Optimizing Returns of Diversified Investment Portfolio with Markowitz Model

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Abstract. In recent years, the underlying assets allocation has become a hot topic, where tremendous investors and analyzers are tried to construct portfolio with well performances (e.g., maximum Sharpe ratio, minimum volatility, maximum Calmar ratio) under the framework of quantitative analysis. As a matter of fact, the portfolio theory utilizes historical data of different underlying assets (e.g., stocks, futures, spots, options as well as cryptocurrencies) to analyze the assets being invested. This paper presents a method to generate the Markowitz model using the Monte Carlo method and combines it with the utility function to obtain a low-risk, high-return investment portfolio. According to the analysis, the allocation of investment business products is illustrated using diversified investment products as an example. Overall, this study provides guidance and suggestions for real-world investment for investors, aiming to avoid risks and achieve relatively high returns. These results shed light on guiding further exploration of portfolio construction.

Keywords: Markowitz model, Monte Carlo method, optimal product allocation, utility function.

1 Introduction

The median-variance technique, developed by Markowitz in 1952, uses the mean and variance to represent the return and risk of an asset, respectively. With this method, the asset distribution issue is transformed into a problem involving quadratic programming, in which the goal is to find the portfolio with the lowest risk for a given return or the portfolio with the highest return for a given risk, leading to the well-known Markowitz portfolio theory [1]. The fundamental disadvantage of Markowitz's portfolio theory is its enormous computer complexity, even though it is quite accurate. Even then, the essential computations could not be handled by computers, which limited the practical uses. Markowitz mean-variance model was made simpler by Sharp's 1963 multi element approach. In addition to providing a novel viewpoint on portfolio diversification, the model greatly reduces its computational price by linking the risk and return of a portfolio to the market portfolio [2]. The capital asset pricing

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model (CAPM), developed by Sharpe, Lintner, and Mossin after a more thorough examination of the Markowitz model, quickly gained popularity for its examination of the relationship between securities returns and market portfolio returns. Later, Ross explored the capital asset pricing model, loosened the necessary assumptions, and proposed an arbitrage pricing model [3]. This has made the model simpler to use. On the basis of intensive research by a number of scholars, Markowitz portfolio theory has been developed and the applicability aspect has been significantly improved, thus gradually establishing the theoretical system of Markowitz portfolio science. When building the Markowitz portfolio model, there are a number of assumptions that may deviate from the reality of investment activities. In order to better suit the actual investment needs, many scholars have improved and studied the model. For example, The signify-semi-variance model was suggested by Mao, which is considered to be more effective in measuring investment risk, and compared it with the mean-semi-variance model [4]. Konno and Yamazaki, on the other hand, devised a model for mean absolute deviation, the computational complexity of the model by employing a mean extreme variation model rather than a variance operation, and carried out an empirical investigation [5]. Lam, Jaaman and Isa note that mean-variance models can be biased if the distribution of returns on assets does not follow a normal distribution. Therefore, it is important to consider both the kurtosis and tilt of the return distribution. Traders that are wary of risk choose portfolios with high skewness and low a tendency towards, and incorporating these factors into the mean-variance model can improve the effectiveness of the model [6]. Markowitz portfolio theory has seen a major improvement in terms of risk assessment, applicability, and investment efficacy thanks to the model.

The efficient frontier curve of Markowitz's portfolio theory is one of his important findings, providing investors with diversified investment choices and important implications for investment decisions. Li and Ng extended Markowitz's single-stage investment choice model to a multi-stage mean-variance model, taking into account the presence of risk-free assets in the portfolio. They obtain the efficient frontier of the mean-variance model when multi-stage and propose an optimal investment strategy in a dynamic investment environment [7]. These studies further refine Markowitz's portfolio theory and improve its applicability and practical effectiveness. Under the mean-variance framework, Wang, Huang and Jou relaxed the special assumptions of perfect markets and asset return probability distributions, and derived dynamic portfolio frontiers and their corresponding dynamic asset allocations to obtain efficient frontier curves for dynamic and static portfolios[8].Whereas Wu, Zeng, and Yao look into the mega-period signify-variance choice of portfolios problem with a variable time horizon, they make the assumption that the conditional distribution of the time horizon is random, derive the ideal investment plan and the efficient frontier, and then use computational methods to introduce some of the efficient frontier's properties [9].

