On Local (*a, d*)-Antimagic Coloring of Some Specific Classes of Graphs

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Abstract. For any graph $G = (V, E)$, the vertex set V and the edge set E, and let w be the edge weight of graph G, with $|V(G)| = p$ and $|E(G)| = q$. A labeling of a graph G is a bijection f from $V(G)$ to the set $\{1, 2, .., p\}$. The bijection f is called an edge antimagic labeling of graph if for any two vertex u and v where $w(uv) = f(u) + f(v), uv \in E(G)$, are distinct. Any local edge antimagic labeling induces a proper edge coloring of G where the edge uv is assigned the color $w(uv)$. The local edge antimagic coloring of graph is said to be a local (a, d)-edge antimagic coloring of G if the set of their edge colors form an arithmetic sequence with initial value a and different d. The local (a, d)-edge antimagic chromatic number $X_{le(a,d)}(G)$ is the minimum number of colors needed to color G such that a graph G admits the local (a, d)-edge antimagic coloring. Furthermore, we will obtain the lower and upper bound of $X_{le(a,d)}(G)$. The results of this research are the exact value of the local (a, d)-edge antimagic chromatic number of ladder graph, cycle graph, octopus graph, tadpole graph, tringular book graph, and helm graph.

Keywords: local (a, d)-antimagic coloring · local edge antimagic coloring · specific graph

1 Introduction

Graph $G(V, E)$ is a graph where $V(G)$ is the non-empty set of vertices and $E(G)$ is the set of edges that connects a pair of vertices, the definition of graph can be see in [7]. In this paper, graph G represents a simple, connected, and finite graph with no loops or multiple edges. Let $|V(G)| = m$ is the number of vertices of G and $|E(G)| = n$ is the number of edges of G.

Graph labeling is a mapping that assigns natural numbers to some set of graph elements of graph G. Labeling is called vertex labeling if the labeling domain is a vertex and labeling is called edge labeling if the labeling domain is

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an edge. Furthermore, if the domain is $V(G) \cup E(G)$, the labeling is called total labeling. Labeling is called antimagic if all the side weights have different values. Antimagic labeling defined as $G(V, E)$ is a mapping $f : V \to \{1, 2, 3, ..., |V(G)|\}$ is a bijection function if for all $u, v \in V(G)$ and the edge weight is $w(uv) = f(u) + f(v)$ [8].

Local antimagic of a graph was first introduced in 2017 by Arumugam et al. A bijection $f : E \to \{1, 2, ..., m\}$ is called a local antimagic labeling if for all $uv \in E$ we have $w(u) \neq w(v)$ where $w(u) = \sum_{e \in E(u)} f(e)$ [5]. The local antimagic labeling have been studied by [5,6,9]. In the same year, Agustin et al. developed a study of local edge antimagic. A bijection function $f : V(G) \to \{1, 2, 3, ..., |V(G)|\}$, is called local edge antimagic labeling if any two incident edges at the same vertices e_1 and $e_2, w(e1) \neq w(e_2)$, where for $e = uv \in G, w(e) = f(u) + f(v)$. So, local edge antimagic labeling is called local edge antimagic coloring if any edge is assigned the color $w(e)$. The local edge antimagic chromatic number, denoted by $\chi_{lea}(G)$ is the minimum number of colors that are taken over by all staining induced by the local edge antimagic labeling of graph G. The concept of local edge antimagic coloring of graphs can be seen in [1,2,4].

Local edge antimagic coloring has developed into Local (a, d)-antimagic coloring. The concept of local (a, d)-antimagic coloring is the same as local edge antimagic coloring. A bijection $f : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ is called an edge antimagic labeling of graph if the element of the edge weight set $w(uv) = f(u)+f(v)$, where $uv \in E(G)$, are distinct. The edge antimagic labeling induces a local edge antimagic coloring of G if each edge of G is colored with a weight of $w(e)$. The antimagic coloring of a graph is said to be a local (a, d)-antimagic coloring of G if the set of edge colors forms an arithmetic sequence with initial values of a and different d. Furthermore, the local (a, d)-antimagic chromatic number $\chi_{le(a,d)}(G)$ is the minimum number of colors needed to color G such that a graph G admits the local (a, d)-antimagic coloring [3]. The local (a, d)-antimagic coloring have been studied by [3,10].

Observation 1.1 [1]. For any graph G, $\chi_{lea}(G) \geq \Delta(G)$, where $\Delta(G)$ is maximum degrees of G

Observation 1.2 [1]. For any graph G, $\chi_{lea}(G) \geq \chi(G)$, where $\chi(G)$ is a chromatic number of vertex coloring of G.

