Applications of Some Extremal and Variational Problems to the Study of Vibrations in Mechanical Systems

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Abstract. The object of the present study is the mathematical side of extremi-related problems. We have studied certain applications of the variational calculus in the domain of vibrations existing in mechanical systems. The most frequently encountered problem in the matter of vibrations is the one of determining the naturally occurring frequencies of a vibrating system. In the practice of engineering, it is the lowest frequency also known as the fundamental one which does present the highest interest. In order to determine the natural frequencies, the usually employed strategy is a combination between Rayleigh's energetical method and the Ritz procedure. Briefly described it does consist in applying a form of vibration or a configuration which should essentially be the intrinsic one of the concerned systems at the moment when it would be situated in the position reflecting the largest possible extent of its translation movement which ought to correspond to its fundamental functioning. We have as well pointed out the advantage which could be taken from applying the integer transformations in the resolution of some extremi-related problems since this choice would render them algebraically shaped and thereby would considerably simplify them.

Keywords: Extremal · Variational problems · Vibrations · Mechanical systems

1 Introduction

The historical trajectory of the iso-perimeter's problem is visible since the Antiquity through the legend which does involve the Carthaginian queen Dido and is easy to follow till Herman Schwartz from Berlin. This was the first ever variational problem encountered in the human history. Dido or Elise is a famous queen from the Antiquity who is known in history as being the founder of Carthage. For two reasons she is also mentioned in the history of mathematics: firstly because she is the first ever woman owning some ascertained knowledge in mathematics and secondly because Carthage had been elevated due to her personal wish. The above mentioned problem is of a mathematical nature and it is only during the XVII-th century that the Swiss scholar Jacques Bernoulli has achieved its scientifically demonstrated solution [1].

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After innumerable wanderings, she has stopped upon the Northern shore of Africa in front of a locality named Byrsa (in Tunis nowadays). Here she has asked from the local inhabitants to be offered a place where she could dwell her own home. However, they have scorned her mockingly by offering her that much of a land surface, which could be contained within an ox, hides (in translation Byrsa does mean, "ox hide"). It is precisely under these circumstances that Dido has proven her wits. She has cut the ox hide into very thin strips. Then she has tied them one to another and surrounded by them a hill from the littoral. Let us suppose the fact that the surface of the ox hide would have been of 4 m^2 from which the cutter strips had a width of 2, 5 mm. The result should be the one of 800 strips long of 2 m each. Therefore, the surrounded perimeter, which is simultaneously the length of the circle consequently drawn around the hill, would have had the value of 1600 m. That is to say approximately 20 ha. For a beginning, it was quite enough.

During the glorious period of the city the length of its defending walls has reached to be 35 Km while in its duly designed and protected harbor could be sheltered a number of warships rising even to 220. The fact that Dido was able to solve that problem is the proof of the fact that in her time she had mastered the knowledge of mathematics: among all of the surfaces, bearing the same perimeter the largest one is the circle. Later known as "the question of the iso-perimeters" this problem has been studied by Bernoulli. Furthermore, since this matter did involve a hill (assimilated to a cone), the area of its lateral surface is larger than the area of its basis (the slanting line is longer than the perpendicular one). The maxima-related problems do stand among the most interesting matters in mathematics. In time, the maxima and minima risen questions have become matters, which by now do involve differential and integer calculations, as well as matters pertaining to linear programming and optimum states' identifying. On the other hand, these procedures are so often made use of in economy, in the theory of probabilities and in many other domains. However, let us return to our subject: the achievement performed then by Dido has been to determine within a plane a closed curve bearing a given length so that it could delimit the largest possible area (that is to say a circle). Now when the variational calculus has been discovered the top scholars were already involved into the study of the particular problems risen by the iso-perimeter's issue and into the one of their analytical generalizations. The theory of the variational calculation does draw its origins from the resolution of the brachistochronical question (that is to say the issue of the shortest time). Jean Bernoulli has formulated it in 1696 (see [2]). The two Bernoulli brothers have both published independently their own respective solutions to it since the methods through which each of these solutions have been obtained were different: Jacques (in paper [1]) and Jean (in paper [2]). Variational calculation may be applied in the mechanics of systems, in hydro-dynamics, in the theory of springiness, in geometrical optics or in the domain of the vibrations, which do pertain to mechanical systems. In the works [3–5] the authors are willingly making use of integer transformations in order to solve some specific problems which do involve vibrations and therefore they do successfully demonstrate the effectiveness of these procedures.

