leq a; 1 \leq j \leq b-2\}$ The cardinality of its vertex and edge are respectively $|V(L_a \triangleright P_b)| = 2ab$ and $|E(L_a \triangleright P_b)| = 2ab + a - 2$. We select subset D as follows

$$D = \begin{cases} \{x_{ij}, y_{ij}; 1 \leq i \leq a, \leq j \leq b-1; j \equiv 1 \pmod 3\} \\ \text{for } b \equiv 2 \pmod 3 \\ \{x_{ij}, y_{ij}; 1 \leq i \leq a, 1 \leq j \leq b-1; j \equiv 1 \pmod 2\} \\ \text{for } b \equiv 1, 3 \pmod 3 \\ \{x_i, y_i; \lfloor\frac{1 \leq i \leq a}{3}\rfloor; i \equiv 2 \pmod 4\} \\ \text{for } b \equiv 1, 3 \pmod 3 \end{cases}$$

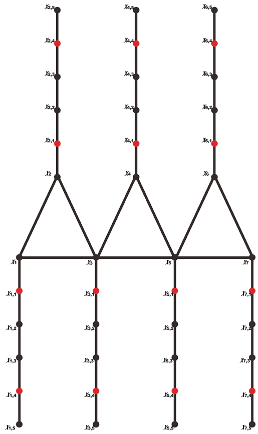

Fig. 3. The resolving efficient dominating numbers of $TS_7 \triangleright P_6$

We Have $|D| = a\lceil \frac{b}{3} \rceil$, for $b \equiv 2 \pmod 3$ and $|D| = a\lfloor \frac{b}{3} \rfloor + 1$, for $b \equiv 1, 3 \pmod 3$.
The steps that D complete the set of efficient domination with the minimum cardinality we show as follows.

First, we will prove that D satisfies the properties from the set of efficient domination. For any $x_{ij}, y_{ij} \in D, d(x_{ij}, y_{ij}) \geq 4$, so that means $|N(x_{ij}, x_i, y_{ij}, y_i \in V(L_a \triangleright P_b) - D) \cap D| = 1$ and each vertex $x_{ij}, x_i, y_{ij}, y_i \in V(L_a \triangleright P_b) - D$ is dominated by exactly one vertex in D. So we say a subset of D is an efficiently dominating set (Fig. 4).

Second, we need to prove that our chosen subset D satisfies the characteristics of the resolving set. Here is a way to find out the distance representing every vertex connected to elements in D that are dissimilar to each other, we can see the distance function of two vertices in the graph $L_a \triangleright P_b$ as follows.

$$d(x_i x_{ijRD}), d(y_i y_{ijRD}) = \begin{cases} j_{RD}, & \text{if } i_{RD} = i \\ i_{RD} + j_{RD} - i, & \text{if } i_{RD} > i \\ |i_{RD} - j_{RD} + i|, & \text{if } i_{RD} < i \end{cases}$$

$$d(x_{ij} x_{ijRD}), d(y_{ij} y_{ijRD}) = \begin{cases} |j_{RD} - j|, & \text{if } i_{RD} = i \\ i_{RD} + j_{RD}, & \text{if } i_{RD} > i \\ |i_{RD} - j_{RD} + i + j|, & \text{if } i_{RD} < i \end{cases}$$

$$d(x_i y_{ijRD}), d(y_i x_{ijRD}) = \begin{cases} j_{RD} + 1, & \text{if } i_{RD} = i \\ i_{RD} + j_{RD} - i + 1, & \text{if } i_{RD} > i \\ |i_{RD} - j_{RD} - i - 1|, & \text{if } i_{RD} < i \end{cases}$$

$$d(x_{ij} y_{ijRD}), d(y_{ij} x_{ijRD}) = \begin{cases} j_{RD} + j + 1, & \text{if } i_{RD} = i \\ i_{RD} + j_{RD} - i + j + 1, & \text{if } i_{RD} > i \\ |i_{RD} - j_{RD} - i - j - 1|, & \text{if } i_{RD} < i \end{cases}$$

We know that every representation of the elements in D must be different from each other considering the subset of D that we have and that we have selected. So, D shows the characteristics of the resolving set.