Observation 1.3 [3]. For any graph G, $\chi_{le(a,d)}(G) \geq \chi_{lea}(G) \geq \chi(G)$

Observation 1.4 [11]. For any graph G, $\chi_{le}(a, d)(G) \geq \chi_{lea}(G) \geq \Delta(G)$

2 Main Result

In this paper, we will study about local (a, d)-antimagic coloring of a some specific classes of graphs and determine the chromatic number of local (a, d)-antimagic coloring of some specific classes of graphs include ladder graph L_n,

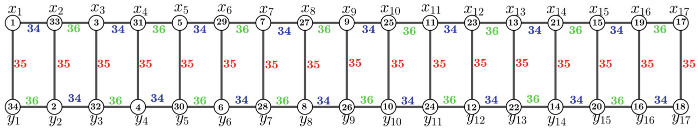

Fig. 1. Local $(2n, 1)$-antimagic coloring of L_{17}

cycle graph C_n, octopus graph O_n, helm graph H_n, tringular book graph Tb_n, and tadpole graph $T_{m,n}$. We also analyse the lower bound and upper bound of the local (a, d)-antimagic coloring of the graphs.

Theorem 2.1. Let L_n be ladder graph with $n \geq 2$, $\chi_{le(2n,1)}(L_n) = 3$.

Proof. The graph L_n has the vertex set $V(L_n) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$ and edge set $E(L_n) = \{x_i x_{(i+1)}, y_i y_{(i+1)} : 1 \leq i \leq n-1\} \cup \{x_i y_i : 1 \leq i \leq n\}$. The vertices cardinality is $|V(L_n)| = 2n$ and the edges cardinality is $|E(L_n)| = 3n - 2$. The local (a, d)-antimagic coloring chromatic number of L_n is $\chi_{le(2n,1)}(L_n) = 3$. First, we will prove that $\chi_{le(a,d)}(L_n) \geq 3$. Based on observation 1.4 we have $\chi_{le(a,d)}(L_n) \geq \chi_{lea}(L_n)$, in Agustin et al. [4] $\chi_{lea}(L_n) = 3$. It concludes that $\chi_{le(a,d)}(L_n) \geq 3$.

To show $\chi_{le(a,d)}(L_n) \leq 3$, by define a bijection $f : V(L_n) \rightarrow \{1, 2, 3, ..., |V(L_n)|\}$ by

$$f(x_i) = \begin{cases} i, & i = odd; 1 \leq i \leq n, \\ 2n - i + 1, & i = even; 1 \leq i \leq n, \end{cases}$$

$$f(y_i) = \begin{cases} 2n - i + 1, & i = odd; 1 \leq i \leq n, \\ i, & i = even; 1 \leq i \leq n, \end{cases}$$

From the labeling function, we can see that f is a local (a, d)-antimagic coloring of L_n and the weights of edge are as follows:

$$w(e) = \begin{cases} 2n, & x_i x_{(i+1)}; i = odd; 1 \leq i \leq n-1, \\ & y_i y_{(i+1)}; i = even; 1 \leq i \leq n-1, \\ 2n+2, & x_i x_{(i+1)}; i = even; 1 \leq i \leq n-1, \\ & y_i y_{(i+1)}; i = odd; 1 \leq i \leq n-1, \\ 2n+1, & x_i y_i; 1 \leq i \leq n, \end{cases}$$

Based on the weights, we have set of edge weights obtained is $W = \{2n, 2n + 1, 2n+2\}$ with the smallest edge is $a = 2n$ and $d = 1$, we have $\chi_{le(2n,1)}(L_n) \leq 3$. It conclude that local (a, d)-antimagic coloring chromatic number of L_n with $n \geq 2$ is $\chi_{le(2n,1)}(L_n) = 3$.
It concludes the proof.

Figure 1 shows an illustration of local (a, d)-antimagic coloring of a ladder graph.

Theorem 2.2. Let C_n be cycle graph with $n \geq 3$, $\chi_{le(n,1)}(C_n) = 3$.

Proof. The graph C_n has the vertex set $V(C_n) = \{x_i : 1 \leq i \leq n\}$ and edge set $E(C_n) = \{x_i x_{(i+1)} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$. The vertices cardinality is $|V(C_n)| = n$ and the edges cardinality is $|E(C_n)| = n$. The local (a, d)-antimagic coloring chromatic number of C_n is $\chi_{le(n,1)}(C_n) = 3$. First, we will prove that $\chi_{le(a,d)}(C_n) \geq 3$. Based on observation 1.4 we have $\chi_{le(a,d)}(C_n) \geq \chi_{lea}(C_n)$, in Agustin et al. [4] $\chi_{lea}(C_n) = 3$. It concludes that $\chi_{le(a,d)}(C_n) \geq 3$.

To show $\chi_{le(a,d)}(C_n) \leq 3$, by define a bijection $f : V(C_n) \rightarrow \{1, 2, 3, ..., |V(C_n)|\}$. In this mapping there are two different cases that occurred in local (a, d)-antimagic coloring of cycle graph with $n \geq 3$, there are cycle graph when n is even and n is odd. These cases are as follows:

Case 1. For n is even.