China's securities market was established relatively late, which makes the research on Markowitz's portfolio theory by domestic scholars relatively late. Mainly based on the theories of Western scholars, empirical and countermeasure-related research has been conducted, and the Markowitz theory system is not yet perfect. However, domestic scholars have conducted relatively more research on the application of Markowitz's portfolio theory in China and the effective frontier of investment portfolios. For example, Li and Xu have conducted a comprehensive evaluation of the Markowitz model and found through empirical research that the application of the Markowitz model to guide the choice of investment strategies is feasible and has important practical significance for diversifying investment risks [10]. Zeng and Tang conducted a study to explore how to determine the effective boundary of a portfolio security under the condition that short selling is not allowed. They find that the efficient frontier is composed of parabolic segments tangent to the point of connection and provide a method for determining the efficient frontier and the variation in the composition of the efficient portfolio [11]. Yang conducted an empirical study on the effective frontier theory of Markowitz's model, selecting stock indices of seven countries as a sample and providing a comparative analysis of the effective frontier under the conditions of "short selling allowed" and "short selling not allowed". The results show that higher potential returns can be achieved under the "short selling allowed" condition [12]. Zhou proposed and solved a mean-conditional value-at-risk model conditional on containing risk-free assets. He found that the effective frontier of the new model and the effective frontier of the original mean-variance model are consistent under certain conditions, and obtained the effective frontier curves of the new model when the borrowing and lending rates are different [13]. Many academics have further amended and enhanced the Markowitz portfolio model by relaxing its presumptions and putting out their own opinions in order to address the rigid assertions of the model. Dai analyses the background of creating the Markowitz model and evaluates the emulate, pointing out that it has some problems, such as the use of variance as a risk measure is unreasonable and needs to be revised to semi-variance [14]. Yu considered the asymmetry of portfolio returns, described the skewness using the ratio of upper and lower semi-variance, proposed a mean-variance-approximate skewness model, and conducted an empirical analysis on the major stock indices in the Chinese stock market [15]. Li and Chen investigate the time incompatibility problem of the mean-variance model for optimal investment strategies in stochastic markets and propose a correction strategy, which can obtain the same mean and variance as the optimal investment strategy [16]. Zhang et al. considered various constraints, such as capital constraint, transaction cost, minimum trading volume, etc., integrated the risk preference coefficient, constructed a risk preference coefficient mean-variance expansion model, and gave the specific implementation process of the genetic algorithm [17]. Yao and Ocean et al. studied the selection of mean-variance portfolios under continuous time conditions, taking inflation into account, and obtained an analytical expression for the mean-variance effective boundary and an effective investment strategy [18].

According to the status of the domestic and international research, the proposed Markowitz portfolio model has gained widespread recognition, and academics have continued to research and develop it, further improving its theoretical system. The Markowitz portfolio model, although widely used in investment activities, has several strict assumptions that do not reflect the real-world scenario. To address this limitation, scholars have made modifications such as incorporating semi-variance and absolute deviation to depict risk, considering factors like kurtosis and skewness, allowing for short selling, and introducing concepts of risk appetite and inflation. Additionally, the Markowitz model has been extended from a static to a dynamic approach, enabling researchers to analyze the efficient frontier curve's characteristics under varying conditions.

With the in-depth study of Markowitz's portfolio theory, scholars have gradually become aware of the impact of parameter uncertainty in asset allocation. Some researchers have explored the impact of parameter uncertainty on optimal portfolios and efficient frontier curves under static conditions and continuous time states, pointing out that parameter uncertainty leads to portfolios that are not optimal, thus imposing opportunity costs. In addition, some scholars have optimized and improved the Markowitz portfolio model in order to solve the parameter uncertainty problem, such as using Bayesian methods and robust optimization methods, etc. These optimization methods have greatly improved the degree of reliability of the model and effectively solved the estimation risk problem.

Although the Markowitz model is widely used in portfolio theory, it still has some problems. It only considers the mean and variance of a portfolio and ignores the correlation between assets. In addition, the model assumes that markets are efficient, whereas in real markets there is a lot of irrational behavior and market noise, which may affect portfolio performance. Therefore, a new asset allocation model is proposed in this paper to better simulate real-life investment scenarios. The model uses Monte Carlo simulation to generate a large number of portfolio stochastic scenarios and considers the effects of a variety of factors, such as market volatility, equity returns, interest rates, etc., to obtain more accurate simulation data. Based on these data, the Markowitz model and maximum Sharpe ratio theory are applied to calculate the expected return, risk level and maximum Sharpe ratio for each portfolio and plot the efficient frontier curve. Finally, by comparing the maximum Sharpe ratios and risk levels of different portfolios, the best asset allocation plan is determined in order to maximize returns while ensuring risk control. The method not only reflects investment risks and uncertainties more accurately, but also improves the scientific and practical feasibility of asset allocation decisions, and has practical application value for investors, asset management companies as well as financial institutions.

