Uncertainty Quantification on Financial Valuation Model

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Abstract. The increasing complexity of financial markets and the introduction of sophisticated products have amplified investment volatility and uncertainty. The present work introduces a general framework for addressing uncertainty in financial multi-stage planning problems. To solve this uncertainty quantification problem, the proposed methodology involves randomization of vectors, dimension reduction via KL-expansion, and distribution transformation using the Maximum Entropy principle. Stochastic solvers, such as Monte Carlo, Generalized Moment methods, or Stochastic Collocation methods, are then employed for forward uncertainty propagation, mapping stochastic inputs to output, and generating the probability distribution. This integrated approach aims to enhance financial analysis and planning by providing a comprehensive understanding of the decision-making context and stochastic factors that influence investment outcomes.

Keywords: market uncertainty, multi-stage stochastic process, probability distribution, financial planning, decision-making

1 Introduction

The globalization of financial markets and the introduction of complex products such as financial derivatives have increased investment volatility and uncertainty. The financial rewards for good decisions and the penalties for bad decisions are both huge. Uncertainty plays a crucial role in this process [1]. There is a great need for an integrated approach to financial analysis and planning that incorporates decision context and random factors. So, I propose a general framework for modeling financial planning problems via multi-stage stochastic process.

In this project, the stochastic modeling of the investor's total wealth is explored, and the inputs are randomized and fit Gaussian distribution [2]. After reducing the dimension, stochastic solvers can be performed to quantify uncertainty, i.e., Monte Carlo, Galerkin method, and Stochastic Collocation method. This project aims at understanding the use of stochastic principles in financial valuation model, and the advantages and disadvantages of these methods. Recently, some researchers also use machine learning and deep learning methods to quantify uncertainty in high dimensions, such as generative adversarial networks [3] and Bayesian neural networks [4].

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2 Stochastic Methods

2.1 Financial Valuation Model

To visualize our multi-stage financial model, a network is displayed in *Figure.1(a)*. Although the network is ideal and not complex enough to reflect reality, it enhances our understanding of the multi-stage financial model.

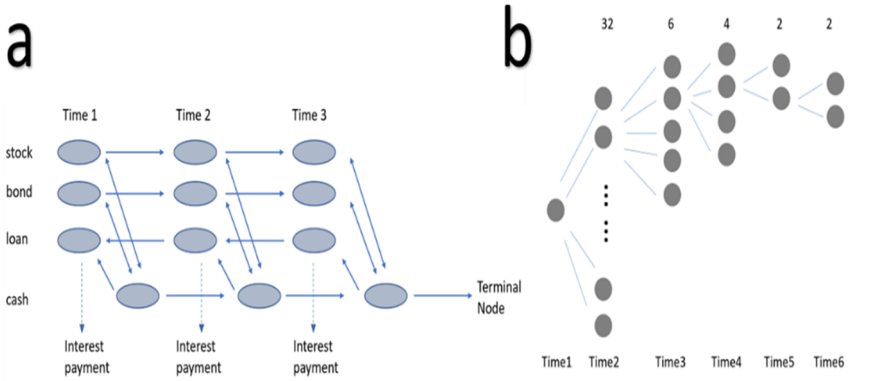

Fig. 1. (a) multi-stage financial planning model, (b) scenario path tree

In our multi-stage stochastic approach, the planning horizon consists of t periods, denoted by T = {0,1,…,t-1}. The first period represents the current date. I focus on the investor's position at the beginning of period t. An asset investment class is defined as the set A = {1,2,…,I}, representing a diversified portfolio of investments such as stocks, bonds, real estate, or cash. S represents the set of scenarios, each scenario represents a set of outcomes for all random coefficients over the entire planning period T. Scenario s is the continuous path through nodes, shown as *Figure.1. (b).* The main decision variable $x_{i,t}^{s}$ represents the amount of assets invested in class i at the beginning of time period t under the scenario. After rebalancing the assets, the buying and selling values of assets are considered, which are respectively regulated by the variables $y_{i,t-1}^{s}$ and $z_{i,t-1}^{s}$, and assume the transaction cost is the constant coefficient ξ_i. Therefore, the total asset is:

$$x_{i,t}s = x_{i,t-1}s + (1-\xi_i)y_{i,t-1}s - (1+\xi_i)z_{i,t-1}s \quad \forall i \in A, t \in T, s \in S \tag{1}$$

For the wealth of the focal investor, the Quantity of Interest (QoI) is:

$$wealth_{T}s = x_{i,t}s - L = x_{i,t-1}s + (1-\xi_i)y_{i,t-1}s - (1+\xi_i)z_{i,t-1}s - L \tag{2}$$

where L is the investor's present value of the liability. To simplify the uncertainty qualification problem, the liability is assumed to be constant.