Third, we need to prove that D is an efficient dominating set with minimum cardinality. If $|D_0| < a\lceil \frac{b}{3} \rceil$, so we have $|D_0| = a\lceil \frac{b}{3} \rceil - 1$ for $a \equiv 0 \pmod 4$ $b \equiv 2 \pmod 3$ and $|D_0| < a\lceil \frac{b}{3} \rceil + 1$, so we have $|D_0| = a\lceil \frac{b}{3} \rceil$ for $a \equiv 0, 2 \pmod 4$ $b \equiv 1, 3 \pmod 3$. Below are possible circumstances that will occur

1) For $b \equiv 2(mod\ 3)$
   - If $\{x_{i,1}\} \notin D_0 \rightarrow \exists\ x_i, x_{i,1}, x_{i,2}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If any vertex $\{x_{i,j} : j \equiv 1(mod\ 3); 4 \leq j \leq b-2\} \notin D_0 \rightarrow \exists\ x_{i,j+1}, x_{i,j}, x_{i,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
2) For $b \equiv 1(mod\ 3)$
   - If vertex $x_1 \notin D_0 \rightarrow \exists\ x_1, x_i$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - Vertex $x_{1,b-1} \notin D_0 \rightarrow \exists\ x_{1,b-1}, x_{1,b-2}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If any vertex $\{x_{i,j} : 2 \leq i \leq a; 2 \leq i \leq b-2; j \equiv 2(mod\ 3)\} \notin D_0 \rightarrow \exists\ x_{i,j+1}, x_{i,j}, x_{i,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If any vertex $\{x_{1,j} : 3 \leq j \leq b-1; j \equiv 1(mod\ 2)\} \notin D_0 \rightarrow \exists\ x_{1,j+1}, x_{1,j}, x_{1,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
3) For $b \equiv 3(mod\ 3)$
   - If vertex $x_{i,1} \notin D_0 \rightarrow \exists\ x_i, x_{i,1}, x_{i,2}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If vertex $x_{i,b-1} \notin D_0 \rightarrow \exists\ x_{i,b-1}, x_{i,b-2}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set.
   - If any vertex $\{x_{i,j}; 1 < j < b-1; i \equiv 2(mod\ 4)\} \notin D_0 \rightarrow \exists\ x_{i,j+1}, x_{i,j}, x_{i,j-1}$ which is not controlled by D_0. Because of that, D_0 can't be called the efficiently dominating set (Fig. 4).

The subset D_0 does not satisfy the efficient dominating set, it contradicts the efficient dominating completion set. Therefore, $|D| = a\lceil \frac{b}{3} \rceil$ must be the minimum cardinality of the resolved efficient domination set of $L_a \triangleright P_b$ when $b \equiv 2(mod\ 3)$ and $|D| = a\lceil \frac{b}{3} \rceil + 1$ must be the minimum cardinality of the resolved efficient domination set of $L_a \triangleright P_n$ when $b \equiv 1(mod\ 3)$. It completed the proof. □

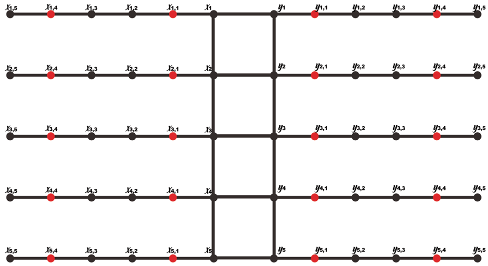

Fig. 4. The resolving efficient dominating numbers of $L_5 \triangleright P_6$

4 Conclusion

In this paper, we have analyze the results of proving the efficient differentiating dominance number values for several special graphs that are operated on comb with path graphs including $BF_a \triangleright P_b, TS_a \triangleright P_b$, and $L_a \triangleright P_b$. The weakness in this paper is that the graphs used are limited to comb product graph operations, in some cases the graphs we use cannot be proven by using graph operations such as corona, amalgamation, etc., and there is no detailed lower bound proof in this paper. This topic is a relatively new topic in research on combining metric dimensions and the study of efficient dominance, so many issues related to this topic have yet to be discovered and it is evident that there are still. We propose several issues or problems that must be found, namely:

Open Problem 1

There is any graph for example with G which is a connected and bounded graph, prove the exact value of efficient domination distinguish elements in G.

Open Problem 2

Show the dominant numbers efficiently from special graphs and other operations, such as corona, cartesian, union, normal, etc.

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