2 Theoretical Framework

The asset portfolio optimization theory is one of the fundamental theories of quantitative investment, mainly used to construct optimal asset portfolios to achieve the purpose of maximizing investment returns or minimizing investment risks. This chapter introduces the basic concepts and theories of asset portfolio optimization, including the Markowitz model, risk parity model, minimum variance model, etc., as well as the advantages, disadvantages, and application ranges of these models. This chapter comprehensively introduces the relevant content of asset portfolio optimization theory, providing theoretical and technical support for the design and implementation of subsequent quantitative analysis systems.

The Markowitz model is among the greatest classic models in asset investments optimization concept. Its basic principle is to maximize the portfolio's feedback while considering the risk. At the heart of the Markowitz model lies the mathematical quantification of both the expected return and risk associated with an investment portfolio, with the ultimate goal of deciding the optimal investment portfolio through mathematical means. This formula derivation process involves incorporating two fundamental concepts, i.e., hazard and anticipated return. Expected return refers to the mean feedback of the investment portfolio during a certain period in the future. The Markowitz model assumes that the frequency distribution of assets is bell-shaped and estimates the future returns of assets by calculating their historical returns. Specifically, the formula for calculating expected return is:

$$\mathrm{E(Rp)} = \sum_{1}^{n} \mathrm{wi}\,\mathrm{E(Ri)} \tag{1}$$

The variables $\mathrm{E(Rp)}$, wi, and $\mathrm{E(Ri)}$ represent the expected profits of the investment combination, the amount of weight given to the portfolio's i-th element, and the i-th entity's anticipated return, respectively.

Risk refers to the uncertainty of the investment portfolio's return, which is typically measured by the standard deviation or variance. The Markowitz model defines risk as the variance of the investment portfolio's return, i.e.,

$$\sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} cov(Ri, Rj) \tag{2}$$

The association between Properties I and J, which indicates the correlation across the two assets, is represented in the equation as the danger of the investment portfolio. The Markowitz model has been widely applied in various financial markets, such as stocks, bonds, and commodities. In order to better respond to market changes and investor demands, the Markowitz model has undergone several improvements, such as risk parity model and minimum variance model. Programming languages such as Python provide rich mathematical calculation and optimization algorithm libraries, making it easy to implement and optimize the Markowitz model. For instance, NumPy library can be used to calculate expected returns and risks, SciPy library can be used to solve for the optimal weights of the investment portfolio, and Pandas library can be used to read and process financial data. In addition, the investment portfolio can be further optimized using artificial learning and deep learning approaches. For example, neural networks can be used to predict asset returns, and genetic algorithms can be used to search for the optimal investment portfolio.

The risk parity model is an optimization model based on the principle of risk parity to construct a portfolio, which can help investors to strike a balance between reward and risks during the investment process. The chapter will provide a detailed introduction to the basic principles, formula derivation, and applications of the risk parity model, as well as how to implement and optimize the risk parity model using programming languages such as Python. We'll start by outlining the potential parity model's foundational ideas. The fundamental tenet of the hazards equality framework is to level off the danger contribution made by each asset to the total risk of the strategy, i.e., to set every asset's risk contribution to be equal. In this method, the portfolio's overall risk can be reduced while the return on the portfolio is increased. This method is different from the risk-return balance of the Markowitz model because it seeks to maximize returns while maintaining risk balance.

