A Graph with Central and Local Reference Set

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Abstract. The local metric set is a subset of vertex set in graph that introduced by Okamoto *et al*. It is referred to a subset where is for every two adjacent vertices in graph has different representation with the subset. Previously, Slater introduced the location set and reference set as another term of resolving set and minimum resolving set, respectively. In this paper, we are developing a new concept refer to locating set and reference set by Slater and local metric set by Okamoto, then combaining it with a central vertices of a simple and connected graph called as central local metric set and central local reference set. We are developed the new concept by observation and literature study of related concept. We used some class of graph, such as complete graph (K_n), complete bipartite graph $(K_{n,m})$, and sun graph (S_n) to apply main definition of central local metric set and find their central local reference set. Then, the result will be formulated in a theorema that apply for general case. The result of this paper show that the central local reference set of K_n and $K_{n,m}$ are equal with their vertex set. Meanwhile, the central local reference set of sun graph is equal with vertex set of its cycle. This is given a implication that since both of radius and diameter of K_n and $K_{n,m}$ are equal so their central local metric dimension equal with their total vertex, meanwhile on sun graph both of radius and diameter different.

Keywords: Local metric, Graph, New development

1 Introduction

The local metric set is a subset of vertex set in graph that introduced by Okamoto et al at 2010. It is referred to a subset where is for every two adjacent vertices in graph has different representation with the subset. Okamoto et al also introduced a local metric dimension and its property as seen as in [1]. There are so many various graphs has been investigated their local metric dimension, as in [2–7]. The improvement of local metric dimensions concept has been developed by combining with other concepts, such as fractional local metric dimension [8], local adjacency metric dimension [9–11], local strong metric dimension [12], and dominant local metric dimension [13]. Meanwhile, in 1975 and 1988, Slater introduced the location set and reference set as another term of resolving set and minimum resolving set, respectively, and it was became a first time

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the idea of resolving set introduced [14]. In this paper, we also develop a new concept refer to locating set and reference set by Slater and local resolving set by Okamoto, then combaining it with a central set of a simple and connected graph called as central local metric set and central local reference set.

Let H is a graph, then $V(H)$ is a vertex set and $E(H)$ is an edge set. A vertex x adjacent with y in H is symbolized by $x \sim y$ or simply $xy \in E(H)$, and the distance of vertex u to the vertex v is symbolized by $d(u, v)$ [15]. The eccentricity of x, symbolized by $e(x)$, is the largest distance of x to all vertices in $V(H) - \{x\}$ where is the largest eccentricity in H is called diameter of H and the smallest eccentricity is called radius of H. The symbol for the diameter and radius are $diam(H)$ and $rad(H)$, respectively. A vertex x is a central vertex if $e(x) = rad(H)$ [16].

Put $U \subseteq V(H)$ is an ordered set and $U = \{u_1, u_2, \dots, u_k\}$, where $k \leq n$, then a representation of x to the subset U is $r(x|U) = (d(x, u_1), d(x, u_2), \dots, d(x, u_k))$, $\forall x \in V(H)$. Further, if $r(x|U) \neq r(y|U)$, for every two adjacent vertices in H, then U is called a local metric set of H and the minimal of metric set is called as local reference set. The cardinality of local reference set of graph H is called as local metric dimension of H, denoted by $lmd(H)$.

Here we defined a new concept of local metric set that contain all vertices of a graphs with eccentricity equal to radius. For illustrated this new concept in real life, suppose H is a graph that represent a district area with the vertices of H are representing a sub-district area and the edges are representing the relationship of two adjacent sub-district. The central vertices of graph H can describe the sub-districts with closest distance to others sub-district in the district. Thus, the concept of a central vertex in graph can be used as a reference for the placement of vital objects in a district that can be easily reached by the local community. For example, the vital object is a telecommunication signal transmitting tower. The placement of telecommunication signal transmitting tower at the central sub-district will help the local community to access telecommunication signal evenly. Besides that, to avoid a conflict of communication, the frequency of each transmitting tower must be different and the unique combination of telecommunication signal transmitting tower frequencies can be represented as a central local metric set.