2.2 Multi-Stage Stochastic Process

To solve the uncertainty qualification problem, randomization of input and output vectors is first employed. The stochastic inputs $y_{i,t-1}^s$ and $z_{i,t-1}^s$ are represented as multi-dimensional random vectors. A statistical summary of these inputs is in *Table.1*. Then, Gaussian distribution is employed to describe the stochastic inputs, KL-expansion is used to reduce the dimension, and subsequently, the distributions of inputs are transformed to a non-Gaussian vector using the Max Entropy principle, with an appropriate probability distribution.

Table 1. Statistical summary of inputs

	Mean	Standard deviation	Lower bound	Upper bound	Correlation length
$y_{i,t-1}^s$	4.87	1.84	0.04	11.44	20
$z_{i,t-1}^s$	4.80	1.61	0.14	9.94	20

Then, use stochastic solvers (Monte Carlo, Galerkin method, and Stochastic Collocation method) to tackle the forward uncertainty propagation problem, which involves mapping the stochastic inputs to the stochastic outputs, resulting in the availability of the probability distribution of the outputs. Figure.2 illustrates the entire workflow.

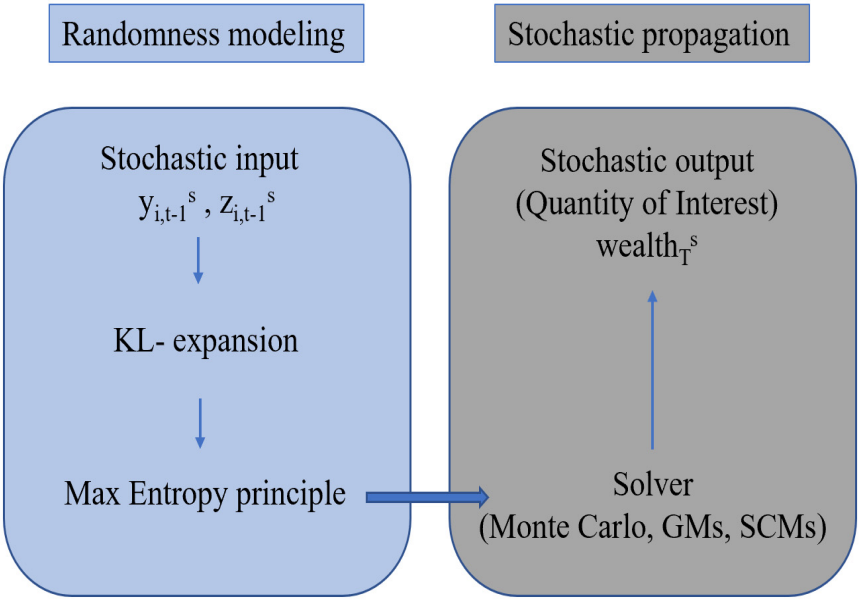

Fig. 2. flowchart of multi-step stochastic model

Randomization of input and output vectors.

People usually use the Gaussian distribution to describe the return on investment for the convenience of analysis. As for asset values yi,t-1s and zi,t-1s , they are both multi-dimensional random vectors and normally represented by Gaussian distribution [5]. To visualize two inputs, their probability density functions are shown in Figure.3. (a-d). In the financial industry, Gaussian random vectors are utilized as a popular choice among various distributions due to their ease of generation. In the approach, the parameters of a Gaussian vector are means and covariance matrix. I assume the mean of each yi,t-1s and zi,t-1s is all 0, $C\left(y_{i,t-1}s, z_{i,t-1}s\right)$ represents the covariance of them, l denotes correlation length structure. The squared-exponential structure is:

$$C\left(y_{i,t-1}s, z_{i,t-1}s\right) = e^{-\frac{y_{i,t-1}s - z_{i,t-1}s}{2l^2}} \tag{3}$$