Next, we will derive the formula of the model of risk redundancy. Firstly, we need to calculate the contribution of every investment to risk. Risk contribution refers to the degree of influence of an asset's risk on the overall portfolio risk, usually measured by the ratio of the asset's volatility to the portfolio volatility. The risk parity concept seeks to level out the risk characteristics of all assets, so we can obtain an equation

$$w1\sigma1/\sigma p = w1\sigma2/\sigma p = \ldots.. = wn\sigma n/\sigma p \qquad (3)$$

We'll start by outlining the risk parity model's foundational ideas. The fundamental tenet of the risk parity model is to make each asset's risk contribution to the overall portfolio risk equal, i.e., to set each asset's risk contribution to be equal. In this method, the risk profile of the whole portfolio can be reduced while the return on the portfolio is increased minimize $\sum$ i,jwiwjσ*σj, Where wi and wj stand for the weight of the i-th and j-th assets, respectively, and σi and σj stand for the volatility of the i-th and j-th assets, respectively. The objective function aims to minimize the covariance between every two assets, thus achieving the goal of balancing risks. There are various tools in Python that can be used to implement the risk parity model, such as NumPy, SciPy, and CVXPy. These tools can help us calculate the risk contribution of each asset, construct the objective function, and solve for the optimal solution using a solver.

The Markowitz and risk parity models, as well as other common asset portfolio optimization models, are constructed under different assumptions and can be chosen accordingly to suit different investment objectives and constraints. One of them, the minimum variance model, optimizes an asset portfolio by reducing the variance of a portfolio of assets while assuming that asset returns follow a normal distribution. The minimum variance approach is more adaptable than the Markowitz model since it does not rely on the presumption that asset returns follow a multivariate normal distribution. Alternatively, there is the maximizing utility model, which is an asset portfolio optimization model constructed on the basis of utility theory. The basic idea of this model is to construct an asset portfolio by maximizing the investor's utility function. The utilitarian operation, which typically takes the shape of a quadratic or exponential function on risk and return, defines how an investor balances between risk and reward.

In addition to different models, there are different optimization algorithms. Traditional optimization algorithms include Newton's method and gradient descent, which require the calculation of derivatives or second-order derivatives and have a high computational complexity. In recent years, optimization algorithms based on machine learning and deep learning have also been widely used, such as neural networks and decision trees, which do not require the calculation of derivatives and have the advantages of fast computational speed and flexibility.

The Markowitz model is a model for maximizing portfolio efficiency through the relationship between the expected return and variance of a portfolio. Its modelling process consists of the following steps:

- Collect asset data: Obtain asset data from various sources, including information such as historical returns and variances of each asset.
- Determine the projected return and asset variation: Considering the past data, calculate each asset's anticipated yield and volatility. The projected profit is likely average return on an asset over a future period; the variance is the extent to which the return on an asset may deviate from the expected return over a future period.
- Calculating the covariance between assets: Calculating the covariance between assets reflects the correlation between different assets. A larger covariance indicates a higher risk correlation between assets.
- Building a portfolio: Build a portfolio based on the assets' predicted returns, variances, and covariances. By changing the weights assigned to each asset, several portfolios can be built. The percentage of each asset in the portfolio is shown by the weights.
- Determine a portfolio's expected return and variance: The anticipated roe and variation in portfolios are determined using information about the assets infects about the portfolio and the anticipated return, volatility, and covariance of each asset.
- Creating a set of portfolios with various risk-return characteristics—the efficient frontier—will allow you to plot the frontier. Each point represents a portfolio; the horizontal coordinate denotes the portfolio's standard deviation (risk), and the vertical coordinate denotes the portfolio's anticipated return.
- Select best-suited portfolio: based on the investor's desire for risk and return requirements, select the optimal portfolio on the effective front line as the final portfolio.

The Efficient Frontier (EFF) is the optimal equilibrium between the benefits and dangers of investing. In the Markowitz model, the efficient frontier is the set of all possible combinations of assets that can minimize risk or maximize return in a given set of risky assets. The efficient frontier can be calculated by the following steps:

- Given a set of risky assets and their historical return data, and calculate the average return and the covariance matrix of returns for each asset.
- construct various possible asset portfolios, i.e., assign different weights to each asset to obtain different expected rates of return and levels of risk
- Calculate the Sharpe ratio, for each asset portfolio.
- Plot the relationship between Sharpe ratio and expected risk for all possible asset portfolios, i.e. the efficient frontier. Where any point on the efficient frontier represents an optimal asset portfolio that minimizes risk or maximizes return.

It is important to note that the calculation of the efficient frontier requires consideration of several factors, such as historical asset return data, expected returns and risk appetite. It also requires an efficient frontier in the calculation process, i.e., ensuring that the sum of the asset weights is one and that the weights are not negative. The importance of the efficient frontier is that it provides a viable asset allocation solution to help investors make the optimal balance between risk and return. Additionally, the efficient frontier can be used to assess the performance of an asset portfolio, for example by comparing the Sharpe ratio of an actual asset portfolio with the efficient frontier to determine whether it is in an optimal state.