2 Method

In this research, we did observation and literature study of graph concept [16], locating or resolving set and reference set [14], local metric dimension [1], eccentricity and central vertex [16], and other related concept to formulate a new concept of a central local metric set and central local reference set. The formal definition of a central local metric set is given in main results. A complete graph, symbolized by K_n, is a graph with every two different vertices are adjacent or simply $x \sim y$, for $x \neq y$ and $\forall x, y \in V(K_n)$ [16]. A complete bipartite graph is a graph where is its vertex set can be divided into two sets U and V, so that $u \sim v$ if and only if u in U and v in V [16]. Further, if $|U| = n$ and $|V| = m$, then a complete bipartite graph is symbolized by $K_{n,m}$ where is the order and size are $n + m$ and size nm, respectively. Sun graph, denoted by S_n, is obtained from a cycle C_n by adding one pendant in each vertex of C_n, so that $|V(S_n)| = 2n$ [17]. The flowchart in Fig.1 described our workflow to find a central local reference set on a graph with specific case, for example on graph K_n for $n = 1, 2, 3, \ldots$ etc.

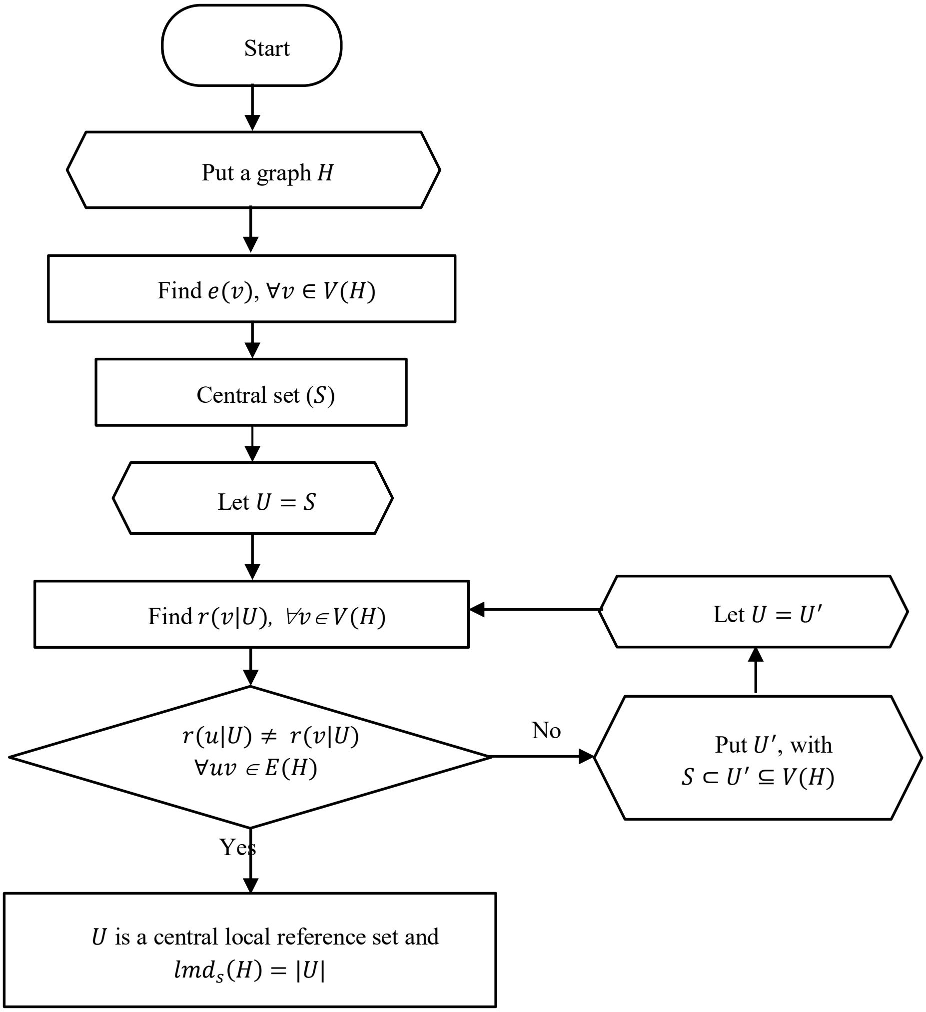

Fig. 1. The flowchart of a central local reference set tracking on a graph H

Then, the result will be formulated in a theorem that apply for general case. In addition, some lemmas are necessary to support the proof of theorems in main results. The following lemma are also used in proving the theorems.