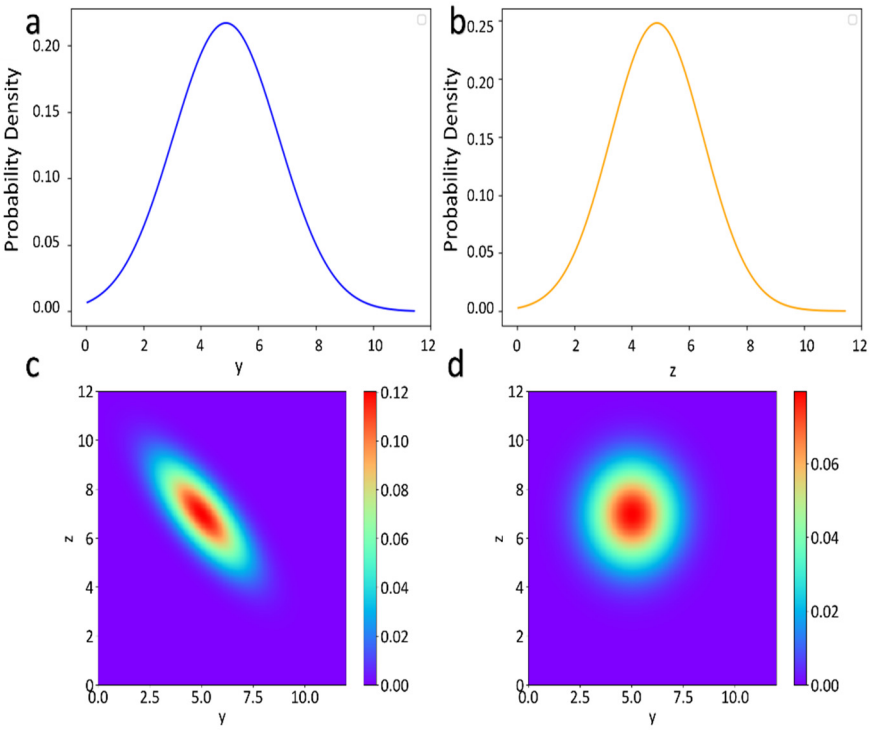

Fig. 3. probability density functions of two inputs (a): y, (b): z. c) 2D gaussian distribution with covariance=1.5. d) 2D gaussian distribution with 0 covariance.

Reduce-order Model.

Since Gaussian vectors yi,t-1s and zi,t-1s are high-dimensional, Karhunen-Lo`eve expansion (KL-expansion) is utilized to construct Gaussian vectors, which helps to

remove redundant and complex data, so that the computational cost of analyzing data could be reduced as well.

According to Mercer's theorem of KL-expansion, the covariance $C\left(y_{i,t-1}s, z_{i,t-1}s\right)$ could be reformulated as:

$$C\left(y_{i,t-1}s, z_{i,t-1}s\right) = \sum_{i=1}^{\infty} \lambda_i \, \varphi_i\left(y_{i,t-1}s\right)\varphi_i^{Trans}(z_{i,t-1}s) \tag{4}$$

where $\{\lambda_i\}$ and $\{\varphi_i\}$ are eigenvalues and corresponding eigenvectors, respectively. Assume a normalized Gaussian field G. Let $\{\eta_i\}$ denote independent Gaussian random variables, $\{\eta_i\}$ of $H \sim N(0,1)$ are independent, centered and reduced, and n represents a truncation degree:

$$\eta_i = \lambda^{-\frac{1}{2}} < G, \varphi i >, \text{ and}$$

$$G = \sum_{i=1}^{\infty} \lambda_i^{1/2} \eta_i \varphi_i \approx \sum_{i=1}^{n} \lambda_i^{1/2} \eta_i \varphi_i = G^{<n>} \tag{5}$$

Once the eigen decomposition of the covariance matrix is computed, the corresponding eigenvalues and eigenvectors are gained. Let ε denote the truncation error which varies with respect to the number of reduced variables q. When q approaches infinite, the initial covariance matrix is recovered. The truncation error can be determined using the following calculation:

$$\varepsilon(q) = 1 - \frac{\sum_{i=q+1}^{\infty} \lambda_i}{\sum_{i=1}^{\infty} \lambda_i} \approx \frac{tr(C) - \sum_{i=1}^{q} \lambda_i}{tr(C)} = 1 - \frac{\sum_{i=1}^{q} \lambda_i}{tr(C)} \tag{6}$$

To reduce the stochastic dimension, KL-expansion is employed on each covariance function. In *Figure.4.(a),* the curve for truncation error as a function of the number of reduced variable q is displayed, indicating that a truncation degree (n) greater than 25 results in a truncation error below 0.1. To account for the assumption of uncorrelated and independent variables for both $y_{i,t-1}{}^s$ and $z_{i,t-1}{}^s$, each random field necessitates a realization, thereby leading to a stochastic dimension of m = 2q = 50.

