Analyzing Double Pendulum Dynamics with Approximate Entropy and Maximal Lyapunov Exponent

Jonathan Ting(✉) and Dan B. Marghitu

Department of Mechanical Engineering, Auburn University, Auburn, AL 36849, USA
jat0076@auburn.edu

Abstract. Two methods were used to study the aperiodicity of a double pendulum based on its chaotic behavior: approximate entropy and maximum Lyapunov exponents. These methods were applied to analyze the aperiodicity of a signal obtained from the angular velocity of the first pendulum. The nonlinear system of differential equations were modeled using Langrage's equation of motion and solved using the computational software MATLAB. Both maximal Lyapunov exponents and approximate entropy values exhibited an increase in magnitude with increasing initial conditions.

Keywords: Double pendulum · Approximate entropy · Lyapunov exponent

1 Introduction

The double pendulum is known for its chaotic behavior; that is, the dynamic behavior of the double pendulum is highly sensitive to its initial conditions which, in turn, makes the double pendulum unpredictable. If the initial condition angles are small, the double pendulum will exhibit regular motion. However, if large angle initial conditions are imposed, the double pendulum will exhibit aperiodic motion.

Two methods will be used to study the aperiodicity of a double pendulum based on its nonlinear behavior: approximate entropy and maximal Lyapunov exponents. These methods will be applied to the signal obtained from the angular velocity of the first pendulum. Both methods are known to measure the predictability of signals obtained from dynamic systems. In previous literature, maximal Lyapunov exponents and Poincare maps were used to study the chaotic motion of a double pendulum with the increasing initial conditions [3]. It was found that the maximum Lyapunov exponent increases with increasing initial conditions [3]. Approximate may offer another way for analyzing the aperiodic behavior of the double pendulum.

Informational entropy, introduced by Claude Shannon, is a quantitative measurement of information in a discrete information source. This measurement is based on the uncertainty correlated to a variable's probability [15]. Signal entropy uses the same reasoning but measures the information in a discrete time signal [2, 4, 14]. A periodic signal would have less information because there is less uncertainty in the signal [2, 4]. Therefore,

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signal entropy is an objective measurement of a discrete time signal periodicity [2, 4, 7, 14, 18].

Signal entropy is a widely researched topic in informational theory; so, there are several forms of signal entropy as well as numerous applications. For example, researchers measured damage condition of a four-story steel truss based on the ambient vibration of the structure using multi-scale entropy [19]. Multi-scale fuzzy entropy, an improvement of multi-scale entropy, was applied on vibrations of roller bearings to detect roller bearing fault [8]. Permutation entropy was used to determine the working status of rotary machines based off vibration signals from rotary machines [17]. The use of approximate entropy, introduced by Pincus, is popular in the realm of biomedical sciences by detecting alterations in a signal from subjects effected with a wide class of diseases [10–12]. For example, approximate entropy was used to study the electroencephalogram of Alzheimer patients [1]. Furthermore, approximate entropy has been used to monitor the vibration of meshed gears [5]. Approximate entropy has been applied to numerous nonlinear, aperiodic signals including physiological signals and mechanical vibrations, and is widely used for studying aperiodic, nonlinear systems.

It is important to note that approximate entropy has several shortcomings. For one, approximate entropy calculations are highly sensitive to the length of the data [18]. The approximate entropy has inherent bias as a due to self-matching in its algorithm [4, 13, 14]. This would yield higher measurements of aperiodicity. Sample entropy, developed my Richman and Moore, is an improvement to approximate entropy by eliminating this bias [4, 13, 14].

There are physiological systems like a pendulum. For example, researchers propose that human gait is analogous to an inverted pendulum, and approximate entropy has been applied to human gait to measure the randomness of a subject gait [5, 9].

Due to the aperiodicity, nonlinear nature of a double pendulum, approximate entropy, in conjunction with maximal Lyapunov exponents, may offer another way in studying the chaotic nature of the double pendulum. To our best knowledge, the approximate entropy has not been applied in studying the nonlinear dynamics of a double pendulum.