To show $\chi_{le(a,d)}(C_n) \leq 3$, by define a bijection $f : V(C_n) \rightarrow \{1, 2, 3, ..., |V(C_n)|\}$ for n is even by

$$f(x_i) = \begin{cases} i, & i = odd; 1 \leq i \leq \frac{n}{2}, \\ & i = even; \frac{n}{2} < i \leq n, \\ n - i + 1, & i = even; 1 \leq i \leq \frac{n}{2}, \\ & i = odd; \frac{n}{2} < i \leq n, \end{cases}$$

From the labeling function, we can see that f is a local (a, d)-antimagic coloring of C_n when n is even and the weights of edge are as follows:

$$f(x_i x_{i+1}) = \begin{cases} n, & i = odd; 1 \leq i < \frac{n}{2}, \\ & i = even; \frac{n}{2} < i \leq n-1, \\ n+2, & i = even; 1 \leq i < \frac{n}{2}, \\ & i = odd; \frac{n}{2} < i \leq n-1, \\ n+1, & i = \frac{n}{2} \\ & x_1 x_n \end{cases}$$

Based on the weights, we have set of edge weights obtained is $W = \{n, n+1, n+2\}$ with the smallest edge is $a = n$ and $d = 1$, we have $\chi_{le(n,1)}(C_n) \leq 3$. It conclude that local (a, d)-antimagic coloring chromatic number of C_n with $n \geq 3$ when n is even $\chi_{le(n,1)}(C_n) = 3$.

Case 2. For n is odd.

To show $\chi_{le(a,d)}(C_n) \leq 3$, by define a bijection $f : V(C_n) \rightarrow \{1, 2, 3, ..., |V(C_n)|\}$ for n is odd by

$$f(x_i) = \begin{cases} i, & i = odd; 1 \leq i \leq n, \\ n - i + 1, & i = even; 1 \leq i \leq n, \end{cases}$$

From the labeling function, we can see that f is a local (a, d)-antimagic coloring of C_n when n is even and the weights of edge are as follows:

$$f(x_i x_{i+1}) = \begin{cases} n, & i = odd; 1 \leq i \leq n-1, \\ n+2, & i = even; 1 \leq i \leq n-1, \\ n+1, & x_1 x_n \end{cases}$$

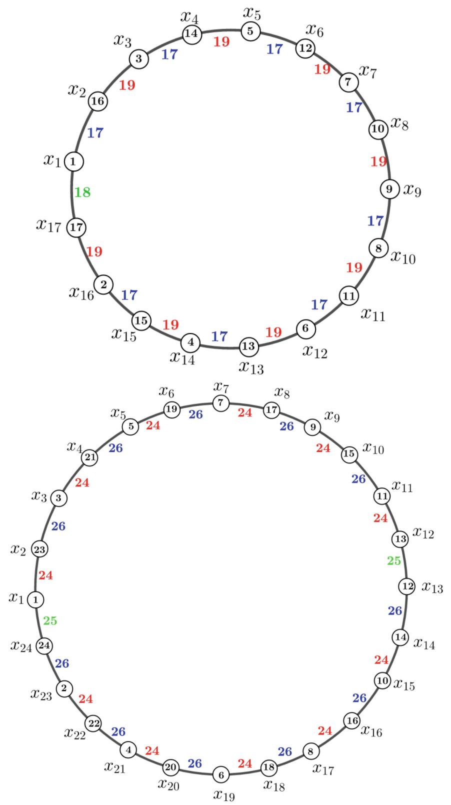

Fig. 2. Local $(n, 1)$-antimagic coloring of C_{17} and C_{24}

Based on the weights, we have set of edge weights obtained is $W = \{n, n+1, n+2\}$ with the smallest edge is $a = n$ and $d = 1$, we have $\chi_{le(n,1)}(C_n) \leq 3$. It conclude that local (a, d)-antimagic coloring chromatic number of C_n with $n \geq 3$ when n is odd $\chi_{le(n,1)}(C_n) = 3$.
It concludes the proof.

From the two cases that have been proven, we have same local (a, d)-antimagic coloring chromatic number of C_n with $n \geq 3$ is $\chi_{le(n,1)}(C_n) = 3$. Figure 2 shows an illustration of local (a, d)-antimagic coloring of a cycle graph.

Theorem 2.3. Let O_n be octopus graph with $n \geq 2$, $\chi_{le(3,1)}(O_n) = 2n$.

Proof. The graph L_n has the vertex set $V(O_n) = \{z\} \cup \{x_i, y_i : 1 \leq i \leq n\}$ and edge set $E(O_n) = \{x_i x_{(i+1)} : 1 \leq i \leq n-1\} \cup \{x_i z, y_i z : 1 \leq i \leq n\}$. The vertices cardinality is $|V(O_n)| = 2n + 1$ and the edges cardinality is $|E(O_n)| = 3n - 1$. The local (a, d)-antimagic coloring chromatic number of O_n is $\chi_{le(3,1)}(O_n) = 2n$. First, we will prove that $\chi_{le(a,d)}(O_n) \geq 2n$. Based on observation 1.4 we have $\chi_{le(a,d)}(O_n) \geq \Delta(O_n)$, then we have $\Delta(O_n) = 2n$. It concludes that $\chi_{le(a,d)}(O_n) \geq 2n$.