2 A Particular Problem Related to the Iso-perimeter Issue

Some problems pertaining to the *variation calculus* do exist into which the function through which the concerned functional should be brought to its extremi may be itself submitted to a few restrictions. Let us therefore denominate such a problem as an iso-perimeter one. Among all of the types of restrictions which could be imposed to the above-mentioned function we will choose to make use of the one which we will describe through what does follow.

Let us define a new functional $J(y(x))$ and another Lagrangean $M\big(x,\ y(x),\ y'(x)\big)$ and let us take into consideration only the functions $y(x)$ for which the concerned functional should bear a given value £.

Therefore, let us consider the new functional:

$$J(y) = \int_a^b M\big(x,\ y(x),\ y'(x)\big)\, dx \tag{1}$$

together with the former functional

$$I(y) = \int_a^b L\big(x,\ y(x),\ y'(x)\big)\, dx \tag{2}$$

Thus, we do come to formulate in what does follow an iso-perimeter problem.

Proposition 2.1. Among all of the curves $y = y(x) \in C^1[a, b]$

for which the functional $J(y)$ does suppose the existence of a given value £ there is one for which the functional $I(y)$ does suppose the existence of an extreme value.

Insofar the Lagrangeans L and M could be concerned let us suppose the fact that they do have continuous partial derivatives of the first and second orders for $a \le x \le b$ as well as for whatever arbitrary values held by $y(x)$ and $y'(x)$. A widely known iso-perimeter issue is the so-called Dido's problem - later denominated Fisher's problem. We are going to utter it below.

Proposition 2.2. Among the closed curves of length £ the request is to find one which could limit the largest surface. The L and M Lagrangeans are:

$$L\big(x,\ y(x),\ y'(x)\big) = y(x) \tag{3}$$

$$M\big(x,\ y(x),\ y'(x)\big) = \sqrt{1 + y'^2(x)} \tag{4}$$

Consequently, the concerned problem does come to the fact of finding a curve $y = y(x)$ for which the functional

$$J(y) = \int_a^b \sqrt{1 + y'^2(x)}\, dx \tag{5}$$

should hold a given value £ and for which the functional

$$y(x) = \int_a^b y(x)\, dx \tag{6}$$

should suppose the existence of an extreme value.

Now due to this context created by Euler we will demonstrate the principle which it does point out by returning towards the generality of the iso-perimeter issue.

Theorem 2.1. Should the curve $y = y(x)$, be extreme for the functional.

$$I(y) = \int_a^b L\big(x,\ y(x),\ y'(x)\big)\, dx \tag{7}$$

under the conditions

$$J(y) = \int_a^b M\big(x,\ y(x),\ y'(x)\big)\, dx = £, \tag{8}$$

$$y(a) = y_a,\ y(b) = y_b$$

and $y = y(x)$ should not be extreme for the functional J then a constant λ would exist so that the curve $y = y(x)$ could be extreme for the functional:

$$\ddot{I}(y) = \int_a^b L\big(x,\ y(x),\ y'(x)\big) - \lambda M\big(x,\ y(x),\ y'(x)\big)\, dx. \tag{9}$$

Demonstration:

Together with the function $y = y(x)$ let us consider a vicinity of the functions which do have the form

$$\{y(x) + \alpha\eta(x) + \beta\gamma(x)\}_{\alpha,\beta}. \tag{10}$$

Each of the functions from this vicinity does have the same limit as the function $y = y(x)$, $\eta(a) = \eta(b) = 0$, $\gamma(a) = \gamma(b) = 0$.

