Analytical Solution of Stress Redistribution in Simple Beams

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Abstract. This paper presents a simple way of derivation of analytical solution for the stress (load) redistribution within a simple beam composed of multiple individual profiles. To demonstrate the main ideas a cantilever beam is considered, consisting of finite number of parallel sub-beams. The external load applied to the beam is shared and redistributed among the individual sub-beams depending on their stiffness (flexural rigidity). The procedure of obtaining the shear force and bending moment diagrams for individual sub-beams is explained in detail, leading to analytical formulas for the shearing and bending load for individual sub-beams, allowing for determination of the corresponding stress. To conclude the work, the possible applications and extensions of the presented approach are discussed. This work is a follow-up of our previous study providing the analytical solution for elastic line (i.e. deflection) of this kind of simple beams.

Keywords: beam · stress · cantilever beam · shear force · bending moment

1 Introduction

The beams are one of essential components in the civil and mechanical engineering. Their dimensioning and control is thus of extreme importance. They are made of various materials, in different shapes and with different supports and anchors. With increasing requirements on their optimization with respect to their reliability, reduced weight or easy production, it became clear that simple beams with homogeneous internal structure, made of single piece of material are in some cases unacceptable. This is why nowadays more sophisticated beams are being used more often than ever. This however also brings some new theoretical questions to be solved concerning their design. Despite of the recent progress of numerical tools for detailed calculation of complex mechanical parts and structures, the classical analytical approach still finds its use. Namely at the initial stages of design, where the appropriate dimensions of the components have to be found, the analytical solutions for stress and deformation are used to estimate the appropriate parameters for the designed parts.

The beams considered in this paper are composed of several individual sub-beams, oriented in parallel to each other. The sub-beams are interconnected in such a way that they all deform in the same way (they follow the same elastic curve) and the load can

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be transferred and redistributed between the sub-beams. In this way a beam can consist of individual sub-beams made of different materials, having different cross-section or orientation. See e.g. Figure 4 for an example.

The example used here to explain the load decomposition is based on a simple cantilever beam shown in the Fig. 1.

The diagrams of the corresponding shear force *T(x)* and bending moment *M(x)* are presented in the Fig. 2 and 3. They can be found e.g. in Czech textbooks [1, 2] or in the well known books [3, 4].

$$\mathrm{T(x)} = \mathrm{F} \tag{1}$$

$$M(x) = F \cdot x \tag{2}$$

In this elementary case the shear force is constant along the beam, while the moment increases linearly. The deformation (deflection) of the simple cantilever beam is described by the function:

$$v(x) = \frac{F}{6EI}\left(x^3 - 3L^2x + 2L^3\right) \tag{3}$$

It is evident, that due to linear increase of the moment *M(x)* towards the anchored end of the beam, the use of a single beam with constant cross-section is not optimal. The critical stress appears in the anchor of the beam, for which the dimension of the beam profile was chosen. Going towards the free end the stress decreases and the material is not optimally used. In order to make better use of the beam material, there is a natural

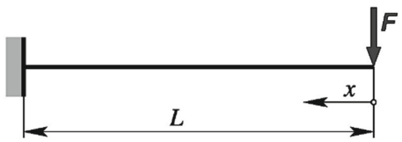

Fig. 1. Simple cantilever beam loaded by an isolated shear force at the beam tip.

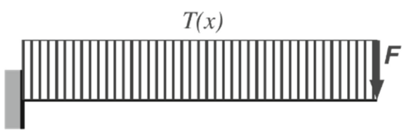

Fig. 2. Shear force diagram for simple cantilever beam.

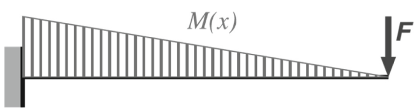

Fig. 3. Bending moment diagram for simple cantilever beam.

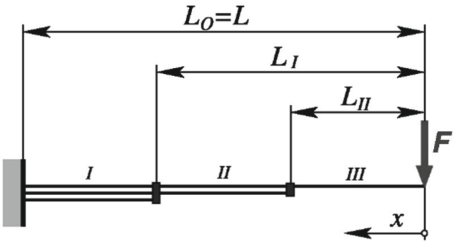

Fig. 4. Cantilever beam with three sub-beams (and three fields) loaded by a single isolated force at the beam tip.

tendency to locally reinforce the beam by attaching additional beams to the original one to help it to share some load. The additional reinforcing beams do not have to be made of the same material and do not need to have the same profile.

The cantilever beam has the same basic shape, however internally it can be built of several sub-beams of different type and different length. An example of the configuration used in this paper is shown in the Fig. 4.

The external load acting on the beam gets internally redistributed to individual sub-beams. The breakup of the loading forces and moments to individual beams determines the local stresses in them and is thus crucial for proper dimensioning and safety evaluation of the whole construction.