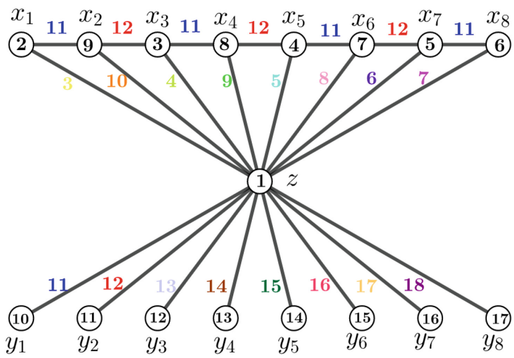

Fig. 3. Local $(3, 1)$-antimagic coloring of O_8

To show $\chi_{le(a,d)}(O_n) \leq 2n$, by define a bijection $f : V(O_n) \rightarrow \{1, 2, 3, ..., |V(O_n)|\}$ by

$$f(x_i) = \begin{cases} \frac{i+3}{2}, & i = odd, \\ n - \frac{i}{2} + 2, & i = even, \end{cases}$$

$$f(y_i) = n + i + 1; i \leq i \leq n$$

$$f(z) = 1$$

From the labeling function, we can see that f is a local (a, d)-antimagic coloring of O_n and the weights of edge are as follows:

$$f(w) = \begin{cases} \frac{i+5}{2}, & x_iz; i = odd; 1 \leq i \leq n \\ n - \frac{i}{2} + 3, & x_iz; i = even; 1 \leq i \leq n, \\ n + i + 2, & y_iz; 1 \leq i \leq n, \\ n + 3, & x_ix_{(i+1)}; i = odd; 1 \leq i \leq n - 1, \\ n + 4, & x_ix_{(i+1)}; i = even; 1 \leq i \leq n - 1 \end{cases}$$

Based on the weights, we have set of edge weights obtained is $W = \{3, 4, 5, ..., 2n + 2\}$ with the smallest edge is $a = 3$ and $d = 1$, we have $\chi_{le(3,1)}(O_n) \leq 2n$. It conclude that local (a, d)-antimagic coloring chromatic number of O_n with $n \geq 2$ is $\chi_{le(3,1)}(O_n) = 2n$.
It concludes the proof.

Figure 3 shows an illustration of local (a, d)-antimagic coloring of a octopus graph.

Theorem 2.4. Let $T_{m,n}$ be tadpole graph with $m \geq 3$ and $n \geq 1$, $\chi_{le(a,1)}(T_{m,n}) = 3$.

Proof. The graph $T_{m,n}$ has the vertex set $V(T_{m,n}) = \{x_i : 1 \leq i \leq m\} \cup \{y_j : 1 \leq j \leq n\}$ and edge set $E(T_{m,n}) = \{x_ix_{(i+1)} : 1 \leq i \leq m - 1\} \cup \{x_1x_m\} \cup \{x_my_1\} \cup \{y_iy_{(i+1)} : 1 \leq j \leq n-1\}$. The vertices cardinality is $|T_{m,n}| =$

$m+n$ and the edges cardinality is $|T_{m,n}| = m+n$. The local (a,d)-antimagic coloring chromatic number of $T_{m,n}$ is $\chi_{le(a,1)}(T_{m,n}) = 3$. First, we will prove that $\chi_{le(a,1)}(T_{m,n}) \geq 3$. Based on observation 1.4 we have $\chi_{le(a,d)}(C_n) \geq \Delta(T_{m,n})$, in Agustin et al. [4] $\chi_{lea}(T_{m,n}) = 3$. It concludes that $\chi_{le(a,d)}(T_{m,n}) \geq 3$.

To show $\chi_{le(a,d)}(T_{m,n}) \leq n+3$, by define a bijection $f : V(T_{m,n}) \rightarrow \{1, 2, 3, ..., |V(T_{m,n})|\}$. In this mapping there are two different cases that occurred in local (a,d)-antimagic coloring of tadpole graph with $n \geq 3$, there are tadpole graph when n is even and n is odd. These cases are as follows:
Case 1. For n is even.
To show $\chi_{le(a,d)}(T_{m,n}) \leq 3$, by define a bijection $f : V(T_{m,n}) \rightarrow \{1, 2, 3, ..., |V(T_{m,n})|\}$ for n is even by

$$f(x_i) = \begin{cases} m + (\frac{n+i+1}{2}), \; i = odd; 1 \leq i \leq n, \\ \\ \frac{n-i}{2} + 1, \qquad i = even; 1 \leq i \leq n \end{cases}$$

$$f(y_j) = m + \tfrac{n}{2} - j + 1, \; \begin{array}{l} m = even, \text{ and} \\ j = odd; 1 \leq j \leq \frac{m}{2}, \\ j = even; \frac{m}{2} < j \leq m. \\ m = odd, \text{ and} \\ j = even; 1 \leq j \leq m. \end{array}$$

$$f(y_j) = \tfrac{n}{2} + j, \; \begin{array}{l} m = even, \text{ and} \\ j = odd; \frac{m}{2} < j \leq m, \\ j = even; 1 \leq j \leq \frac{m}{2}. \\ m = odd, \text{ and} \\ j = even; 1 \leq j \leq m. \end{array}$$