Lemma 1. [15] Let H is a connected graph. If $U \subseteq V(H)$, then $r(u_i|U) \neq r(u_j|U)$, $\forall u_i, u_j \in V(U)$, $i \neq j$.

3 Result and Discussion

3.1 Central Local Metric Dimension

A central local metric dimension is a new concept that developed by combining local metric set and central vertex. Formal definition of central set, central local metric set, central local reference set, and central local metric dimension are given by definition 1, 2, 3 and 4, respectively.

Definition 1. Let $S \subseteq V(H)$, S is a central set of H if all element of S are central vertices in H or $S = \{s|e(s) = rad(H), s \in V(H)\}$.

Definition 2. Let $U \subseteq V(H)$ and S is a central set of H, U is called a central local metric set of H if U is a local metric set and $S \subseteq U$.

Definition 3. The minimal central local metric set of H is called a central local reference set of H.

Definition 4. A central local metric dimension of H, symbolized by $lmd_s(H)$, is cardinality of central local reference set of H.

Let H is a graph shown in Fig.2. The eccentricity of all vertices on H is listed on Table 1. Let, S is a central set of H, then we get $S = \{c, h\}$. Let $U = \{c, f, h\}$, the representation of all vertices of H respect to U are:

$$r(a|U) = (2,4,2) \quad r(d|U) = (1,3,2) \quad r(g|U) = (2,1,2)$$
$$r(b|U) = (1,3,2) \quad r(e|U) = (1,1,1) \quad r(h|U) = (1,2,0)$$
$$r(c|U) = (0,2,1) \quad r(f|U) = (2,0,2) \quad r(i|U) = (2,1,3).$$

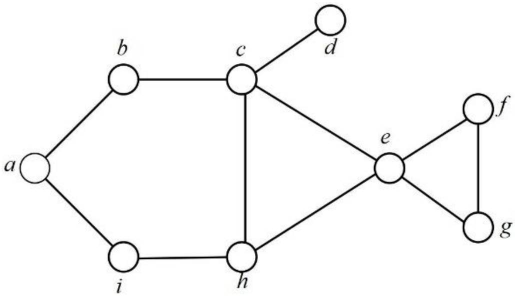

Fig. 2. A graph *H*

Table 1. The eccentricity of all vertices of *H*

v	$e(v)$	v	$e(v)$
a	4	*e*	3
b	3	*f*	4
c	2	*g*	4
d	3	*h*	2
		i	3

We get there are only two vertices has equal representation, these are $r(b|U) = r(d|U)$ but $bd \notin E(H)$. So, it can be conclude that for every $u, v \in V(H), uv \in E(H)$ applies $r(u|U) \neq r(v|U)$. Thus U is a central local metric set of H.

Central Local Metric Dimension of $\boldsymbol{K_n}$

Here, the formal definition of central local metric set is apply to construct a central local reference set of complete graph. The symbolization of each vertex and edge of K_n is illustrated by Fig. 3, where is the vertex is symbolized by x_i, for $1 \leq i \leq n$. Lemma 3 is given to support the proof of the next theorem.

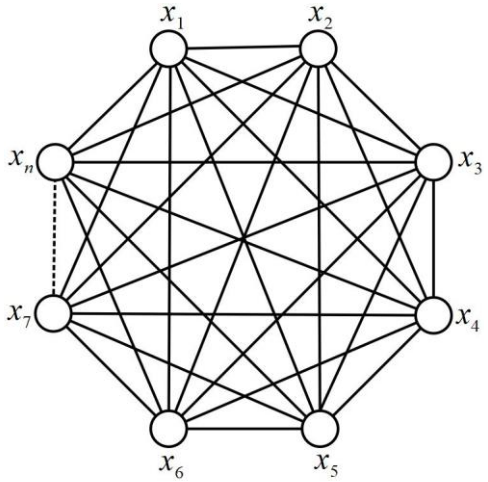

Fig. 3. Ilustration of K_n

Lemma 2. Let S is a central set of complete graph K_n, then $S = V(K_n)$.