2 Materials and Methods

The local deformation (deflection) of all sub-beams is always identical, leading to a single elastic curve for all sub-beams. The corresponding stress can however be different (in general) for each sub-beam. The loading shear force (and bending moment) breakup to individual sub-beams might be a difficult task. The analytical approach presented here relies on a quite simple idea. Considering the local deformation (deflection) of a beam is known and material characteristic (Young's modulus E) and profile characteristic (second area moment I) of the beam are given (see e.g. [5], the force needed to achieve the required deformation can be computed. Following this idea, applied to multiple beams sharing the same deformation, the forces in individual beams can be found.

In the case of cantilever beam, loaded by an isolated force, the stress in individual sub-beams can be evaluated analytically. Here we assume that the only the normal (shear) force can be transferred between the sub-beams and only by isolated internal forces in the places, where the clips connecting the sub-beams are installed. The schematic picture of this configuration is presented in Fig. 1, showing three individual sub-beams placed in parallel to each other (with possible gaps between them), connected by two clips.

2.1 Shear Forces

The total local shear force acting on the whole cantilever beam in the considered case is constant along the beam. The shear force diagram doesn't depends on the internal structure of the beam and its possible splitting to individual sub-beams. This global picture changes however significantly when each of the sub-beams is evaluated separately.

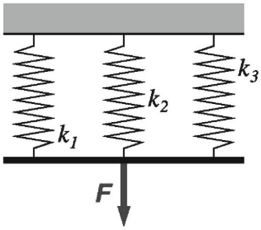

Fig. 5. Springs attached in parallel have a joint equivalent stiffness $k = k_1 + k_2 + k_3$.

Considering that the shear force for each sub-beam can only be modified due to an isolated (inter-beam) force at the position of the connecting clip, it's evident, that the resulting shear force diagram will always be represented by a piece-wise constant function with the force being constant away from clips, having only jumps (discontinuities) at the place of clips. The breakup (decomposition) of the local shear to individual sub-beams follows the above briefly described idea of equal deformation condition.

Roughly speaking, from the point of view of elastic deformation, the beam deflection is analogical to deformation of springs attached in parallel, loaded by a joint force (see Fig. 5).

In the case of the mechanical spring analog shown in the Fig. 5 the total force F is redistributed to individual springs 1, 2, and 3 proportionally to their stiffness as $F_1{:}F_2{:}F_3 = k_1{:}k_2{:}k_3$, where $F = F_1 + F_2 + F_3$.

The deformation of the cantilever beams was explained in detail in the previous work [6], showing (similarly as in (3)) that the beam deflection is proportional to the shear force, divided by the flexural rigidity of the beam (defined as the product *EI* of the Young's modulus E and second area moment I), i.e.:

$$v(x) = \frac{T(x)}{E(x)I(x)} \tag{4}$$

Assuming now (just for simplicity) that each (*i-th*) sub-beam has constant E_i and I_i along its length and also the acting shear force $T_{K,i}$ is constant along the considered (*K-th*) beam field, the formula becomes even easier. The equal deformation condition states that each individual sub-beam is subject to the same deflection as the whole beam leads to:

$$\frac{F}{(EI)_I} = \frac{T_{I,1}}{E_1 I_1} = \frac{T_{I,2}}{E_2 I_2} = \frac{T_{I,3}}{E_3 I_3} \tag{5}$$

where T stands for local total shear force (equal to the isolated external force F in this case). For example in the field I the shear force T_I splits into three parts according to the number of beams sharing the load in that field.

$$T = T_I = T_{I,1} + T_{I,2} + T_{I,3} \tag{6}$$

The general expression for the compatibility (conservativity) condition in the *K-th* field can be written as:

$$F = T_K = \Sigma_{i=1}^{N-K+1} T_{K,i} \tag{7}$$

Here N is the total number of fields for the cantilever beam with the structure shown in the Fig. 4 for $N = 3$.

The $(EI)_I$ stands for an equivalent flexural rigidity of the whole in the field I, defined as the sum of the flexural rigidities of the individual beams participating in the load sharing within the considered beam field I:

$$(EI)_I = E_1 I_1 + E_2 I_2 + E_3 I_3 \tag{8}$$

Again, the general expression for the equivalent stiffness of the *K-th* field can be written as:

$$(EI)_K = \Sigma_{i=1}^{N-K+1} E_i I_i \tag{9}$$

This leads to a simple formula for the size of the shared shear force for each individual sub-beams. For example the force shared by the sub-beam *1* in the field I, is given by

$$T_{I,1} = F\frac{E_1 I_1}{(EI)_I} = F\frac{E_1 I_1}{E_1 I_1 + E_2 I_2 + E_3 I_3} \tag{10}$$