From the labeling function, we can see that f is a local (a,d)-antimagic coloring of $T_{m,n}$ when n is even and the weights of edge are as follows:

$$f(w) = \begin{cases} m+n+1, & x_i x_{(i+1)}; i = odd; 1 \leq i \leq n-1, \\ & y_1 y_m, \\ & y_j y_{(j+1)}; j = \frac{m}{2} (m = even), \\ m+n+2, & x_i x_{(i+1)}; i = even; 1 \leq i \leq n-1, \\ & x_1 y_m, \\ & y_j y_{(j+1)}; j = odd; 1 \leq j \leq m-1, \\ m+n, & y_j y_{(j+1)}; j = even; 1 \leq j \leq m-1 \end{cases}$$

Based on the weights, we have set of edge weights obtained is $W = \{m+n, m+n+1, m+n+2\}$ with the smallest edge is $a = m+n$ and $d = 1$, we have $\chi_{le(m+n,1)}(T_{m,n}) \leq 3$. It conclude that local (a,d)-antimagic coloring chromatic number of $T_{m,n}$ with $n \geq 3$ when n is even $\chi_{le(m+n,1)}(T_{m,n}) = 3$.
Case 2. For n is odd.

To show $\chi_{le(a,d)}(T_{m,n}) \leq 3$, by define a bijection $f : V(T_{m,n}) \rightarrow \{1, 2, 3, ..., |V(T_{m,n})|\}$ for n is odd by

$$f(x_i) = \begin{cases} m + (\frac{n+i+1}{2}), \ i = even; 1 \leq i \leq n, \\ \\ \frac{n-i}{2} + 1, \qquad i = odd; 1 \leq i \leq n \end{cases}$$

$$f(y_j) = m + \frac{n+1}{2} - j + 1, \begin{array}{l} m = even \text{ and} \\ j = odd; \frac{m}{2} < j \leq m, \\ j = even; 1 \leq j \leq \frac{m}{2}, \\ m = odd \text{ and} \\ j = even; 1 \leq j \leq m, \end{array}$$

$$f(y_j) = \frac{n+1}{2} + j, \begin{array}{l} m = even \text{ and} \\ j = odd; 1 \leq j \leq \frac{m}{2}, \\ j = even; \frac{m}{2} < j \leq m, \\ m = odd \text{ and} \\ j = odd; 1 \leq j \leq m \end{array}$$

From the labeling function, we can see that f is a local (a, d)-antimagic coloring of $T_{m,n}$ when n is even and the weights of edge are as follows:

$$f(w) = \begin{cases} m + n + 1, \ x_i x_{(i+1)}; i = even; 1 \leq i \leq n - 1, \\ \qquad\qquad\quad x_1 y_m, \\ \qquad\qquad\quad y_j y_{(j+1)}; j = odd; 1 \leq j \leq m - 1, \\ m + n + 2, \ x_i x_{(i+1)}; i = odd; 1 \leq i \leq n - 1, \\ \qquad\qquad\quad y_1 y_m, \\ \qquad\qquad\quad y_j y_{(j+1)}; j = \frac{m}{2} (m = even), \\ m + n + 3, \ y_j y_{(j+1)}; j = even; 1 \leq j \leq m - 1 \end{cases}$$

Based on the weights, we have set of edge weights obtained is $W = \{m + n + 1, m + n + 2, m + n + 3\}$ with the smallest edge is $a = m + n + 1$ and $d = 1$, we have $\chi_{le(m+n+1,1)}(T_{m,n}) \leq 3$. It conclude that local (a, d)-antimagic coloring chromatic number of $T_{m,n}$ with $n \geq 3$ when n is odd $\chi_{le(m+n+1,1)}(T_{m,n}) = 3$.

From the two cases that have been proven, we have same local (a, d)-antimagic coloring chromatic number of $T_{m,n}$ with $n \geq 3$ but the value of a is different follow as:

$$a = \begin{cases} m + n, \qquad i = even, \\ m + n + 1, \ i = odd. \end{cases}$$

It concludes the proof.

Figure 4 shows an illustration of local (a, d)-antimagic coloring of a tadpole graph with n is even.

Fig. 5 shows an illustration of local (a, d)-antimagic coloring of a tadpole graph with n is odd.

Theorem 2.5. Let H_n be helm graph with $n \geq 3$, $\chi_{le(2n+2,1)}(H_n) = n + 3$.

