Analysis of a Cantilevered Beam Under Parametric Uncertainties

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Abstract. This paper presents the analysis of a cantilevered beam under parametric uncertainties using FEMCAS a new software developed for structural analysis. The theoretical approach is described, including linear analysis module with parametric uncertainties - Monte Carlo simulation, the Perturbation method and the Neumann method. The perturbation method involves a Taylor series extension of the stiffness matrix and is a numerically less expensive alternative to the Monte Carlo method. The Neumann method applied in finite element analysis is known in the literature as Spectral Stochastic Finite Element Method and models the uncertainties using a modified first-order stochastic perturbation method together with a truncated Karhunen-Loeve expansion instead of the Taylor series. In the absence of parametric uncertainties, a modal analysis and forced vibration analysis were performed. Under parametric uncertainties, good correlation between Monte Carlo, Perturbation and Neumann methods was obtained. However, Perturbation and Neumann method obtained much lower simulation times which made those solutions much cheaper from computational perspective.

Keywords: Parametric Uncertainties · Monte Carlo · Perturbation · Neumann · Modal Analysis · FEA · FEM · Simulation

1 Introduction - Parametric Uncertainties

The parametric uncertainties are frequently considered in the analyses of various situations, like control of nonlinear processes, dynamics and mass balance, optimization of trajectories of robots or various physical predictions. For example, paper [1] approaches a procedure for the design of a robust controller for a nonlinear process, taking into account the various issues arising in the design. There are used the main theoretical results from the Literature about this topic. An extended model is set-up, linking performance and robustness to the control law, having as result a state feedback control law which guarantees robust performance.

A generalized uncertainty principle is described in paper [2] that reviews some of the physical predictions of the GUP, and focuses on the bounds that present experimental tests can put on the value of the deformation parameter β. There are described a theoretical value computed for β, and comment on the vast parameter region.

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This research studies the behavior of a cantilever beam in the presence of uncertainties using a novel software based on stochastic finite element method.

2 Theoretical Considerations

2.1 Monte Carlo Simulation

Degradation of stiffness parameters is an aspect that is not considered by current simulations, which consider the model to be geometrically and materially perfect. This is not the case in physical models because they include imperfections in fabrication and operation. These imperfections can be quantified using statistical computation and modelled as uncertainties. Geometrical and material uncertainty is considered for bending stiffness parameters. The bending stiffness parameters can be modeled as $\mathrm{EI} = \overline{\mathrm{EI}} + \widetilde{\mathrm{EI}}$. E is the modulus of elasticity (material property), I is the moment of inertia of the cross-section of the beam (geometrical property). The mean values $\overline{\mathrm{EI}}$ are assumed to be much larger than the root mean square of the variability of the random field represented by $\widetilde{\mathrm{EI}}$. The random field is assumed to be Gaussian distributed with zero mean having standard deviation, σ_{EI}, much smaller than the corresponding mean value. This implies that the stiffness parameters form random fields with positive value.

Monte Carlo simulation consists in the numerical accumulation of a population corresponding to the random quantities in the physical problem, solving the problem associated with each member of the respective population and obtaining a population corresponding to the random response quantities. This population can then be used to obtain the statistics of the response variables.

In order to include uncertainties, random fields will be generated for each finite element and assembled in the general matrix. Monte Carlo simulation consists of solving the equations of motion, which includes random variables divided by elements, several times. The higher the number of solvers, the higher the accuracy is. A minimum of 1000 runs is required for acceptable accuracy. This is computational expensive, especially if the simulation models will be large (high nodes and elements). This method has a high accuracy compared to other stochastic calculation methods.

2.2 The Perturbation Method

The perturbation method applied in finite element analysis is known in the literature as PFEM (Probabilistic Finite Element Method). We consider the equation of motion in matrix form:

$$[\mathrm{M}]\left\{\ddot{\mathrm{U}}\right\} + [\mathrm{C}]\left\{\dot{\mathrm{U}}\right\} + [\mathrm{K}]\{\mathrm{U}\} = \{\mathrm{F}\} \tag{1}$$

where, [M] is the mass matrix, [C] is the damping matrix, [K] is the stiffness matrix, {F} is the applied external force vector, {U} is the displacement vector, $\left\{\dot{\mathrm{U}}\right\}$ is the velocity vector and $\left\{\ddot{\mathrm{U}}\right\}$ is the acceleration vector.

Geometrical and material uncertainty is taken into account for bending stiffness parameters. The bending stiffness parameters are shown below:

$$\mathrm{EI} = \overline{\mathrm{EI}} + \widetilde{\mathrm{EI}} \tag{2}$$

where E is the modulus of elasticity (material property), I is the moment of inertia of the cross-section of the beam (geometrical property). The mean values $\overline{\mathrm{EI}}$, are assumed to be much larger than the root mean square of the variability of the random field represented by $\widetilde{\mathrm{EI}}$. The random field is assumed to be Gaussian distributed with zero mean, with the standard deviation σ_{EI} being much smaller than the corresponding mean value. This implies that the stiffness parameters form random fields with positive value.