In the field II, the load is only shared by the sub-beams *1* and *2*, leading to the force in sub-beam *1*:

$$T_{II,1} = F\frac{E_1 I_1}{(EI)_{II}} = F\frac{E_1 I_1}{E_1 I_1 + E_2 I_2} \tag{11}$$

In general, considering the beam field number K, formed by n_K sub-beams (as in (7) and (9) it's possible to use $n_k = N - K + 1$) the loading shear force in the sub-beam i becomes:

$$T_{K,i} = F\frac{E_i I_i}{(EI)_K} = \frac{E_i I_i}{\Sigma_{i=1}^{n} E_i I_i},\ \mathrm{i} = 1, \ldots, \mathrm{n_k} \tag{12}$$

The Fig. 6 shows the breakup of the shear force for three individual sub-beams.

2.2 Bending Moments

The bending moment diagram for the whole beam (disregarding its internal structure) shows in the considered case a simple linear shape, starting from zero at the tip of the cantilever beam. The maximum moment (in absolute value) is reached in the fixed anchor end. Due to the assumption, that only the shear forces can be transferred and only where the connecting clips are, the resulting sub-beams moment diagrams will be piece-wise linear. The slope changes can only appear where the clips are located and they respect the corresponding jumps in the shear force diagrams. In general, the bending moment $M(x)$ depends on the shear force $T(x)$ as:

$$\frac{dM(x)}{dx} = T(x) \tag{13}$$

which leads immediately to the similar relations for each sub-beam i, in the field K.

$$\frac{DM}{dx} = T_{K,i}\ \ \mathrm{i} = 1, \ldots, \mathrm{n_k} \tag{14}$$

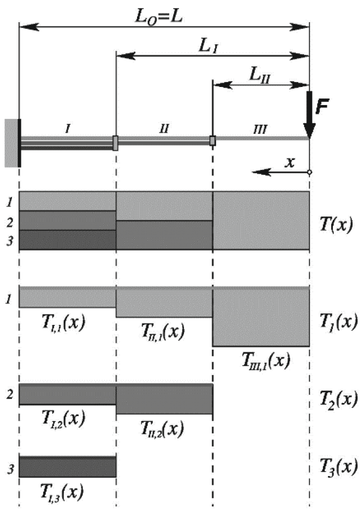

Fig. 6. Shear Force diagram and its breakup for individual sub-beams.

Following now for example the sub-beam 1, from the free end, i.e. from the field *III*, the moment will be:

$$M_{III,1}(x) = T_{III,1} \cdot x \tag{15}$$

where there is only single sub-beam in the field *III* and thus $T_{III,1} = F$. In the next field *II* the linear moment curve for the sub-beam *1* continues, however the part of the moment shared by the sub-beam *2* must be subtracted.

$$M_{II,1}(x) = F \cdot x - T_{II,2} \cdot (x - L_{II}) = T_{III,1} \cdot L_{II} + T_{II,1} \cdot (x - L_{II}) \tag{16}$$

Here the moment for the sub-beam *2* in the field *II* is

$$M_{II,2}(x) = T_{II,2} \cdot (x - L_{II}) \tag{17}$$

It is important to note (and verify) that the sum of moments for all sub-beams (at any given point) is equal to the total moment, i.e. that here sum of (16) and (17) gives exactly *F.x*.

The situation repeats in the last (for this simple case) field *I*, giving the moment in the sub-beam *1* in the form:

$$M_{I,1}(x) = T_{III,1} \cdot L_{II} + T_{II,1} \cdot (L_I - L_{II}) + T_{I,1} \cdot (x - L_I) \tag{18}$$

For the sub-beams *2* and *3* its possible to write:

$$M_{I,2}(x) = T_{II,2} \cdot (L_I - L_{II}) + T_{I,2} \cdot (x - L_I) \tag{19}$$

$$M_{I,3}(x) = T_{I,3} \cdot (x - L_I) \tag{20}$$

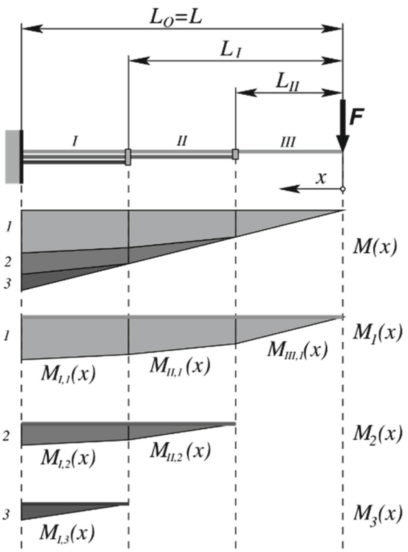

Fig. 7. Bending moment diagram and its breakup for individual sub-beams.