Should we calculate the value held by the functional $I(y)$ in an arbitrary spot of this vicinity we would find a function which does depend upon α and β:

$$\begin{aligned} I(y(x) + \alpha\eta(x) + \beta\gamma(x)) = \int_a^b & L\big(x,\ y(x) + \alpha\eta(x) + \beta\gamma(x),\ y'(x) \\ & + \alpha\eta'(x) + \beta\gamma'(x)\big)dx = I(\alpha,\ \beta). \end{aligned}$$

But α and β are not independent because of the fact that:

$$J(y(x) + \alpha\eta(x) + \beta\gamma(x)) = \int_a^b M\big(x,\ y(x) + \alpha\eta(x) + \beta\gamma(x), y'(x)$$

$$+\alpha\eta'(x)+\beta\gamma'(x)\big)\,dx = J(\alpha,\ \beta).$$

Therefore:

$$J(\alpha,\ \beta) = £.$$

Should we suppose that J does depend upon β we could make use of the theorem of the implicit functions so that we would come to the three following situations:

- β could be expressed as a o function of α that is to say $\beta = \beta(\alpha)$;
- should $\alpha = 0$ then $\beta = 0$ that is to say $\beta(0) = 0$;
- the derivative of β would be:

$$\beta'(\alpha) = \frac{d\beta}{d\alpha} = -\frac{\frac{\partial J}{\partial \alpha}}{\frac{\partial J}{\partial \beta}}. \tag{11}$$

For $\alpha = 0$, $\beta = 0$ the arbitrary spot of the vicinity does come to be reduced to a curve $y = y(x)$ which is leading towards an extreme point the functional I. This fact does mean that $\alpha = 0$ is an extreme value for a function $I(\alpha,\ \beta) = I(\alpha,\ \beta(\alpha))$ and that according to the requirements of an extreme we should have:

$$\int_a^b \frac{\partial L}{\partial y}\eta(x)\,dx + \int_a^b \frac{\partial L}{\partial y'}\eta'(x)\,dx + \frac{d\beta}{d\alpha}\int_a^b \frac{\partial L}{\partial y}\gamma(x)\,dx + \frac{d\beta}{d\alpha}\int_a^b \frac{\partial L}{\partial y'}\gamma'(x)\,dx = 0. \tag{12}$$

Let us integer through parts and we should obtain:

$$\int_a^b \frac{\partial L}{\partial y'}\eta'(x)\,dx = \frac{\partial L}{\partial y'}\,\eta\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\eta(x)\,dx \tag{13}$$

For as long as $\eta(a) = \eta(b) = 0$, the result would be:

$$\int_a^b \frac{\partial L}{\partial y'}\eta'(x)\,dx = -\int_a^b \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\eta(x)\,dx. \tag{14}$$

Similarly

$$\int_a^b \frac{\partial L}{\partial y'}\gamma'(x)\,dx = \frac{\partial L}{\partial y'}\,\gamma\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\gamma(x)\,dx \tag{15}$$

and for as long as $\gamma(a) = \gamma(b) = 0$ the result would be:

$$\int_a^b \frac{\partial L}{\partial y'}\gamma'(x)\,dx = -\int_a^b \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\gamma(x)\,dx. \tag{16}$$

Should we take into consideration the relationships (13) and (14) the extreme requirement (12) would become:

$$\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\,dx + \frac{d\beta}{d\alpha}\left\{\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\gamma(x)\,dx\right\} = 0, \tag{17}$$

and should we take into consideration the fact that

$$\beta'(\alpha) = \frac{d\beta}{d\alpha} = -\frac{\frac{\partial J}{\partial \alpha}}{\frac{\partial J}{\partial \beta}}, \tag{18}$$

we would obtain:

$$\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\,dx - \frac{\frac{\partial J}{\partial \alpha}}{\frac{\partial J}{\partial \beta}}\left\{\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\gamma(x)\,dx\right\} = 0. \tag{19}$$

