Laplacian Spectrum of Identity Graph of Commutative Ring $\mathbb{Z}_{2p}$

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Abstract. Research on the spectrum of a graph with an algebraic structure still attracts much attention. Let $\mathbb{Z}$ be a commutative ring. The $I(\mathbb{Z})$ is a graph with set $\mathbb{Z}$ as its vertices and two vertices $x, y \in \mathbb{Z}$ are adjacent if and only if $x.y = 1$, and also every vertex adjacent with 1. In 2020, research about Laplacian eigenvalues of a graph has been studied, that focuses on a zero divisor graph of ring $\mathbb{Z}_n$. This paper aims to determine the Laplacian spectrum of the identity graph of commutative ring $\mathbb{Z}_{2p}$, which can be constructed with investigate the eigenvalues of $I(\mathbb{Z}_{2p})$. We find the set of all eigenvalues of $I(\mathbb{Z}_{2p})$ are integers.

Keywords: graph · spectrum · identity graph · commutative ring

1 Introduction

Algebraic graph theory is the science that uses techniques of algebra in graph theory, other types of graphs have also developed, one of which is spectrum graphs. Research on the spectrum of a graph with an algebraic structure still attracts much attention. The research of the spectrum concept of the graph was first introduced by Norman Bigg [1] in 1974, researchers continue this study with other spectrum concepts. For example, Ayyaswamy and Balachandran in [2] have studied the detour spectrum of a complete graph, the double graph of a complete graph, the cartesian product of a complete graph, and also the lexicographic product of a complete graph. Chattopadhyay et al. in [3] studied the Laplacian eigenvalues of a graph, that is on a zero divisor graph of the ring $\mathbb{Z}_n$. Abdussakir et al. [4] studied the characteristic polynomials and spectrum of a graph on the dihedral group, that is for the Laplacian matrix, signless Laplacian matrix, detour matrix, and also for adjacency matrix. Torktaz and Ashrafi [5] studied about signless Laplacian of graph of a finite group with certain classes. Murni et al. [6] studied about Laplacian spectrum and anti-adjacency of the graph of a group. Pirzada et al. in [7, 8] studied about Laplacian spectrum, normalized Laplacian spectrum of a graph, and also signless Laplacian of a graph that focuses on the zero divisor graph $\Gamma(\mathbb{Z}_n)$.

Let $G = (V(G), E(G))$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G)$ are respectively as vertex set of G and as an edge set of G. The adjacency matrix $A(G) = [a_{ij}]$ is $n \times n$ matrix, with $a_{ij} = 1$ if $v_i v_j = 1$ and $a_{ij} = 0$ if $v_i v_j \neq 1$ CITATION Big741 \l 1057

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[1]. Let $L(G)$ be a Laplacian matrix of G that is presented by $L(G) = D(G) - A(G)$, where the degree matrix $D(G)$ is a diagonal matrix of vertex degree with $d_{ij} = \deg_G(v_i)$ and $d_{ij} = 0$, where $i \neq j$. The Laplacian eigenvalues of G are the eigenvalues of $L(G)$ and the Laplacian characteristic polynomial of G is the characteristic polynomial of $L(G)$.

The spectrum of square matrix G that is denoted by $Spec(G)$ is the set of all the eigenvalues of G, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $m(\lambda_1), m(\lambda_2), \ldots, m(\lambda_n)$ are respectively as distinct eigenvalues of G with their respective multiplicities, then we will denote spectrum of graph G by

$$Spec(G) = \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ m(\lambda_1) & m(\lambda_2) & \ldots & m(\lambda_n) \end{bmatrix} \tag{1}$$

for graph G, the *Laplacian spectrum of* G which is denoted by $Spec_L(G)$ is the spectrum from the Laplacian matrix of G.