2 Approximate Entropy

Approximate entropy is an algorithmic technique that estimates the aperiodicity of a discrete time series [4, 14]. The algorithm separates a signal with N number of data points into vectors of length m. The number of separated vectors is:

$$\text{Number of separated vectors} = \mathrm{N} - \mathrm{m} + 1 \tag{1}$$

The algorithm then checks if the separated vectors match with each other. If difference between the components of the vectors is within a set tolerance r the vectors are considered matching and thus a match is counted. Usually, the value of r is determined as 0.15 times the standard deviation of the signal [14].

Take a signal with data points $x(1)$, $x(2)$, …, $x(\mathrm{N})$. The data points are formed into vectors sequences V (1) through $V(N - m)$. Individual vectors are then compared with all the separated vectors pertaining to the set tolerance. For example, the first vector, V (1), is compared with all the separated vectors including itself. Then, V (2) is compared with all the separated vectors the V (3) and so forth until the final vector $V(N - m + 1)$ is compared with the rest of the vectors. This relationship must be satisfied for the vectors to be considered matching:

$$|V(j) - V(i)| \leq r, \tag{2}$$

where j is an integer ranging from $1 \leq j \leq N - m + 1$ and i is an integer ranging from $1 \leq i \leq N - m + 1$. For the first vector, $j = 1$ because V (1) is being compared with the rest of the vectors. n_j is how many times the vector matches with the other separated vectors. A_j is calculated as:

$$A_j = \frac{n_j}{N - m + 1}, \tag{3}$$

which is known as the probability of the vector reoccurring in the signal within the tolerance. The process of calculating A_j is repeated for the rest of the vectors. All the values of A_j are then summed to calculate C_m

$$C_m = \frac{\sum_{k=1}^{N-m+1} \frac{n_j}{N-m+1}}{N - m + 1}. \tag{4}$$

The entire process is repeated; instead, the vector is separated in length of $m + 1$. $C_{\mathrm{m+1}}$ is calculated as:

$$C_{m+1} = \frac{\sum_{i=1}^{N-(m+1)+1} \frac{n_{m+1}}{N-(m+1)+1}}{N - (m + 1) + 1} = \frac{\sum_{i=1}^{N-m} \frac{n_m}{N-m}}{N - m}. \tag{5}$$

Taking the values of C_{m+1} and C_m, approximate entropy be calculated as:

$$\text{approximate entropy} = -\ln\left(\frac{C_{m+1}}{C_m}\right). \tag{6}$$

If the signal is periodic, the C_m and C_{m+1} values would be close together which, in turn, decreases the logarithmic value. So, the less periodic the signal, the greater the sample entropy value because the values of C_m and C_{m+1} are farther from each other. For example, a sine wave, with an initial vector length of 2 and a tolerance 0.15 times the standard deviation, has an approximate entropy value of 0.140. On the other hand, a HÃl'non Map has an approximate entropy value of 0.560 under the same conditions. It is important to note that the approximate entropy bias is negligible when the data has a large number of sample [14].

For this study, an initial vector length of 4 and a tolerance of 0.15 times the standard deviation of the data was set before applying the approximate entropy algorithm.

3 Maximal Lyapunov Exponent

The Lyapunov exponent is a quantity that characterizes the divergence of two trajectories with infinitely close initial conditions. The magnitude of divergence between the two trajectories at any point of time is denoted as $\delta(t)$. $\delta(t)$ can be quantified as:

$$\delta(t) \approx \delta_0 e^{\lambda t}, \psi \tag{7}$$

where λ is the Lyapunov exponent. A negative Lyapunov exponent would mean that the points will eventually converge to a single value with increasing time. A positive Lyapunov would mean that the points would diverge from each other. As a result, a positive Lyapunov exponent means that the system is sensitive to its initial conditions and is hard to predict while a negative Lyapunov exponent is a non-chaotic system.