On the other hand, should we integer through parts and take into consideration the fact that $\eta(a) = \eta(b) = 0$ we would obtain:

$$\frac{\partial J}{\partial \alpha} = \int_a^b \left[\frac{\partial M}{\partial y} - \frac{d}{dx}\left(\frac{\partial M}{\partial y'}\right)\right]\eta(x)\,dx. \tag{20}$$

Similarly, should we take into consideration the fact that $\gamma(a) = \gamma(b) = 0$, we would obtain:

$$\frac{\partial J}{\partial \beta} = \int_a^b \left[\frac{\partial M}{\partial y} - \frac{d}{dx}\left(\frac{\partial M}{\partial y'}\right)\right]\gamma(x)\,dx. \tag{21}$$

When we should take into consideration the relationships (20) and (21) the relationship (19) would become:

$$\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\,dx - \frac{\int_a^b \left[\frac{\partial M}{\partial y} - \frac{d}{dx}\left(\frac{\partial M}{\partial y'}\right)\right]\eta(x)\,dx}{\int_a^b \left[\frac{\partial M}{\partial y} - \frac{d}{dx}\left(\frac{\partial M}{\partial y'}\right)\right]\gamma(x)\,dx} \cdot \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\gamma(x)\,dx = 0. \tag{22}$$

Should we make use of the denotation:

$$\lambda = \frac{\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\gamma(x)\,dx}{\int_a^b \left[\frac{\partial M}{\partial y} - \frac{d}{dx}\left(\frac{\partial M}{\partial y'}\right)\right]\gamma(x)\,dx}, \tag{23}$$

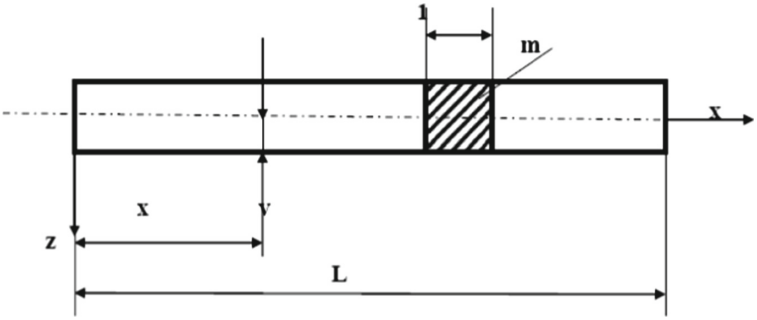

Fig. 1. Straight bar geometry.

The relationship (22) may be written as:

$$\int_a^b \left\{ \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) \right] - \lambda \left[\frac{\partial M}{\partial y} - \frac{d}{dx}\left(\frac{\partial M}{\partial y'} \right) \right] \right\} \eta(x)\, dx = 0. \tag{24}$$

Should we take into consideration the fact that $\eta(x)$ does satisfy to the requirements of the fundamental lemma then the Eq. (24) would lead us to the equation:

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) - \lambda \left[\frac{\partial M}{\partial y} - \frac{d}{dx}\left(\frac{\partial M}{\partial y'} \right) \right] = 0, \tag{25}$$

which may be written under the form:

$$\frac{\partial}{\partial y}(L - \lambda M) - \frac{d}{dx}\left[\frac{\partial L}{\partial y'}(L - \lambda M) \right] = 0. \tag{26}$$

Finally let us remark the fact that the Eq. (26) is Euler's equation for the functional $\ddot{I}(y)$ where:

$$\ddot{I}(y) = \int_a^b L\big(x,\ y(x),\ y'(x)\big) - \lambda M\big(x,\ y(x),\ y'(x)\big)\, dx, \tag{27}$$

Therefore, the theorem is demonstrated.