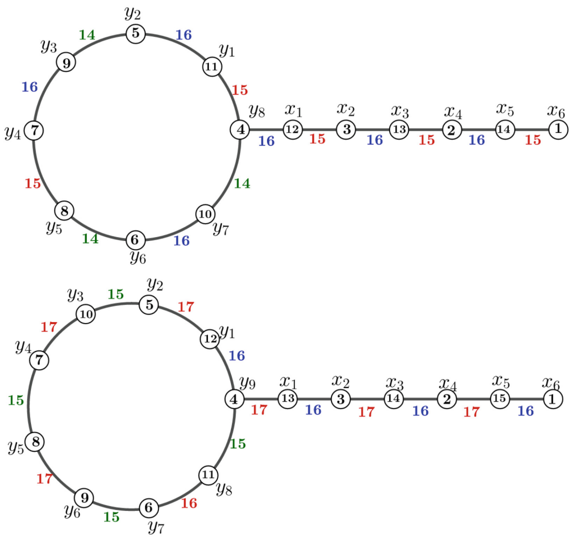

Fig. 4. Local $(m+n,1)$-antimagic coloring of $T_{8,6}$ and $T_{9,6}$

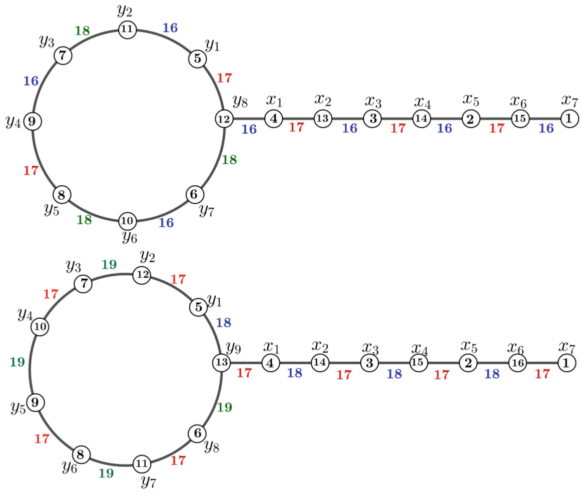

Fig. 5. Local $(m+n+1,1)$-Antimagic Coloring of $T_{8,7}$ and $T_{9,7}$

Proof. The graph C_n has the vertex set $V(H_n) = \{x\} \cup \{x_i, y_i : 1 \leq i \leq n\}$ and edge set $E(H_n) = \{x_i x_{(i+1)} : 1 \leq i \leq n-1\} \cup \{x_n x_1\} \cup \{xx_i, x_i y_i : 1 \leq i \leq n\}$. The vertices cardinality is $|V(H_n)| = 2n+1$ and the edges cardinality is $|E(H_n)| = 3n$. The local (a,d)-antimagic coloring chromatic number of H_n is $\chi_{le(2n+2,1)}(H_n) = n+3$. First, we will prove the lower bound. Helm graph consists of wheel graph (W_n) with n vertex and n pendant vertex. Based on observation 1.4 we have $\chi_{le(a,d)}(H_n) \geq \chi_{lea}(H_n)$, in Agustin et al. [4] $\chi_{lea}(W_n) = n+2$ and every edge of the pendant vertev has one color, so antimagic coloring of helm graph $\chi_{lea}(H_n) = n + 3$. It concludes that $\chi_{le(a,d)}(C_n) \geq n + 3$.

To show $\chi_{le(a,d)}(H_n) \leq n + 3$, by define a bijection $f : V(H_n) \rightarrow \{1, 2, 3, ..., |V(H_n)|\}$. In this mapping there are two different cases that occurred in local (a, d)-antimagic coloring of helm graph with $n \geq 3$, there are helm graph when n is even and n is odd. These cases are as follows:

Case 1. For n is even.

To show $\chi_{le(a,d)}(H_n) \leq 3$, by define a bijection $f : V(H_n) \rightarrow \{1, 2, 3, ..., |V(H_n)|\}$ for n is even by

$$x = n + 1$$

$$f(x_i) = \begin{cases} i, & i = odd; 1 \leq i \leq \frac{n}{2}, \\ & i = even; \frac{n}{2} < i \leq n, \\ n - i + 1, & i = even; 1 \leq i \leq \frac{n}{2}, \\ & i = odd; \frac{n}{2} < i \leq n, \end{cases}$$

$$f(y_i) = \begin{cases} 2n - i + 2, & i = odd; 1 \leq i \leq \frac{n}{2}, \\ & i = even; \frac{n}{2} < i \leq n, \\ n - i + 1, & i = even; 1 \leq i \leq \frac{n}{2}, \\ & i = odd; \frac{n}{2} < i \leq n, \end{cases}$$

From the labeling function, we can see that f is a local (a, d)-antimagic coloring of H_n when n is even and the weights of edge are as follows:

$$f(w) = \begin{cases} 2n + 2, & x_i y_i; 1 \leq i \leq n, \\ 3n + 4, & y_i y_{(i+1)}; \begin{array}{l} i = odd; 1 \leq i < \frac{n}{2}, \\ i = even; \frac{n}{2} < i \leq n - 1, \end{array} \\ 3n + 2, & y_i y_{(i+1)}; \begin{array}{l} i = even; 1 \leq i < \frac{n}{2}, \\ i = odd; \frac{n}{2} < i \leq n - 1, \end{array} \\ 3n + 3, & y_i y_{(i+1)}; i = \frac{n}{2}, \\ & y_1 y_n, \\ 3n - i + 3, & x y_i; \begin{array}{l} i = odd; 1 \leq i \leq \frac{n}{2}, \\ i = even; \frac{n}{2} < i \leq n, \end{array} \\ 2n + i + 2, & x y_i; \begin{array}{l} i = even; 1 \leq i \leq \frac{n}{2}, \\ i = odd; \frac{n}{2} < i \leq n, \end{array} \end{cases}$$

Based on the weights, we have set of edge weights obtained is $W = \{2n + 2, 2n + 3, ..., 3n + 4\}$ with the smallest edge is $a = 2n + 2$ and $d = 1$, we have $\chi_{le(n,1)}(H_n) \leq n + 3$. It conclude that local (a, d)-antimagic coloring chromatic number of H_n with $n \geq 3$ when n is even $\chi_{le(2n+2,1)}(H_n) = n + 3$.