Proof:

Let $V(K_n) = \{x_1, x_2, x_3, \dots, x_n\}$, $E(K_n) = \{x_i x_j | 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$, and S is a central set.

Since the every two different vertices on K_n is adjacent, thus $d(x_i, x_j) = 1$ for $\forall x_i, x_j \in V(K_n)$,and $i \neq j$. Then the eccentricity of every vertex on K_n is $e(x_1) = e(x_2) = e(x_3) = \cdots = e(x_n) = 1$. Since the eccentricity of all vertices are equal, then $rad(K_n) = diam(K_n)$, so the central set of K_n is $S = V(K_n)$. ■

Theorem 1. Let K_n is a complete graph, then $lmd_s(K_n) = n$.

Proof:

Define graph K_n with $V(K_n) = \{x_1, x_2, x_3, \dots, x_n\}$ and $E(K_n) = \{x_i x_j | 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$.

Let $S = V(K_n)$, then respect to Lemma 2, S is a central set of K_n with $|S| = n$. Since $S = V(K_n)$, then based on Lemma 1 for every $v_i, v_j \in S$ where $i \neq j$ apply $r(v_i|S) \neq r(v_j|S)$. So that, for every $x_i, x_j \in V(K_n)$ where $x_i x_j \in E(K_n)$ apply $r(x_i|S) \neq r(x_j|S)$, for $1 \leq i \leq n$, $1 \leq j \leq n, i \neq j$. Thus, S is local metric set of K_n. Since S is also a central set on K_n, then S is a central local reference set and $lmd_s(K_n) = n$. ■

The example of a central local reference set on K_4 is shown on Fig 4 with $S = \{x_1, x_2, x_3, x_4\}$ and $lmd_s(K_4) = 4$.

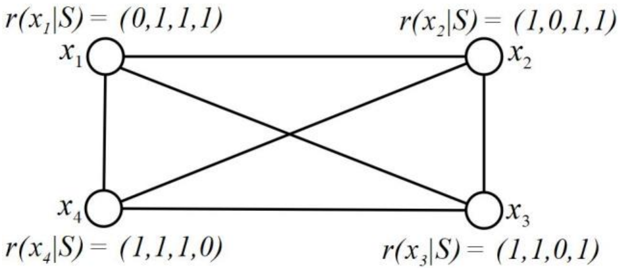

Fig. 4. a central local reference set on K_4

Central Local Metric Dimension of $\boldsymbol{K_{n,m}}$

Here, the formal definition of central local metric set is apply to construct a central local reference set of complete bipartite graph. The symbolization of each vertex and edge of $K_{n,m}$ is illustrated by Fig 5, where is both part of the vertex set is symbolized by x_i and y_j, for $1 \leq i \leq n$ and $1 \leq j \leq m$, respectively. Lemma 4 is given to support the proof of the next theorem.

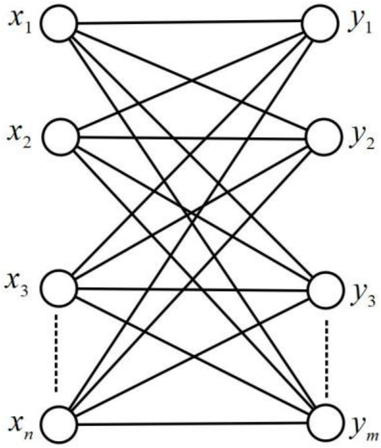

Fig. 5. Ilustration of $K_{n,m}$

Lemma 3. Let S is a central set of complete bipartite graph $K_{n,m}$ for $n, m \geq 2$, then $S = V(K_{n,m})$.

Proof:

Let $V(K_{n,m}) = \{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_m\}$, $E(K_{n,m}) = \{x_i y_j | 1 \leq i \leq n, 1 \leq j \leq m\}$ for $n, m \geq 2$, and S is a central set.