To examine if the truncated Gaussian vector is efficient, given a correlation length, the Frobenius norm of the difference between the approximated and original covariance matrices with respect to q is shown as the convergence of the covariance matrix in *Figure.4.(b)*. The plot shows the error converges when the truncated number is about 50, which verifies the results in *Figure.4.(a)*. The error changes as the covariance changes. A larger covariance usually leads to a more significant concave error-truncated number curve (shown in *Figure.4.(c)*).

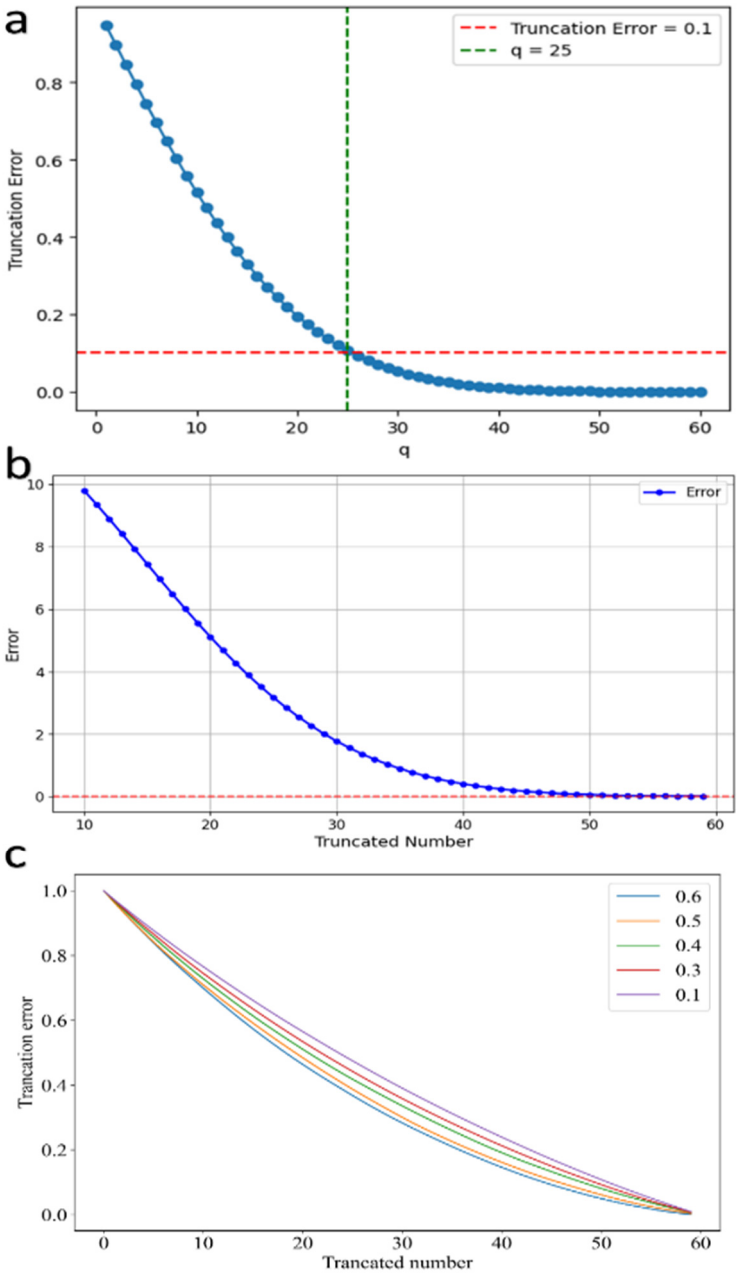

Fig. 4. (a) truncation error curve, (b) Frobenius norm of the error, (c) truncation error at different levels of covariances.

Transformation to Non-Gaussian Field.