The identity graph concept was first introduced by Kandasamy and Smarandache in [9], they studied the identity graph of various special groups, semi-groups, and also various rings of the graph. Let R be a commutative ring with a unit such that $a.1 = 1.a$, for every $a \in R$ where 1 is the identity element of R. The set of units in R forms the vertices of the simple graph such that two distinct vertices $x, y \in R$ are adjacent if and only if $x.y = 1$, and assume that 1 is adjacent with every unit in R. This graph is called the identity graph or unit graph of R that denoted by $I(R)$ [9]. Some research on identity graphs has been done before, for example, Ibrahim and Essa [10] studied the identity graph of the rings, and they investigate its properties, clique number, diameter, and girth. Herawati et al. in [11] have studied about identity graph of a group that focuses on its characteristics. Sowaity et al. [12] studied the identity graph of multigroup that was derived as a graph. Abdussakir et al. [13] also studied about identity graph, that is of a cyclic group and ring with unity, their research is to determine the eccentric connectivity index.

In this paper, we are inspired to continue research from Chattopadhyay, et al. [3] with the different graph is the identity graph and different ring. In another word, we will determine the Laplacian spectrum of the identity graph of the ring. The ring that will be discussed here is a commutative ring of integers modulo $2p$, where p is a prime number. All graphs in this study will be considered simple (without loop and undirected), and finite. In Sect. 2, we provide the steps that have been carried out in this research. In Sect. 3, we are discussing the Laplacian spectrum of $I(\mathbb{Z}_{2p})$ by presenting some theorem and their proof, techniques of algebra will be used in graph theory, especially to present an identity ring in the form of a graph. In the process of calculating eigenvalues and characteristic polynomials, we use the help of Maple 18 computing software.

2 Methodology

To find the Laplacian spectrum of $I(\mathbb{Z}_{2p})$, where p is a prime, we do the following steps:

1. Drawing the identity graph $I(\mathbb{Z}_{2p})$ for $p \in (2, 3, 5, 7, 11, 13)$.
2. Construct all matrices of each $I(\mathbb{Z}_{2p})$ in step 1.

3. Determine characteristic polynomials and find eigenvalues.
4. Determine the pattern of the Laplacian spectrum of $I(\mathbb{Z}_{2p})$ and formulate conjectures.
5. Presenting conjectures into theorems along with their proofs.

3 Results and Discussion

Let $\mathbb{Z}_{2p}$ be a commutative ring of modulo p, where p is a prime number,$p \geq 5$. We know that $\mathbb{Z}_{2p} = \{0, 1, 2, \ldots, 2p-1\}$ and 1 is the identity element of $\mathbb{Z}_{2p}$ such that $x.1 = x = 1.x$, $x \in \mathbb{Z}_{2p}$. By using the definition of an identity graph, two vertices x and y are called adjacent if and only if $x.y = 1$, and also for every vertex adjacent with 1. The identity graph of $\mathbb{Z}_{2p}$ is denoted by $I(\mathbb{Z}_{2p})$ and so $I(\mathbb{Z}_{2p}) = \{v_i | i = 1, 3, 5, \ldots, 2p-1\} - \{p\}$, where v_i is vertex of $I(\mathbb{Z}_{2p})$. Let $p \in \{2, 3\}$, ring $\mathbb{Z}_{2p}$ has an element x as units. So, the identity graph of $\mathbb{Z}_{2p}$ is an edge with 1 and x as vertices of $I(\mathbb{Z}_{2p})$. For the next discussion, we focus on $p \geq 5$.

Theorem 3.1 Let p be a prime number, $p \geq 5$. The identity graph $I(\mathbb{Z}_{2p})$ has $\frac{p-3}{2}$ triangles with 1 as a common vertex and a single edge.

Proof: The identity graph $I(\mathbb{Z}_{2p})$ has $\{1, 3, 5, \ldots, 2p-1\} - \{p\}$ as its vertex set, this means $I(\mathbb{Z}_{2p})$ has $p-1$ vertices . According to the definition of an identity graph, every vertex is adjacent to 1 such that there are $p-2$ vertices adjacent to 1. $2p-1$ as a greater vertex will be only one edge connecting it, that is the edge which connects $2p-1$ to 1. Then, $p-3$ vertices will be paired and every pair are adjacent to 1, hence this pair form $\frac{p-3}{2}$ triangles. Therefore, we have $\frac{p-3}{2}$ triangles and a single edge on the identity graph$I(\mathbb{Z}_{2p})$. Identity graph $I(\mathbb{Z}_{2p})$ presented in the Fig. 1

From theorem 3.1 we get $\deg(1) = p-2$, $\deg(2p-1) = 1$, and the degree for other vertices is 2. This result will be used to find the characteristic polynomial of $I(\mathbb{Z}_{2p})$.