3 Determining of Natural Frequencies

Our goal is now to determine the pulsation which does correspond to the fundamental vibration modality of a straight bar of which for a length unit the mass is m while the bending rigidity has the constant value of EI_y (see Fig. 1).

Solution: We will apply the Rayleigh method. In the differential equation of the axis of the bent bar:

$$EI_y \frac{\partial^4 v}{\partial x^4} = p(x), \tag{28}$$

we will consider $p(x)$ as being the bar's inertial force itself – according to the principle of d'Alembert – so that the differential equation of the bar's free vibrations should have the form:

$$EI_y \frac{\partial^4 v}{\partial x^4} + m \frac{\partial^2 v}{\partial t^2} = 0, \tag{29}$$

to which we will associate for example the limiting requirements:

$$\begin{array}{l} x = 0,\ x = L\ \Rightarrow v = 0 \\ x = \frac{L}{2}\ \Rightarrow \frac{\partial v}{\partial x} = 0 \end{array}. \tag{30}$$

In order to obtain the pulsation which does correspond to the fundamental vibration modality of a straight bar we will equate the expression of its maximum kinetic energy with the one of its maximum deforming's potential energy. For the considered bar the deforming energy E_p is:

$$E_p = \frac{1}{2} \int_0^L EI_y \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx, \tag{31}$$

while the kinetic energy is:

$$E_c = \frac{1}{2} \int_0^L m \left(\frac{\partial v}{\partial t} \right)^2 dx. \tag{32}$$

Assuming the fact that the concerned vibration is a harmonic one that is to say:

$$v(x, t) = V(x) \cos(\omega t) \tag{33}$$

from the Rayleigh requirement $\left(E_p\right)(E_c)_{max_{max}}$ the pulsation should result as expressed under the form:

$$\omega^2 = \frac{\int_0^L EI_y \left(\frac{\partial^2 V}{\partial x^2} \right)^2 dx}{\int_0^L mV^2\, dx}. \tag{34}$$

In order to determine from the relationship (12) the value of ω^2 we should take into consideration a form of $V(x)$ which could satisfy to the limiting requirements (11) but not necessarily to the movement Eq. (10). Such a form is:

$$V(x) = 1 - \cos\left(\frac{2\pi x}{L} \right), \tag{35}$$

which, when substituted in the relationship (34) would lead us, with $\frac{EI_y}{m} = k^2L^4$, at the expression of the fundamental pulsation under the form:

$$\omega^2 = k^2L^4 \frac{\int_0^L \left\{ \frac{\partial^2\left[1-\cos\left(\frac{2\pi x}{L}\right)\right]}{\partial x^2} \right\}^2 dx}{\int_0^L \left[1-\cos\left(\frac{2\pi x}{L}\right)\right]^2 dx}. \tag{36}$$

By performing the calculation in the relationship (36) the approximate value of the fundamental pulsation (of the slightest pulsation) should result as being: $\omega_1 = 22,792$ k.

4 Extreme Spots of Functionals

Our goal is now to determine the extreme spot of the concerned functional as well as its nature under the requirements that:

$$F : D \to R,$$

$$F[y,\ z] = \int_0^{\frac{\pi}{2}} \left[(y')^2 + (z')^2 + 2yz \right] dx, \tag{37}$$
$$D = \left\{ (y,\ z) \in C^1\left(\left[0,\ \tfrac{\pi}{2}\right]\right) \middle| y(0) = z(0) = 0,\ y\left(\tfrac{\pi}{2}\right) = 1,\ z\left(\tfrac{\pi}{2}\right) = -1 \right\}.$$

Solution:
The Euler-Lagrange system is:

$$\begin{cases} y'' - z = 0 \\ z'' - y = 0 \end{cases}, \tag{38}$$

It will be solved by two different methods.