4 Mathematical Model

A double pendulum is depicted in Fig. 1. The particle 1 at A has the mass m_1 and the particle 2 at B has the mass m_2. The length of OA is L_1 and the length of AB is L_2. The gravitational acceleration is g. For this study, the mass and the length of both pendulums is 1 kg and 1 m, respectively.

The first generalized coordinate $q_1(t)$ denotes the radian measure of the angle between the vertical axis and OA and the second generalized coordinate $q_2(t)$ denotes the radian measure of the angle between the vertical axis and AB.

A cartesian reference frame xyz with the unit vectors $[\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}]$ is selected. The unit vector $\boldsymbol{j}$ is vertical and upward and the unit vector $\boldsymbol{i}$ is horizontal and to the right. The position vector of the particle at A is

$$\boldsymbol{r}_1 = \boldsymbol{r}_A = L_1 \sin q_1(t)\boldsymbol{i} - L_1 \cos q_1(t)\boldsymbol{j}. \tag{8}$$

The position vector of the particle at B is

$$\boldsymbol{r}_2 = \boldsymbol{r}_B = \left[L_1 \sin q_1(t) + L_2 \sin q_2(t)\right]\boldsymbol{i} - \left[L_1 \cos q_1(t) + L_2 \cos q_2(t)\right]\boldsymbol{j}. \tag{9}$$

The velocities of the particles at A and B are:

$$\boldsymbol{v}_A = \frac{d\boldsymbol{r}_A}{dt} = L_1 \dot{q}_1 \cos q_1(t)\boldsymbol{i} + L_1 \dot{q}_1 \sin q_1(t)\boldsymbol{j}, \tag{10}$$

$$\boldsymbol{v}_B = \frac{d\boldsymbol{r}_B}{dt} = \dot{\boldsymbol{r}}_B = \left[L_1 \dot{q}_1 \cos q_1(t) + L_2 \dot{q}_2 \cos q_2(t)\right]\boldsymbol{i} + \left[L_1 \dot{q}_1 \sin q_1(t) + L_2 \dot{q}_2 \sin q_2(t)\right]\boldsymbol{j}. \tag{11}$$

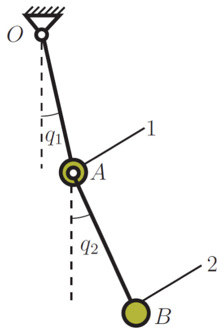

Fig. 1. Double pendulum.

Kinetic energy
The kinetic energy of particle 1 is

$$T_1 = \frac{1}{2} m_1 \boldsymbol{v}_A \cdot \boldsymbol{v}_A = \frac{1}{2} m_1 L_1^2 \dot{q}_1^2, \tag{12}$$

and the kinetic energy of the particle 2 is

$$T_1 = \frac{1}{2} m_2 \boldsymbol{v}_B \cdot \boldsymbol{v}_B = \frac{1}{2} m_2 \big[L_1 \dot{q}_1 \cos q_1(t) + L_2 \dot{q}_2 \cos q_2(t) \big]^2$$
$$+ \frac{1}{2} m_2 \big[L_1 \dot{q}_1 \sin q_1(t) + L_2 \dot{q}_2 \sin q_2(t) \big]^2. \tag{13}$$

The total kinetic energy is

$$T = T_1 + T_2. \tag{14}$$

Generalized forces
The forces that act on 1 at A is the gravity force

$$\boldsymbol{F}_A = -m_1 g \boldsymbol{j}. \tag{15}$$

The gravity force acts on mass 2 at B

$$\boldsymbol{F}_{\boldsymbol{A}} = -m_2 g \boldsymbol{j}. \tag{16}$$

There are two generalized forces. The generalized force associated to q_1 is

$$Q_1 = \boldsymbol{F}_A \cdot \frac{\partial \boldsymbol{r}_A}{\partial q_1} + \boldsymbol{F}_B \cdot \frac{\partial \boldsymbol{r}_B}{\partial q_1}. \tag{17}$$