Table 1 indicates that the two parameters yi,t-1s and zi,t-1s are constrained to a specific range, whereas the Gaussian distribution has unbounded support. Therefore, it is necessary to introduce a non-Gaussian vector to accommodate the bounded parameters.

Assume S=[ζlo, ζhi] supports non-Gaussian vector, ζ denotes non-Gaussian variables. For each variable in vector S, there are two constraints:

$$\mathrm{E}\{\log(\zeta - \zeta_{lo})\}<c_1 \text{ and } \mathrm{E}\{\log(\zeta_{hi} - \zeta)\}<c_2 \tag{7}$$

where the values of c1 and c2 are not important. In this method, logarithmic is employed to trigger an impulse near the boundness.

Using Shannon's information theory, entropy is a measure of the uncertainty or randomness associated with a probability distribution. Let **x** be a vector-valued random variable defined by a probability distribution $f_{\mathbf{x}}(X)dx$. An entropy of a distribution is:

$$\mathscr{E}(f_{\mathbf{x}}) = -\,\mathbb{E}\{\log(f_{\mathbf{x}}(X))\} \tag{8}$$

According to the Max Entropy principle, the most reasonable probability distribution to assume is the one that maximizes entropy subject to the constraints imposed by the available information. Thus, I maximize a Lagrange equation as follows:

$$f^* = \arg\max \mathscr{L}(f) = \mathscr{E}(f_{\mathbf{x}}) - \lambda_1(\mathbb{E}\{\log(\zeta-\zeta_{lo})\} - c_1) - \lambda_2(\mathbb{E}\{\log(\zeta_{hi}-\zeta)\} - c_2) \tag{9}$$

where above variables are the set of all possible probability density functions and satisfying the constraints. After solving this equation, notice that the Beta distribution is the most reasonable distribution, which maximizes entropy and satisfies constraints. The probability density function and cumulative density function are displayed in *Figure.5*. Therefore, let $\mathscr{F}$ denotes the cumulative probability function of Beta distribution, and the non-Gaussian vector transformed from the Gaussian vector is:

$$\zeta(x) = (\zeta_{hi} - \zeta_{lo})\mathscr{F}^{-1}_{\mathbb{B}(1-\lambda_1,1-\lambda_2)}\{\mathscr{F}_{\mathcal{N}(0,1)}(\xi(x))\} + \zeta_{lo} \tag{10}$$

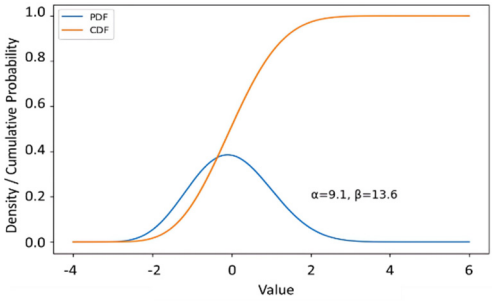

Fig. 5. probability density function and cumulative density function of beta distribution after Max Entropy principle

3 Uncertainty Propagation: Stochastic Solvers

To propagate uncertainties through large-scale models or when the dimension is fairly high, stochastic solvers are devised. Three propagation techniques are employed in this project: Monte Carlo, SGM, or SCM.

3.1 Monte Carlo Approach

Although the KL-expansion technique was used to reduce the stochastic dimension to m = 50 in this project, in real-world applications, complex problems may have a significantly larger stochastic dimension, making the computational cost prohibitively expensive. However, the Monte Carlo approach has a convergence rate that is independent of the stochastic dimension. Hence, it is not affected by the curse of dimensionality.

Our problem is solved NMC times, and as NMC → ∞, the PDF of output should be precise, and the moments are converged. For each solution, I sample m random variables:

$$\eta_j(\Theta_i) \text{ of } H \sim N(0,1),\ 1 \le i \le N_{MC},\ 1 \le j \le m \tag{11}$$

The Quantity of Interest (QoI), the wealth of investor in our financial problem, is:

$$wealth_{Ts} = \frac{1}{n}\sum_{i=1}^{n}[\,(1-\xi_i)y_{i,t-1}s(H) - (1+\xi_i)z_{i,t-1}s(H)] - L \quad \forall i \in A, t \in T, s \in S \tag{12}$$

where n is the number of elements in the bound. Based on sampling variables and KL-expansion, let ω_i, μ_i be element averages of nodal values, the realization of QoI is:

$$wealth_{T}{}^{s} = \frac{1}{n}\sum_{i=1}^{n}[\omega_i(\eta(\Theta_i)) + \mu_i(\eta(\Theta_i))]\text{-L} \tag{13}$$