Since the graph is $K_{n,m}$, so the farthest of u to the others all vertices in $K_{n,m}$ is 2, $\forall x_i, y_i \in V(K_{n,m})$. Thus, the eccentricity of every vertices in $K_{n,m}$ is $e(x_1) = e(x_2) = \cdots = e(x_n) = e(y_1) = e(y_2) = \cdots = e(y_m) = 2$. Since the eccentricity of all vertices are equal, then $rad(K_{n,m}) = diam(K_{n,m})$, so the central set of $K_{n,m}$ is $S = V(K_{n,m})$. ■

Theorem 2. Let $K_{n,m}$ is a complete bipartite graph for $n, m \geq 2$, then $lmd_s(K_{n,m}) = n + m$.

Proof:

Define graph $K_{n,m}$ with $V(K_{n,m}) = \{x_1, x_2, \dots, x_n\} \cup \{y_1, y_2, \dots, y_m\}$ and $E(K_{n,m}) = \{x_i y_j | 1 \leq i \leq n, 1 \leq j \leq m\}$, for $n, m \geq 2$.

Let $S = V(K_{n,m})$, respect to the lemma 3, S is a central set of $K_{n,m}$ and $|S| = n + m$. Since $S = V(K_{n,m})$, respect to the lemma 1, $\forall v_i, v_j \in S$, $i \neq j$, apply $r(v_i|S) \neq r(v_j|S)$. Then, for every $x_i, y_j \in V(K_{n,m})$ where $x_i y_j \in E(K_{n,m})$ give a result $r(x_i|S) \neq r(y_j|S)$, $1 \leq i \leq n, 1 \leq j \leq m$. Thus, S is local metric set of $K_{n,m}$. Since S is also a central set of $K_{n,m}$, then S is a central local reference set and $lmd_s(K_{n,m}) = n + m$.■

The example of a central local reference set on $K_{2,3}$ is shown on Fig 6 with $S = \{x_1, x_2, y_1, y_2, y_3\}$ and $lmd_s(K_{2,3}) = 5$.

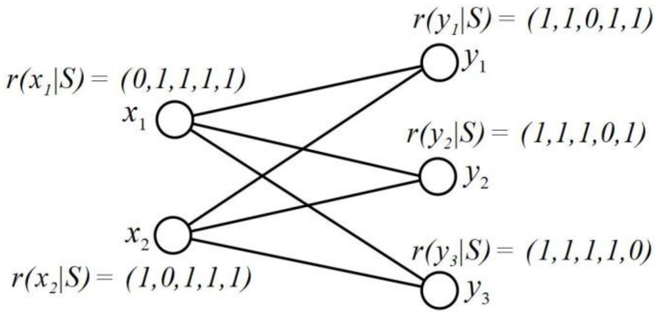

Fig. 6. a central local reference set on $K_{2,3}$

3.2 Central Local Metric Dimension of S_n

Let $S_n = C_n \odot K_1$ is a sun graph, to support discussion for the next lemma and theorem, the vertices of sun graph are symbolized by u_i and v_i, $1 \leq i \leq n$. The vertices of u_i is a vertices on C_n and the vertices of v_i is a vertices on K_1 as illustrated by Fig 7. Sun graph S_n has order $2n$. Lemma 5 is given to describe the central set of S_n.

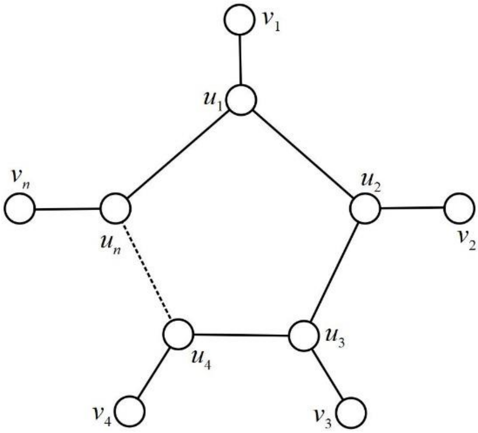

Fig. 7. Illustration of S_n

Lemma 4. Let S is a central set of sun graph S_n, then $S = \{u_i | 1 \leq i \leq n\}$.

Proof:

Let $V(S_n) = \{u_i | 1 \leq i \leq n\} \cup \{v_i | 1 \leq i \leq n\}$, $E(S_n) = \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{u_i v_i | 1 \leq i \leq n\} \cup \{u_n u_1\}$, and S is a central set of S_n.