Returning to the equation of motion (1), the stiffness matrix [K], includes the bending stiffness parameters EI. The degradation of stiffness parameters can be quantified using Eq. (2). In finite element analysis, the stiffness matrix includes values split into elements and nodes. To include uncertainties, random fields will be generated for each individual element which will be introduced into the overall matrix. Mass, damping, and external forces are considered deterministic.

The perturbation method involves Taylor series extension of the stiffness matrix and is a numerically less expensive alternative to the Monte Carlo method.

In mathematics, a Taylor series is a representation of a function as an infinite sum of terms computed from the derivative values of that function at a point. The stiffness matrix can be represented in Taylor series according to the relation [3]:

$$\mathrm{K} = \mathrm{K}_0 + \sum_{i=1}^{N} \mathrm{K}_i^{\mathrm{I}} a_i + \ldots \tag{3}$$

where K_0 is the mean of matrix K, $\mathrm{K}_i^{\mathrm{I}}$ is the first partial derivative of K with respect to the random variable a_i:

$$\mathrm{K}_i^{\mathrm{I}} = \left.\frac{\partial \mathrm{K}}{\partial a_i}\right|_{\{a\}=0} \tag{4}$$

To solve Eq. (1), it is also necessary to represent the displacement, velocity and acceleration vector in Taylor series:

$$\mathrm{U} = \mathrm{U}_0 + \sum_{i=1}^{N} \mathrm{U}_i^{\mathrm{I}} a_i + \ldots; \ \dot{\mathrm{U}} = \dot{\mathrm{U}}_0 + \sum_{i=1}^{N} \dot{\mathrm{U}}_i^{\mathrm{I}} a_i + \ldots;$$
$$\ddot{\mathrm{U}} = \ddot{\mathrm{U}}_0 + \sum_{i=1}^{N} \ddot{\mathrm{U}}_i^{\mathrm{I}} a_i + \ldots \tag{5}$$

where U_0, $\dot{\mathrm{U}}_0$, $\ddot{\mathrm{U}}_0$ are the mean values and $\mathrm{U}_i^{\mathrm{I}}$, $\dot{\mathrm{U}}_i^{\mathrm{I}}$, $\ddot{\mathrm{U}}_i^{\mathrm{I}}$, $\mathrm{U}_{ij}^{\mathrm{II}}$, $\dot{\mathrm{U}}_{ij}^{\mathrm{II}}$, $\ddot{\mathrm{U}}_{ij}^{\mathrm{II}}$ are the first and second derivatives of the random variables a_i and a_j, evaluated at $\{a\} = 0$.

Considering the Taylor series representation up to order 1, the displacement, velocity and acceleration vectors are obtained by successive solvers as follows:

0-order equations:

$$\mathrm{M}\ddot{\mathrm{U}}_0 + \mathrm{C}\dot{\mathrm{U}}_0 + \mathrm{K}_0\mathrm{U}_0 = \mathrm{F} \tag{6.1}$$

First order equations:

$$\mathrm{M}\ddot{\mathrm{U}}_i^{\mathrm{I}} + \mathrm{C}\dot{\mathrm{U}}_i^{\mathrm{I}} + \mathrm{K}_0\mathrm{U}_i^{\mathrm{I}} = -\mathrm{K}_i^{\mathrm{I}}\mathrm{U}_0 \tag{6.2}$$

2.3 Neumann Method

The Neumann method applied in finite element analysis is known in the literature as SSFEM (Spectral Stochastic Finite Element Method) [4]. Uncertainties are modeled using a modified first-order stochastic perturbation method together with a truncated Karhunen-Loeve expansion instead of the Taylor series [5]. The Taylor series is used only for the displacement vector expansion.

The Neumann method involves the Karhunen-Loeve (K-L) series extension of the stiffness matrix and is a numerically less expensive alternative to the Monte Carlo method, with higher accuracy than the perturbation method.

The continuous random field model will be discretized using the Karhunen-Loeve (K-L) series. Geometric and material uncertainty is taken into account for the bending stiffness parameters. The bending stiffness parameters are shown below [5]:

$$\mathrm{EI_x}(\mathrm{y}, \theta) = \overline{\mathrm{EI_x}} + \sum_{\mathrm{r}=1}^{\mathrm{NKL}} \sqrt{\lambda_\mathrm{r}}\,\mathrm{f_r}(\mathrm{y})\mathrm{a_r}(\theta) \tag{7}$$

where $\mathrm{a_r}(\theta)$ are orthonormal random variables of mean 0 with θ belonging to the space of random events; NKL is the number of K-L series; $\mathrm{f_r}$ and λ_r are functions and eigenvalues of the covariance $C(y_1, y_2)$ [4, 5]:

$$\int_{-L/2}^{L/2} C(y_1, y) f(y_1) dy_1 = \lambda f(y) \tag{8}$$

where$C(y_1, y) = \sigma_{EI_x}^2 e^{-|y-y_1|/l_{cor}}$, L is the length of the bar, l_{cor} is the correlation length, $l_{cor}/L = 1$, $\sigma_{\mathrm{EI_x}}^2$ is the variance of the random field.