Again, the sum of all moments in individual sub-beams must be equal to the total moment defined by (2), for the whole cantilever beam. By summing up the (18), (19) and (20) gives

$$M_I(x) = T_{III,1} \cdot L_{II} + \left(T_{II,1} + T_{II,2}\right) \cdot (L_I - L_{II}) + \left(T_{I,1} + T_{I,2} + T_{I,3}\right) \cdot (x - L_I) \quad (21)$$

which can be simplified using the consistency condition (6), resp. (7)

$$F = T_{III,1} = T_{II,1} + T_{II,2} = T_{I,1} + T_{I,2} + T_{I,3} \quad (22)$$

leading to

$$M_I(x) = F \cdot x = F \cdot L_{II} + F(L_I - L_{II}) + F \cdot (x - L_I) \quad (23)$$

The schematic picture of the moments breakup is shown in Fig. 7.

2.3 Stresses

The normal stress σ (due to bending) is dominant for beam-like structures. It can easily be evaluated (see e.g. [7] or [3]) from known bending moment *M(x)* and section modulus *W(x)* as

$$\sigma(x) = \frac{M(x)}{W(x)} \quad (24)$$

It's good to note that the moment diagrams for the whole beam as well as for the individual sub-beams are monotone. It means that they reach their maximum in the anchored end of the beam. So the value of the maximum moment (for each sub-beam) can be obtained by setting $x = L$ in (18), (19) and (20). This gives for the sub-beam *1:*

$$M_{0,1}(x) = T_{III,1} \cdot L_{II} + T_{II,1} \cdot (L_I - L_{II}) + T_{I,1} \cdot (L - L_I) \tag{25}$$

and for the sub-beams *2* and *3* it gives

$$M_{0,2}(x) = T_{II,2} \cdot (L_I - L_{II}) + T_{I,2} \cdot (L - L_I) \tag{26}$$

$$M_{0,3}(x) = T_{I,3} \cdot (L - L_I) \tag{27}$$

The explicit formulas for the shear forces $T_{K,i}$ were given in the Sect. 2.1, so the whole solution can be written in a closed analytical form.

When considering the constant beam profile characteristics as and constant material properties along the sub-beams, the maximum of stress appears exactly at the same place as for the moment, i.e. in the anchored end in this case.

The tangential shear stress is usually marginal in the thin, long beams. In case it will be important it can easily be computed from known shear force *T(X)* and sectional area of the beam *A(x)* as:

$$\tau(x) = \frac{T(x)}{A(x)} \tag{28}$$

The known stress can further be used for dimensioning and stress safety control of individual sub-beams.

3 Results and Discussion

The proposed analytical solution of the shear force and bending moment decomposition in cantilever beam to individual sub-beams was presented. The approach is based on the assumption of equal deformation of all sub-beams in the group. The simplifying assumption of the inter-beam force transfer by isolated forces at the places where the clips are attached, makes the solution easy, leading to closed form analytical solution for shear forces, bending moments and resulting stresses in all the individual sub-beams.

The presented work is still in progress. There are many important verifications, validations and extensions to be done before it can be used for any practical design calculations.

The future extensions of the presented work will focus on the following topics:

- *validation* – The presented analytical solution can (and should) be compared with experimental results, or at least with more accurate (e.g. finite-element) numerical solutions. The influence of the interconnection of individual beams in the should be studied in detail.
- *extension* – The examples presented here were based on cantilever beam loaded by a single force at its free end. Both the loading and the type (shape) of the beam can be modified and generalized within the presented framework. The principle of equal deformation is independent the loading type, however the critical issue can be the possibility to perform the calculations analytically, providing closed form solution.

- *implementation* – There raised several questions concerning practical implementation possibilities of the beams. Among them namely the methods of interconnection of sub-beams, treatment of contact surfaces between the beams, material and structural properties of sub-beams.
- *applications* – The motivation for the presented study came from solving practical case of a moving arm of a mechanism. This might be a potentially challenging area of applications, considering the rapid evolution of mechanization and robotics. There are however many other application areas, including e.g. biomechanics or civil and transport engineering.

4 Conclusions

It should be pointed out and understood that the presented approach is simplified and it heavily depends on the assumption of isolated inter-beam shear force transfer at the place of clips. The practical situation can differ from the described one depending especially on the way how the sub-beams are interconnected. If the actual technical realization of the sub-beams attachment and connection is different, the force, moment and stress redistribution might be different as well. The specific case considered here however represents the worst case scenario. This is why the proposed method of load redistribution is suitable mainly for the primary dimensioning and preliminary design calculations of the beam. The final control, using all the complete detailed geometry should be done numerically, using e.g. suitable FEM tools.

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