In the process of executing the Monte Carlo simulation, 10,000 random samples are generated utilizing the fitted Beta distribution. It is assumed that the displacement distribution at the end node adequately represents all other output variables. The convergence results for the first and second moments are illustrated in Figure.6. The graphical representation demonstrates that convergence is achieved when NMC → 3,000, which constitutes a reasonable quantity of realizations.

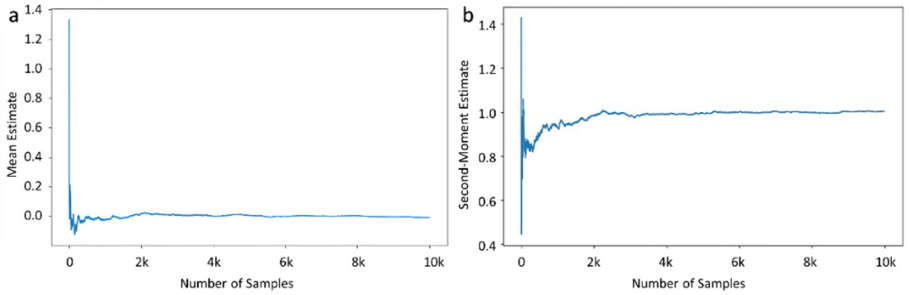

Fig. 6. the convergence results for the first and second moments using the Monte Carlo simulation

3.2 GMs or SCMs Approach

The Galerkin Methods (GMs) or Stochastic Collocation Methods (SCMs) are commonly utilized to solve for the coefficients of the polynomial chaos expansion (PCE). Their equations are similar:

$$Wealth(\Xi) = \sum_{i=1}^{N_{pce}} wealth_i \cdot \Psi_i(\Xi) \tag{14}$$

When using GMs, the PCEs are incorporated into the model equations to determine the equations that the chaos coefficients of the output must satisfy. GMs are considered intrusive and satisfy the model in a weak form. On the other hand, SCMs involve sampling the solution map to enforce the governing equations on a finite set of points (known as collocation points). SCMs are non-intrusive, and the model is satisfied in a discrete (or strong) sense.

To test the two methods and compare the results, the Monte Carlo method is utilized as a benchmark. Evaluating the convergence was employed.

In GMs method, let K be the PCE coefficients of the stiffness matrix, and let F be the PCE coefficients of the force vector, the deterministic linear system is:

$$K \cdot wealth = F \tag{15}$$

It is extended to a larger linear system. The main idea is to derive the PCE coefficients of the stiffness matrix and the force vector, rather than solving for the PCE of the output wealth.

$$K(\Xi) = \sum_{i=1}^{N_{pce}} k_i \Psi_i(\Xi) \quad F(\Xi) = \sum_{i=1}^{N_{pce}} f_i \Psi_i(\Xi) \tag{16}$$

The global linear system can be obtained by projecting the original equation to each of the polynomials $\Psi_i(\Xi)$. By solving the resulting extended global linear system, the PCE coefficients {wealth i}i≥1 can be determined.

The results for the Monte Carlo simulation and the Galerkin method with polynomial degrees Q=2,4,6,8 were plotted in Figure.7, and it contains the displacement distribution on the middle and the end node. The Galerkin method for all polynomial

degrees converges to the Monte Carlo simulation well, although some noise exists. When polynomial degree Q=8, the convergence result is the best.