The generalized force associated to q_2 is

$$Q_2 = \boldsymbol{F}_A \cdot \frac{\partial \boldsymbol{r}_A}{\partial q_2} + \boldsymbol{F}_B \cdot \frac{\partial \boldsymbol{r}_B}{\partial q_2}. \tag{18}$$

The Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1,$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2. \tag{19}$$

5 Results

The largest Lyapunov exponent and approximate entropy theories were applied to the signals obtained from the angular velocity of the first pendulum. The differential equation obtained from the inverted pendulum was solved using the MALAB function ode113 [20]. Ode113 was used to solve all the instances of angular velocity from 0 to 50 s in increments of 0.01 s. The angular velocity initial conditions for both pendulums are 0 rad/s and the positional initial condition of the first pendulum is 0 radians for all instances.

The goal in analyzing the angular velocity of a double pendulum is to find a trend in approximate entropy values with initial conditions ranging from 45 degrees to 135 degrees. These findings will be compared to findings from maximal Lyapunov exponents. Our analysis confirms that there is an increase in maximal Lyapunov exponent when the position initial conditions of q_1 were increased. The other initial conditions were zero. Figure 2 shows the maximal Lyapunov exponent values with varying initial conditions from 45 to 135 degrees in 0.5- degree increments.

A linear regression fit was applied to the maximal Lyapunov exponent data points to help determine a trend. Based on the positive slope calculated by the linear fit, there is an increase in maximal Lyapunov exponent. As a result, the pendulum maximal Lyapunov exponent is increasingly with increasing initial angle of the inner pendulum. Figure 3 shows the approximate entropy values with the same varying initial conditions. The linear fit for the approximate entropy values also shows that there is an increase in approximate entropy values with increasing initial conditions. However, there is a smaller rate of change in approximate entropy values than maximal Lyapunov values.

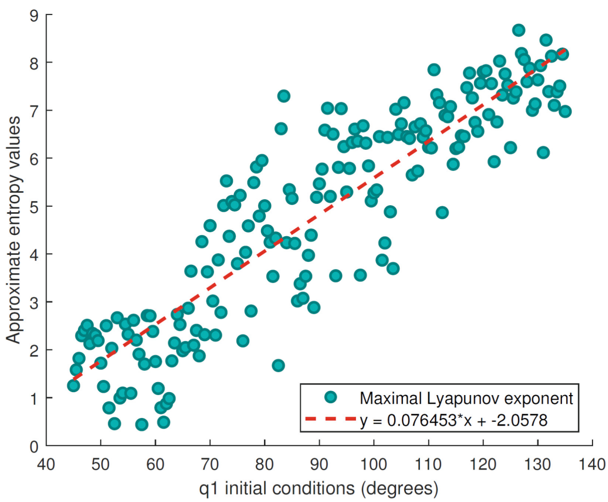

Fig. 2. Maximum Lyapunov exponents with initial angle $q_1(0)$

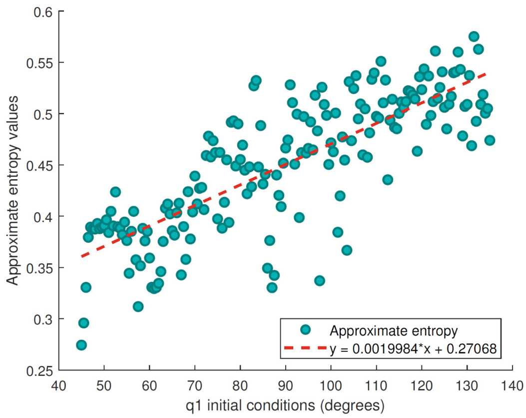

Fig. 3. Approximate Entropy with initial angle $q_1(0)$

6 Conclusions

The purpose of this paper was to study the chaotic behavior of a double pendulum using approximate entropy values and Lyapunov exponents. Approximate entropy values and maximal Lyapunov exponents increased with increasing initial conditions for the inner angle of the first pendulum. In the future studies, sample entropy may offer a more accurate representation of the dynamics of a double pendulum.

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