Equation (8) can be solved analytically for the one-dimensional case [4, 5].

The elementary matrix becomes:

$$\mathrm{K_e}(\theta) = \mathrm{K_{e0}} + \sum_{\mathrm{r}=1}^{\mathrm{NKL}} \mathrm{K_{e,r}}\mathrm{a_r}(\theta) \tag{9}$$

Putting the element matrices together we get the global equation:

$$\mathrm{M}\ddot{\mathrm{U}}(\mathrm{t}, \theta) + \mathrm{C}\dot{\mathrm{U}}(\mathrm{t}, \theta) + \mathrm{K}_0\mathrm{U}(\mathrm{t}, \theta) + \left(\sum_{\mathrm{r}=1}^{\mathrm{NKL}} \mathrm{K_{e,r}}\mathrm{a_r}(\theta)\right)\mathrm{U}(\mathrm{t}, \theta) = \mathrm{F}(\mathrm{t}) \tag{10}$$

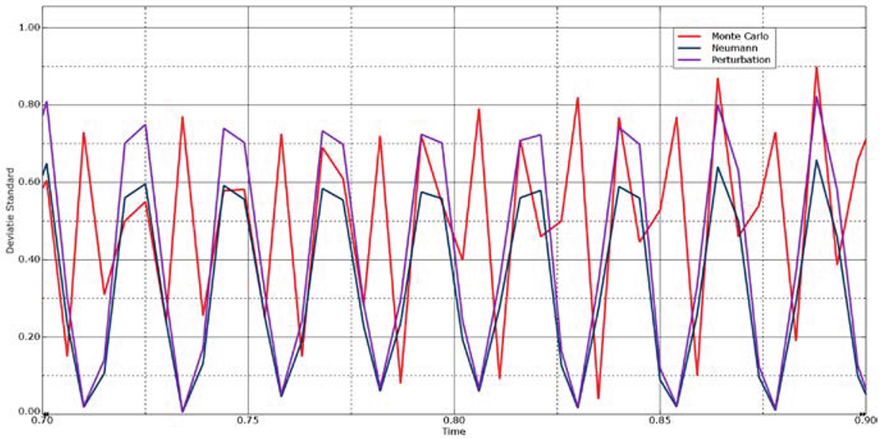

Fig. 1. Standard deviation of displacement response in vertical direction

where F is the vector of external forces.

Applying the first-order perturbation method the displacement vector is represented in Taylor of order one:

$U = U_0 + \sum_{i=1}^{NKL} U_i^I a_i;\ \dot{U} = \dot{U}_0 + \sum_{i=1}^{NKL} \dot{U}_i^I a_i;\ \ddot{U} = \ddot{U}_0 + \sum_{i=1}^{NKL} \ddot{U}_i^I a_i$ (11).

whereU_0, $\dot{U}_0$,$\ddot{U}_0$ are mean values, andU_i^I,$\dot{U}_i^I$,$\ddot{U}_i^I$, the first derivative relating to the random variable a_i evaluated at$\{a\} = 0$.

Considering the Taylor series representation up to order 1, the displacement, velocity and acceleration vectors are obtained by successive solvers as follows:

0-order equations:

$$M\ddot{U}_0 + C\dot{U}_0 + K_0U_0 = F \tag{12.1}$$

First-order equations:

$$M\ddot{U}_i^I + C\dot{U}_i^I + K_0U_i^I = -K_i^IU_0 \tag{12.2}$$

3 Simulation

Uncertainties Are Introduced into the Stiffness Matrix in the Vertical Direction. A Field of Random Variables with Gaussian Distribution with Mean Zero and Standard Deviation 0.01 is Generated.

The results shown in Fig. 1 represent the standard deviation of the displacement at the point of force application in the vertical direction. A good correlation between Monte Carlo, Neumann and Perturbation is observed.

Table 1. Simulation time results, depending on the methods used

	Monte Carlo	Perturbation	Neumann
Simulation time (s)	276	99	22

The analysis time is greatly improved in the case of perturbation and Neumann methods as shown in Table 1.

4 Conclusions

A new structural analysis software is proposed. The software is based on finite element method and can analyze structures in 3-dimensional space, structures that can be approximated by beams of different cross-sections.

The element of novelty for the software is the presence of a module that includes parametric uncertainties. Degradation of stiffness parameters is an aspect that is not considered by current simulations, which assume the model to be geometrically and materially perfect. This is not the case in physical models because they include imperfections in fabrication and operation. These imperfections can be quantified using statistical computation and modelled as uncertainties. There are three types of uncertainties that can be analyzed using FEMCAS software: Monte Carlo, Perturbation, and Neumann method.

A force vibration analyses under parametric uncertainties in FEMCAS is proposed. Good correlation between Monte Carlo, Perturbation and Neumann methods is obtained. However, Perturbation and Neumann methods lead to much lower simulation times which made those solutions much cheaper from computational perspective.

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