S_n is a cycle graph C_n with adding one pendant on every vertices of C_n, so $\forall u_i, v_i \in V(S_n)$ the eccentricity of u_i and v_i are $e(u_i) = n-1$ and $e(v_i) = n$, $1 \leq i \leq n$, respectively. Since $n-1$ is the smallest eccentricity, thus $rad(S_n) = n-1$ and u_i is a central vertex of S_n. So, the central set of sun graph S_n is $S = \{u_i | 1 \leq i \leq n\}$. ∎

Theorem 3. Let S_n is a sun graph, then $lmd_s(S_n) = n$

Bukti:

Define graph S_n with $V(S_n) = \{u_i | 1 \leq i \leq n\} \cup \{v_i | 1 \leq i \leq n\}$, $E(S_n) = \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{u_i v_i | 1 \leq i \leq n\} \cup \{u_n u_1\}$.

Let $S \subseteq V(G)$, $S = \{u_i | 1 \leq i \leq n\}$, respect to lemma 4, S is a central set of S_n and $|S| = n$. S_n is a sun graph where is the vertices u_i adjacent with u_{i+1} and $u_i, u_{i+1} \in S$, so respect to lemma 1, apply $r(u_i|S) \neq r(u_{i+1}|S)$, $\forall u_i, u_{i+1} \in V(S_n)$, $1 \leq i \leq n-1$. The same reason is applied to vertex u_n that adjacent with u_1, where $u_n, u_1, \in S$, so $r(u_n|S) \neq r(u_1|S)$. Next, vertex u_i adjacent with v_i, and $u_i \in S$ but $v_i \notin S$, so $r(u_i|S) \neq r(v_i|S)$. Thus, S is local metric set of S_n. Since S is a central set of S_n, then S is a central local reference set and $lmd_s(S_n) = n$.

The example of a central local reference set on S_3 is shown on Fig 8 with $S = \{u_1, u_2, u_3\}$ and $lmd_s(S_3) = 3$.

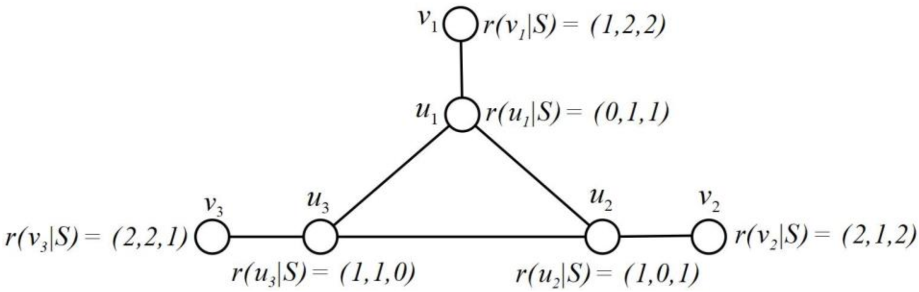

Fig. 8. a central local reference set on S_3

4 Conclusion

This paper discussed about a new development concept of central local reference set of some graphs. The minimal cardinality of central local reference set of graph h defined as central local metric dimension, denoted by $lmd_s(H)$. We applied the definition of central local metric dimension of complete graph K_n, complete bipartite graph $K_{n,m}$ and sun graph S_n. The result is $lmd_s(K_n) = n$, $lmd_s(K_{n,m}) = n + m$, and $lmd_s(S_n) = n$, these are indicated that the central local reference set of K_n and $K_{n,m}$ are equal with their vertex set. Meanwhile, the central local reference set of sun graph is equal with vertex set of its cycle. This is giving an implication that since both of radius and diameter of K_n and $K_{n,m}$ are equal so their central local reference set equal with their total vertex, meanwhile on sun graph both of radius and diameter different.

The concept of central local metric dimension is interesting to explore more. In the future, we would extend our work to find the characteristic of central local metric dimension of graph H and graph G operation H.