Theorem 3.2 The characteristic polynomial of identity graph $I(\mathbb{Z}_{2p})$ is a polynomial of degree $p-1$ that is.

$$p(\lambda) = \lambda(\lambda-1)^{\frac{p-1}{2}-1}(\lambda-3)^{\frac{p-1}{2}-1}(\lambda-p+1).$$

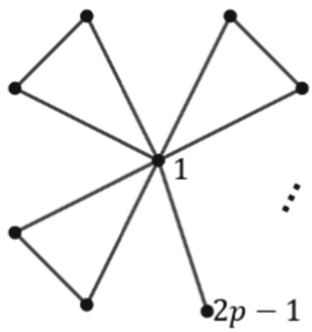

Fig. 1. Identity Graph $I(\mathbb{Z}_{2p})$

Proof: In the identity graph $I(\mathbb{Z}_{2p}) = \{v_i | i = 1, 3, 5, \ldots, 2p-1\} - \{p\}$, where v_i is the vertex of $I(\mathbb{Z}_{2p})$, we know that every vertex will adjacent to 1 and two distinct vertices $x, y \in I(\mathbb{Z}_{2p})$ adjacent if and only if $xy = 1$. So, we have $(p-1) \times (p-1)$ adjacency matrix $A(I(\mathbb{Z}_{2p})) = [a_{ij}]$, where a_{ij} obtained from the relation of v_i as a column and row in $A(I(\mathbb{Z}_{2p}))$ according to the definition of adjacency matrix as follows:

$$a_{ij} = \begin{cases} 1, & i,j = 1,\ i \neq j \ and\ \ i.j = 1 \\ 0, & otherwise \end{cases},$$

and the degree matrix $D(I(\mathbb{Z}_{2p})) = [d_{ij}]$ is $(p-1) \times (p-1)$ diagonal matrix that is clear. $\deg(1) = p-2$, $\deg(2p-1) = 1$, and the degree for other vertices is 2. So, we get $[d_{ij}]$ as follows (Table 1).

$$d_{ij} = \begin{cases} p-2, & d_{1,1} \\ 2 & i = j, i, j \neq 1 \ and \ i, j \neq p-1 \\ 1, & d_{p-1,p-1} \\ 0, & i \neq j \end{cases}$$

Thus, the Laplacian matrix $L(I(\mathbb{Z}_{2p})) = D(I(\mathbb{Z}_{2p})) - A(I(\mathbb{Z}_{2p})) = [l_{ij}]$, where

$$l_{ij} = \begin{cases} p-2, & d_{1,1} \\ 2 & i = j, i, j \neq 1 \ and \ i, j \neq p-1 \\ 1, & d_{p-1,p-1} \\ -1, & i, j = 1, i \neq j \ and \ i.j = 1 \\ 0, \ otherwise & \end{cases}$$

Using cofactor expansion for $\det(L(I(\mathbb{Z}_{2p})) - \lambda I)$, it will be found

$$p(\lambda) = \lambda(\lambda-1)^{\frac{p-1}{2}-1}(\lambda-3)^{\frac{p-1}{2}-1}(\lambda-p+1).$$

Table 1. Characteristic Polynomial of $I(\bar{\mathbb{Z}}_{2p})$

Graph	**Characteristic Polynomial**
$I(\mathbb{Z}_{10})$	$\lambda(\lambda-1)(\lambda-3)(\lambda-4)$
$I(\mathbb{Z}_{14})$	$\lambda(\lambda-1)^2(\lambda-3)^2(\lambda-6)$
$I(\mathbb{Z}_{22})$	$\lambda(\lambda-1)^4(\lambda-3)^4(\lambda-10)$
$I(\mathbb{Z}_{26})$	$\lambda(\lambda-1)^5(\lambda-3)^5(\lambda-12)$
$\vdots$	$\vdots$
$I(\mathbb{Z}_{2p})$	$\lambda(\lambda-1)^{\frac{p-1}{2}-1}(\lambda-3)^{\frac{p-1}{2}-1}(\lambda-p+1)$

Theorem 3.3 The Laplacian spectrum of identity graph $I(\mathbb{Z}_{2p})$ is.