The Maximum Sharpe Ratio (MSR) assesses how risk and return are traded off, an asset portfolio optimization indicator. It is described as the median deviation of the profits on a maximized investing of all assets with equal risk. The maximum Sharpe ratio computes a portfolio's performance by dividing the excess return by the asset portfolio's the average deviation. The projected benefits on the asset investments less the risk-free rate is referred to as the excess return, and the standard deviation is the return volatility of the asset investments. Typically, the risk-free rate is the return on a risk-free asset, such as government debt. The method of calculation is as follows:

- First, the expected return and covariance matrix for all assets is calculated according to the Markowitz model.
- Then, the weights of each asset are calculated such that the sum of the weights of all assets is 1, i.e. a set of asset portfolios is obtained.
- Next, calculate the risks and anticipated earnings (standard deviation) of the portfolio.
- Finally, calculate the Sharpe ratio of the portfolio, i.e. the proportion of risk to predicted extra gain.
- Repeat steps 2 to 4 to calculate the Sharpe ratio for the asset portfolio with different weights.
- The asset group with the most Sharpe ratios should be located, which is the optimal asset portfolio.

It should be noted that in the actual calculation, it may be necessary to backdate or forecast the historical data on the expected asset returns and covariance matrix. The impact of factors such as transaction costs and liquidity also need to be considered. The advantage of the maximum Sharpe ratio is its ability to maximize the return on a portfolio of assets subject to risk control. It is therefore widely used in portfolio optimization, especially among investors with a high-risk appetite. The maximum Sharpe ratio can be used to construct an efficient frontier, i.e., maximizing the Sharpe ratio of a portfolio at all levels of risk, to obtain a curve where each point on the curve represents the maximum Sharpe ratio that can be achieved at a given level of risk. The peak on the frontier of efficiency is the ideal group, and the asset weights that correspond to it are those of the portfolio with the greatest Sharpe Ratio.

3 Design and Implementation of the System

As the subject of this paper is a diversified portfolio of securities, the paper selects investment products from stocks and futures and analyses the portfolio separately. The logic of the selection is as follows. In order to reflect the characteristics of the market, in the selection of stocks, this paper gives preference to leading stocks in the industry, choosing stocks that have been established for a long time and have a high market capitalization, such as Apple and Amazon. In the selection of futures, this paper selects futures with high liquidity, such as gold futures. Underlying return data considers the performance from 2016 to 2021 as long-term performance. In the processing of the data, we only keep the closing price of each asset for the trading day and for futures, we remove the duplicate data because having duplicate data rows in the data source will affect the results of the calculation.

This paper uses the percentage change of each asset relative to the starting price as the simple return and sums the daily simple returns to obtain the annualized return for each year and calculates the average of all annualized returns and the average annualized return, where the average of all annualized returns is the result of summing the annualized returns for each year and dividing by the total number of trading days. The standard deviation of log returns (STD) for each year and the standard deviation of log returns (std) for the entire time period are also calculated to measure the risk of the investment. Finally, the results of these calculations can be used to assess the risk and return performance of a portfolio.

This paper uses a Monte Carlo simulation approach to generate a large number of portfolios, measuring the performance of each portfolio by calculating the expected return, expected risk and Sharpe ratio. The direct Monte Carlo method has good applicability to finite element models and each cycle is independent of the other, making it suitable for parallel computation theory. Each portfolio in this paper consists of four elements: expected return, expected risk, Sharpe ratio and portfolio weights. At the end of the simulation, the portfolios with the highest Sharpe ratio and the lowest risk can be found and their optimised weights can be obtained. By adding the Capital Allocation Line (CAL) to the previously obtained graphs, the optimal only one risky asset portfolio can be more intuitively seen.