Case 2. For n is odd.

To show $\chi_{le(a,d)}(H_n) \leq 3$, by define a bijection $f : V(H_n) \rightarrow \{1, 2, 3, ..., |V(H_n)|\}$ for n is odd by

$x = n + 1$

$$f(x_i) = \begin{cases} i, & i = odd; 1 \leq i \leq n, \\ n - i + 1, & i = even; 1 \leq i \leq n, \end{cases}$$

$$f(y_i) = \begin{cases} 2n - i + 2, & i = odd; 1 \le i \le n, \\ n + i + 1, & i = even; 1 \le i \le n, \end{cases}$$

From the labeling function, we can see that f is a local (a, d)-antimagic coloring of H_n when n is even and the weights of edge are as follows:

$$f(w) = \begin{cases} 2n + 2, & x_i y_i; 1 \le i \le n, \\ 3n + 4, & y_i y_{(i+1)}; i = even; 1 \le i \le n - 1, \\ 3n + 2, & y_i y_{(i+1)}; i = odd; 1 \le i \le n - 1, \\ 3n + 3, & y_1 y_n, \\ 3n - i + 3, & xy_i; i = odd; 1 \le i \le n, \\ 2n + i + 2, & xy_i; i = even; 1 \le i \le n, \end{cases}$$

Based on the weights, we have set of edge weights obtained is $W = \{2n + 2, 2n + 3, ..., 3n + 4\}$ with the smallest edge is $a = 2n + 2$ and $d = 1$, we have $\chi_{le(2n+2,1)}(H_n) \le n+3$. It conclude that local (a, d)-antimagic coloring chromatic number of H_n with $n \ge 3$ when n is odd $\chi_{le(2n+2,1)}(H_n) = n + 3$.
It concludes the proof.

From the two cases that have been proven, we have same local (a, d)-antimagic coloring chromatic number of H_n with $n \ge 3$ is $\chi_{le(2n+2,1)}(H_n) = n+3$. Figure 6 shows an illustration of local (a, d)-antimagic coloring of a helm graph.

Theorem 2.6. Let Tb_n be tringular book graph with $n \ge 2$, $n + 1 \le \chi_{le(3,1)}(Tb_n) \le n + 2$.

Proof. The graph Tb_n has the vertex set $V(Tb_n) = \{x\} \cup \{y\} \cup \{x_i : 1 \le i \le n\}$ and edge set $E(Tb_n) = \{xy\} \cup \{xx_i, yx_i : 1 \le i \le n\}$. The vertices cardinality is $|V(Tb_n)| = n + 2$ and the edges cardinality is $|E(Tb_n)| = 2n + 1$. The local (a, d)-antimagic coloring chromatic number of Tb_n is $\chi_{le(3,1)}(Tb_n) = n + 2$. First, we will prove that $\chi_{le(a,d)}(Tb_n) \ge n + 2$. Based on observation 1.4 we have $\chi_{le(a,d)}(Tb_n) \ge \Delta(Tb_n)$, then we have $\Delta(Tb_n) = n + 1$. It concludes that $\chi_{le(a,d)}(Tb_n) \ge n + 1$.

To show $\chi_{le(a,d)}(Tb_n) \le n + 2$, by define a bijection $f : V(Tb_n) \to \{1, 2, 3, ..., |V(Tb_n)|\}$ by

$$f(x_i) = \begin{cases} 1, & x, \\ 2, & y, \\ i + 2, & 1 \le i \le n, \end{cases}$$

From the labeling function, we can see that f is a local (a, d)-antimagic coloring of Tb_n and the weights of edge are as follows:

$$f(w) = \begin{cases} 3, & xy, \\ i + 3, & xx_i; 1 \le i \le n, \\ i + 4, & yx_i; 1 \le i \le n, \end{cases}$$