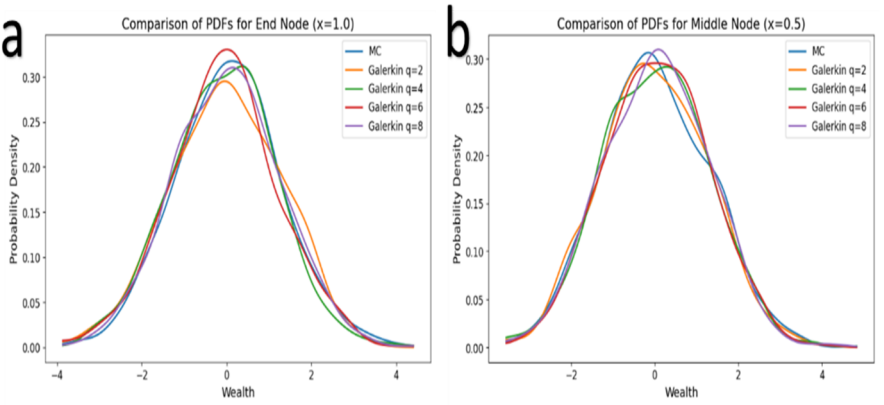

Fig. 7. the convergence results for the Monte Carlo simulation and the Galerkin method with polynomial degrees Q=2,4,6,8 for (a): end node(x=1.0), (b): middle node(x=0.5)

In SCMs method, the PCE coefficients $\{Wealth_i\}_{i\geq 1}$ can be determined by:

$$Wealth_i = \int Wealth(\xi)\Psi_i(\xi)p\Xi(\xi)d\xi \tag{17}$$

Due to the high number of dimensions involved in stochastic inputs, it is often required to use quadrature rules to calculate the integration mentioned earlier. This implies that a substantial number of deterministic simulations need to be performed, which can be extremely time-consuming, particularly if each simulation requires a considerable amount of time to execute. In this project, the Gaussian-Hermite rules are employed due to the Gaussian distribution of stochastic inputs. The convergence result with respect to quadrature depth for second-order expansion is illuminated in Figure.8. It is shown that the peak of SCMs method lines is almost twice that of the Monte Carlo line, so the SCMs method does not converge well.

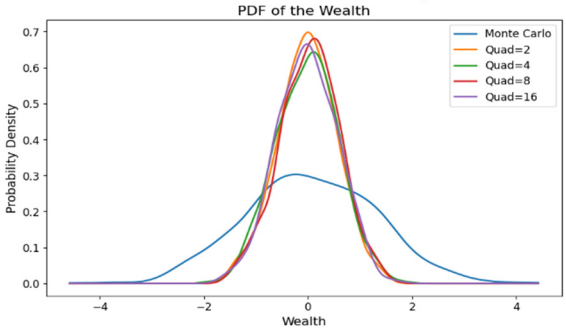

Fig. 8. the convergence result with respect to quadrature depth for second-order expansion

4 Conclusion

Monte Carlo, Galerkin method, and Stochastic Collocation method are all techniques for solving stochastic problems. Each method has its advantages and disadvantages, making them more suitable for specific problems or situations. In this project, Monte Carlo and Galerkin method works well for the financial model, while the Stochastic Collocation method is not suitable. Their advantages and disadvantages are as follows.

As a general method that is easy to implement, the Monte Carlo method is employed as the benchmark for the comparison of two other methods. It is also not sensitive to the dimensionality of the problem. The convergence rate is independent of the smoothness of the function. It could also handle non-linear and non-Gaussian problems. However, the convergence rate is normally slow, and it requires many samples for accurate results.

For Galerkin, it provides rapid convergence for smooth functions with low effective dimensions, and handles a wide range of problems, including linear and nonlinear systems. It also provides a compact representation of the solution in terms of orthogonal polynomial basis functions. On the other hand, it assumes specific probability distributions for input variables (usually Gaussian) and is sensitive to the choice of polynomial basis. It is also computationally expensive for high-dimensional problems due to the curse of dimensionality. Solving a system of equations that may become ill-conditioned is required as well.

The stochastic Collocation method (SCM) can handle non-linear problems and non-Gaussian input distributions. The convergence rate is generally faster than Monte Carlo for smooth functions. It also can handle higher effective dimensions and non-smooth functions better than the Galerkin method and does not require solving a system of equations. However, it is sensitive to the choice of collocation points and interpolation strategy and can be affected by numerical issues like the curse of dimensionality or ill-conditioning in the interpolation step, so it may not converge as well as the Galerkin method for certain problems.

In summary, each method has its strengths and weaknesses. The choice of the method depends on the problem at hand, its characteristics, and the desired level of accuracy. In this project, the Galerkin method achieves better convergence. It captures the structure of the financial model more efficiently, while it is computationally expensive. Overall, the choice between the Galerkin method and the Stochastic Collocation method is a trade-off between computational efficiency, convergence, and accuracy.

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