In reality, market sentiment is also an indicator that must be taken into account and the utility function is used in this paper to measure investors' preferences for asset portfolios and to test whether the utility of a portfolio at maximum Sharpe ratio is high enough. Utility function = return expectation - 1/2 * risk aversion coefficient * square of the standard deviation of risk. In this formula, the larger the risk aversion coefficient, the more risk averse the investor is, and the smaller the value of the portfolio's utility function, as it has to take higher risks. Therefore, the vertical coordinates in Fig. 2 represent the utility function values and the horizontal coordinates represent the risk aversion coefficients. By calculating the utility function values for each of the three portfolios under different risk aversion coefficients and plotting them on the graph, it can help investors understand the expected return and risk performance of different investment strategies under different risk preferences. The three portfolios shown are: 1. the dummy (control) portfolio, 2. the max sr (maximum Sharpe ratio) optimized portfolio and 3. the min risk (min risk) optimized portfolio. A yellow horizontal line is also drawn to indicate the risk-free rate of return.

4 Results & Discussion

The portfolio return and variance of the portfolio with the largest Sharpe ratio is obtained by sampling and generating an efficient frontier curve through the Monte Carlo method and then taking CAL from it according to the mean-variance model, as shown in Fig. 1. By establishing this CAL, the optimal portfolio ratio for each group and the optimal annualized rate of profits for the portfolio as a whole can be determined. By referring to the utility function, we are able to evaluate the portfolios in a comprehensive manner in relation to market sentiment. The utility function allows us to know that the utility of each portfolio is negatively correlated with investor risk aversion and the utility in the case of the maximum Sharpe ratio is shown in Fig. 2. By combining the above methods, the annualised return, variance, Sharpe ratio and utility of the portfolio at the maximum Sharpe ratio can be solved. The final results are: annualized return is 33.41%; risk is 9.7%; Sharpe ratio is 3.185832; and utility is 31.99%. The ratios of each investment within the portfolio are shown in the Table. 1. The change in the annualized return of the portfolio is shown in the Fig. 3.

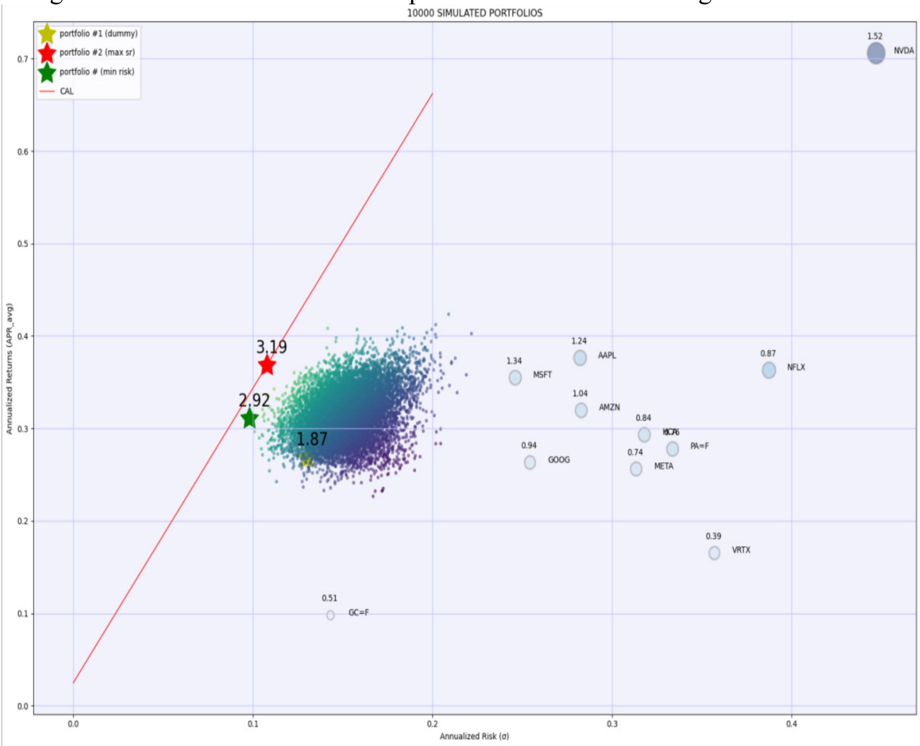

Fig. 1. CAL at maximum Sharpe ratio.