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References

1. Okamoto, F.; Phinezy, B.; Zhang, P. The Local Metric Dimension of a Graph. *Math. Bohem.* **2010**, *135*, 239–255, doi:10.21136/MB.2010.140702.
2. Budianto, W.T.; Kusmayadi, T.A. The Local Metric Dimension of Starbarbell Graph, Km ⊙ Pn Graph, and Möbius Ladder Graph. *J. Phys. Conf. Ser.* **2018**, *1008*, doi:10.1088/1742-6596/1008/1/012050.
3. Khoiriah, S.; Atmojo Kusmayadi, T. The Local Metric Dimension of Generalized Broken Fan Graph and Edge Corona Product of Star Graph and Path Graph. *J. Phys. Conf. Ser.* **2019**, *1306*, doi:10.1088/1742-6596/1306/1/012015.
4. Pratama, J.A.B.Y.; Kusmayadi, T.A. On the Local Metric Dimension of Dipyramidal Graph and King Graph. *AIP Conf. Proc.* **2021**, *2326*, doi:10.1063/5.0039319.

5. F, F. V.; Cynthia, V.J.A. Local Metric Dimension of Certain Wheel Related Graphs. *Int. J. Math. Comput. Sci.* **2021**, *16*, 1303–1315.
6. Rodríguez-Velázquez, J.A.; Barragán-Ramírez, G.A.; García Gómez, C. On the Local Metric Dimension of Corona Product Graphs. *Bull. Malaysian Math. Sci. Soc.* **2016**, *39*, 157–173, doi:10.1007/s40840-015-0283-1.
7. Levin, S.; Darmaji, D. Dimensi Metrik Lokal Pada Amalgamasi Graf Lengkap Dengan Graf Roda Dan Graf Kincir. *J. Sains dan Seni ITS* **2020**, *9*, A1–A5, doi:10.12962/j23373520.v9i1.52129.
8. Benish, H.; Murtaza, M.; Javaid, I. The Fractional Local Metric Dimension of Graphs. **2018**.
9. Fernau, H.; Rodríguez-Velázquez, J.A. On the (Adjacency) Metric Dimension of Corona and Strong Product Graphs and Their Local Variants: Combinatorial and Computational Results. *Discret. Appl. Math.* **2018**, *236*, 183–202, doi:10.1016/J.DAM.2017.11.019.
10. Badri, A.Y.; Darmaji Local Adjacency Metric Dimension of Sun Graph and Stacked Book Graph. *J. Phys. Conf. Ser.* **2018**, *974*, doi:10.1088/1742-6596/974/1/012069.
11. Pratama, F.A.; Susilowati, L.; Utoyo, M.I. DIMENSI METRIK KETETANGGAAN LOKAL GRAF HASIL OPERASI K-COMB. *Contemp. Math. Appl.* **2019**, *1*, 1–18, doi:10.20473/CONMATHA.V1I1.14771.
12. Sutardji, N.A.; Susilowati, L.; Purwati, U.D. Dimensi Metrik Kuat Lokal Graf Hasil Operasi Kali Kartesian. *Contemp. Math. Appl.* **2019**, *1*, 64–73, doi:10.20473/CONMATHA.V1I2.17383.
13. Umilasari, R.; Susilowati, L.; Slamin, S.; Prabhu, S. On the Dominant Local Metric Dimension of Corona Product Graphs. *IAENG Int. J. Appl. Math.* **2022**, *52*, 1098–1104.
14. Chartrand, G.; Eroh, L.; Johnson, M.A.; Oellermann, O.R. Resolvability in Graphs and the Metric Dimension of a Graph. *Discret. Appl. Math.* **2000**, *105*, 99–113, doi:10.1016/S0166-218X(00)00198-0.
15. Susilowati, L.; Sa'adah, I.; Fauziyyah, R.Z.; Erfanian, A.; Slamin The Dominant Metric Dimension of Graphs. *Heliyon* **2020**, *6*, e03633, doi:10.1016/J.HELIYON.2020.E03633.
16. Chartrand, G.; Lesniak, L.; Zhang, P. *Graphs & Digraphs 6th Edition*; sixth.; CRC Press: Boca Raton, FL, 2016; ISBN 9781498735803.
17. Liu, M.; Yuan, Y.; Das, K.C. The Sun Graph Is Determined by Its Signless Laplacian Spectrum. *Czechoslov. Math. J.* **2020**, *70*, 21–31, doi:10.21136/CMJ.2019.0159-18.