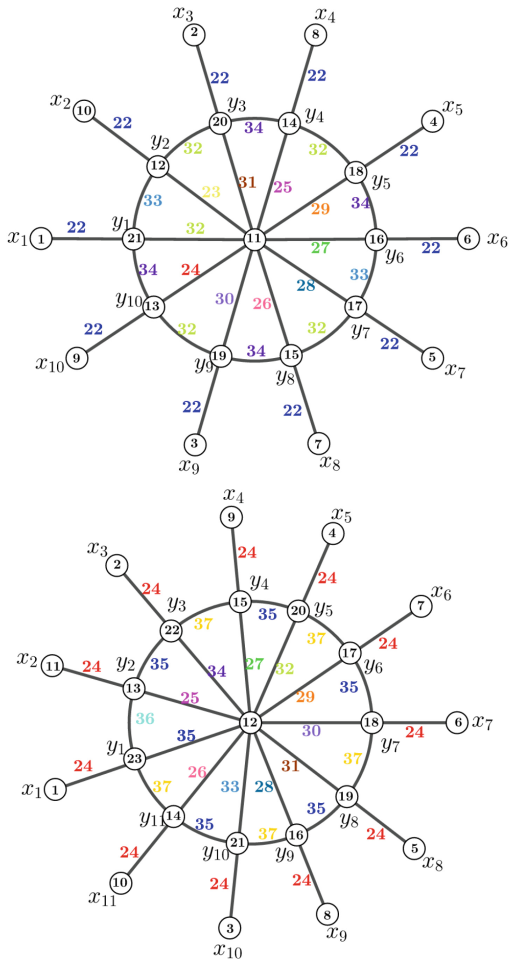

Fig. 6. Local $(2n + 2, 1)$-Antimagic Coloring of H_{10} and H_{11}

Based on the weights, we have set of edge weights obtained is $W = \{3, 4, 5, ..., n + 4\}$ with the smallest edge is $a = 3$ and $d = 1$, we have $\chi_{le(3,1)}(Tb_n) \leq n + 2$. It conclude that local (a, d)-antimagic coloring chromatic number of Tb_n with $n \geq 2$ is $n + 1 \leq \chi_{le(3,1)}(Tb_n) \leq n + 2$.
It concludes the proof.

Figure 7 shows an illustration of local (a, d)-antimagic coloring of a tringular book graph.

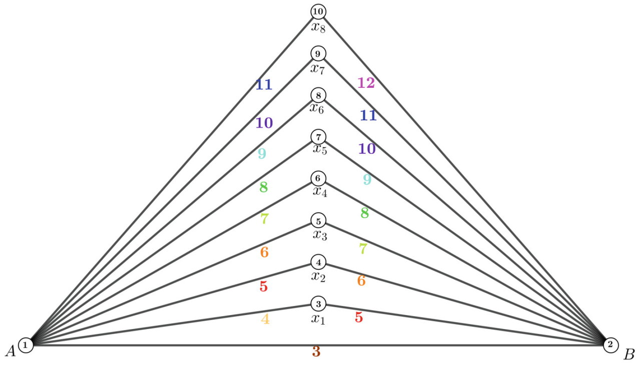

Fig. 7. Local $(n+2,1)$-antimagic coloring of Tb_8

3 Concluding Remarks

In this paper, we study the local (a, d)-antimagic coloring of some specific classes of graph, namely ladder graph, cycle graph, octopus graph, helm graph, tringular book graph, and tadpole graph by finding the local (a, d)-antimagic coloring chromatic numbers that reach lower bound. However, due to there is still little research related to the topic local (a, d)-antimagic coloring, so we propose open problem.

Open Problem

Determine the exact value of the local (a, d)-antimagic coloring chromatic number of all types of graphs regardless of what has been researched.

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References

1. I. H. Agustin, M. Hasan, Dafik, R. Alfarisi, R. M. Prihandini, "Local edge antimagic coloring of graphs," Far East Journal of Mathematical Sciences (FJMS), vol. 102, pp. 1925–1941, 2017.
2. I. H. Agustin, M. Hasan, Dafik, R. Alfarisi, A. I. Kristiana, R. M. Prihandini, "Local Edge Antimagic Coloring of Comb Product of Graphs," Journal of Physics: Conference Series, vol. 1008, 2018.
3. I. H. Agustin, Dafik , E. Y. Kurniawati, Marsidi , N. Mohanapriya, A. I. Kristiana, "On the local (a, d)-antimagic coloring of graphs," 2022.
4. S. Aisyah, R. Alfarisi, R. M. Prihandini, A. I. Kristiana, R. Dwi, "On the Local Edge Antimagic Coloring of Corona Product of Path and Cycle," CAUCHY -Jurnal Matematika Murni dan Aplikasi, vol. 6, pp. 40–48, 2019.

5. S. Arumugam, K. Premalatha, M. Baca, A. S. Fenovcikova, "Local Antimagic Vertex Coloring of a Graph," Graphs and Combinatorics, vol. 33, pp. 275–285, 2017.
6. S. Arumugam, Y. C. Lee, K. Permalatha, T. M. Wang, "On Local Antimagic Vertex Coloring for Corona Products of Graphs," arXiv: Combinatorics, 2018.
7. J. L. Gross, J. Yellen, and P. Zhang, Handbook of graph Theory Second Edition. CRC Press Taylor and Francis Group, 2014.
8. N. Hartsfield and G. Ringel, Pearls in Graph Theory Academic Press. United Kingdom, 1994.
9. N. H. Nazula, Slamin, Dafik, "Local antimagic vertex coloring of unicyclic graphs(Local antimagic vertex coloring of unicyclic graphs," Indonesian Journal of Combinatorics, vol. 2, pp. 30–34, 2018.
10. R. Sundaramoorthy and N. M. Chettiar, "On (a, d)-edge local antimagic coloring number of graphs," Turkish Journal of Mathematics, vol. 46, pp. 1994–2002, 2022.
11. R. Izza , Dafik, A. I. Kristiana, On local(a, d)-edge antimagic coloring of some graphs., in press.