$$Spec_L(I(\mathbb{Z}_{2p})) = \begin{pmatrix} 0 & 1 & 3 & p-1 \\ 1 & \frac{p-1}{2} - 1 & \frac{p-1}{2} - 1 & 1 \end{pmatrix}.$$

Proof: From theorem 3.2, the characteristic polynomial of identity graph $I(\mathbb{Z}_{2p})$ is.

$$p(\lambda) = \lambda(\lambda - 1)^{\frac{p-1}{2}-1}(\lambda - 3)^{\frac{p-1}{2}-1}(\lambda - p + 1).$$

By solving the characteristic polynomial $p(\lambda) = 0$, we have $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$, and $\lambda_4 = p-1$ as eigenvalues of $I(\mathbb{Z}_{2p})$ with its respectively multiplicities are $m(\lambda_1) = 1$, $m(\lambda_2) = \frac{p-1}{2} - 1$, $m(\lambda_3) = \frac{p-1}{2} - 1$, and $m(\lambda_4) = 1$. Therefore, using the definition of the spectrum of a graph we have

$$Spec_L(I(\mathbb{Z}_{2p})) = \begin{pmatrix} 0 & 1 & 3 & p-1 \\ 1 & \frac{p-1}{2} - 1 & \frac{p-1}{2} - 1 & 1 \end{pmatrix}.$$

Example 3.4 Let $\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the ring of integers modulo 10. We have $\{1, 3, 7, 9\}$ as a unit set of $\mathbb{Z}_{10}$ that will be vertices of identity graph $I(\mathbb{Z}_{10})$. From the definition of an identity graph, we get Identity Graph $I(\mathbb{Z}_{10})$ presented in Fig. 2.

Based on theorem 3, we get adjacency matrix $I(\mathbb{Z}_{10})$ is

$$\mathbf{A}(\mathbf{I}(\mathbb{Z}_{10})) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and degree matrix $I(\mathbb{Z}_{10})$ is

$$\mathbf{D}(\mathbf{I}(\mathbb{Z}_{10})) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Laplacian matrix $L(I(\mathbb{Z}_{10})) = D(I(\mathbb{Z}_{10})) - A(I(\mathbb{Z}_{10}))$ can be presented as

$$\mathbf{L}(\mathbf{I}(\mathbb{Z}_{10})) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

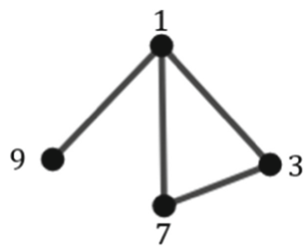

Fig. 2. Identity Graph $I(\mathbb{Z}_{10})$

By using cofactor expansion, we will get the determinant of $(\lambda I - L(I(\mathbb{Z}_{10})))$ as follows

$$\det(\lambda I - L(I(\mathbb{Z}_{10}))) = \begin{vmatrix} \lambda-3 & 1 & 1 & 1 \\ 1 & \lambda-2 & 1 & 0 \\ 1 & 1 & \lambda-2 & 0 \\ 1 & 0 & 0 & \lambda-1 \end{vmatrix} = \lambda^4 - 8\lambda^3 + 19\lambda^2 - 12\lambda$$

Then, the characteristic polynomial $p(\lambda) = \lambda^4 - 8\lambda^3 + 19\lambda^2 - 12\lambda$. By solving $p(\lambda) = 0$ will find the different eigenvalues of $L(I(\mathbb{Z}_{10}))$. That is $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$ and $\lambda_4 = 4$ with their multiplicities are $m(\lambda_1) = 1, \mathrm{m}(\lambda_2) = 1, \mathrm{m}(\lambda_3) = 1$ and $m(\lambda_4) = 1$.

So, by using the definition of the spectrum of a graph, we can present the Laplacian spectrum of $I(\mathbb{Z}_{10})$ by

$$Spec_L(I(\mathbb{Z}_{10})) = \begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

4 Conclusion

From the results and discussion of this paper can be seen that every eigenvalue of matrices of the identity graph of ring $\mathbb{Z}_{2p}$ are integer. This research focuses on the ring $\mathbb{Z}_{2p}$, for the next research can be continued to the other ring.

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