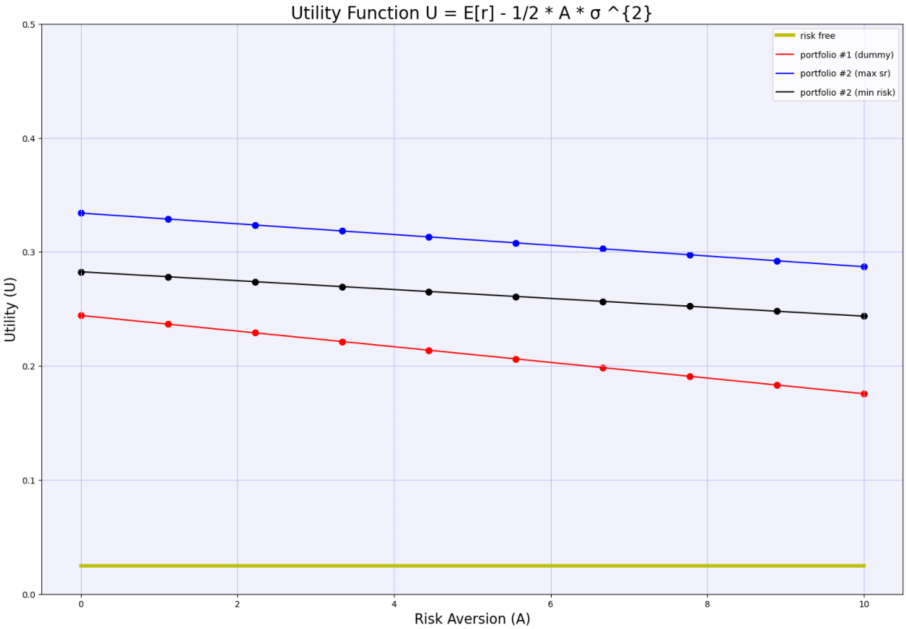

Fig. 2. Utility function of each condition.

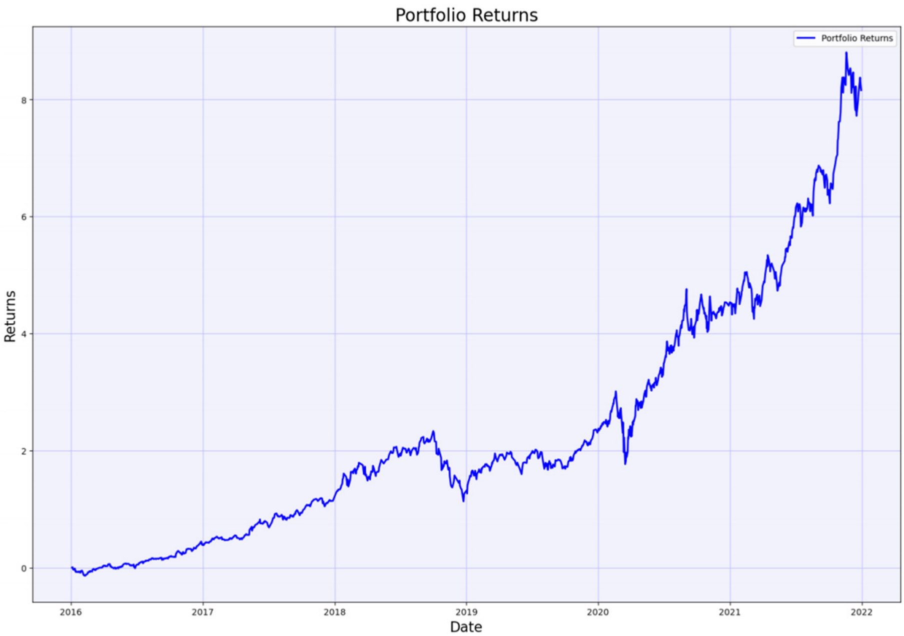

Fig. 3. Annualized return of the portfolio.

Table 1. Percentage of individual investments in the portfolio at maximum Sharpe ratio.

AAPL	MSFT	AMZN	GOOG	META	NFLX	NVDA	HCA	VRTX	GC=F	PA=F
0.0136	0.2649	0.0711	0.0103	0.0058	0.2389	0.1208	0.1857	0.0323	0.0288	0.0273

5 Conclusion

To sum up, in the practical implementation of the investment business, the assumptions of the Mean-variance analysis and the utility function are not nearly as idealistic as those discussed in this paper are still relevant. In portfolio selection, attention is paid to dispersion and risk diversification, to the purchase ability and substitutability of products, and to the degree of correlation between portfolio and market sentiment, such as the impact on market participants of news such as a change in company management. When making investments, it's crucial to continually screen and consider, and to fully engage with the market in order to select the right product for investors.

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