4.1 Method 1. Applying the Laplace Transformation

Should we apply to the equations of this system the unilateral Laplace transformation in respect to the variable x we would obtain for the unknown $y(x)$ and $z(x)$ the system of algebraic equations expressed through the formers' Laplace images $\tilde{y}(s)$ and $\tilde{z}(s)$ under the form:

$$\begin{cases} s^2\tilde{y}(s) - sy(0) - y'(0) - \tilde{z}(s) = 0 \\ s^2\tilde{z}(s) - sz(0) - z'(0) - \tilde{y}(s) = 0 \end{cases}. \tag{39}$$

When solving this system elementarily we should obtain the solution of the differential equation' system (38) expressed through Laplace images:

$$\begin{cases} \tilde{y}(s) = \frac{y(0)s^3 + y'(0)s^2 + z(0)s + z'(0)}{s^4 - 1} \\ \tilde{z}(s) = \frac{z(0)s^3 + z'(0)s^2 + y(0)s + y'(0)}{s^4 - 1} \end{cases}. \tag{40}$$

Should we apply to the equations of system (40) the reversed Laplace transformation in respect to the variable x we would obtain the solution of the differential equation' system (38) under the form:

$$\begin{cases} y(x) = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x, \\ z(x) = C_1 e^x + C_2 e^{-x} - C_3 \cos x - C_4 \sin x, \end{cases} \tag{41}$$

where:

$$C_1 = \frac{1}{4}\big[y(0) - y'(0) + z(0) - z'(0)\big],\ C_2 = \frac{1}{4}\big[y(0) + y'(0) + z(0) + z'(0)\big],$$
$$C_3 = \frac{1}{2}\big[y(0) - z(0)\big],\ C_4 = \frac{1}{4}\big[y'(0) - z'(0)\big]. \tag{42}$$

4.2 Method 2. The Classical Method

Let us remark the fact that the solutions of the Euler-Lagrange system (38) do bear the form (41) which we have previously obtained through the first method. From $(y,\ z) \in D$ we do obtain: $C_1 = C_2 = C_3 = 0,\ C_4 = 1$, which - when substituted into the system (38) - is leading us towards the conclusion that the line through which the extreme is accomplished is provided by:

$$\begin{cases} \bar{y} = \sin x \\ \bar{z} = -\sin x \end{cases}. \tag{43}$$

The Legendre requirements are:

$$D_1 = F_{y'y'} = 2,\ D_2 = \begin{vmatrix} F_{y'y'} & F_{y'z'} \\ F_{z'y'} & F_{z'z'} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4, \tag{44}$$

The consequent result is that the minimum value for the functional is reached through the extreme spot of $(sinx,\ -sinx)$. The minimum value is easy to obtain:

$$F(\sin x, -\sin x)_{\min} \tag{45}$$

5 Conclusions

The main inconvenience of the analytical studies which do concern the extremum issues is constituted by the lacunae that exist in the study of the existence of such extremi. It is Weierstrass who for the first time ever has invoked this problem. A. Hurwitz has provided a new demonstration by making use of the trigonometric lines. G. Cramer, S. Lhuilier and especially the all-comprehensive Steiner have also explored this path: one among the first demonstrations provided by this latter has proved itself to be useful even in our own days: it is known as the quadrilateral's method. As a conclusion: through the present study we have attempted to slightly contribute insofar are concerned the extremi-related issues and their respective applications within the technical field. We have as well demonstrated the opportunity of making use of the integer transformations for the purposes of the operational calculation.

References

1. Bernoulli Jakob (Jacques): Propositiones arithmeticae de seriebus infintis aerumque summa finita, Basel, 1689 - 1704.
2. Bernoulli Johann (Jean): Lectiones mathematicae de methodo integralium alisque, (1742).
3. Băgnaru D., Rinderu P.: Vibrations of the mechanical systems, LOTUS Publishing House Craiova, (1998).
4. Bolcu D., Băgnaru D.: Applications of the integral transformations in Mechanics, Universitaria Craiova Publishing House, (2004).
5. Stănescu M.M., Băgnaru D. Gh.: Extremal and variational problems, Universitaria Craiova Publishing House, (2017).