Suitability of *S*-Shaped Functions for Fitting Smooth *S*-Hydrograph

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Abstract. In hydrologic analysis of floods, after having derived a unit hydrograph (UH) of a specified duration, one may wish to change the time step and define a new UH of desired duration. The conventional/classical S-hydrograph method used for the same always results in erroneous or fluctuating altered duration UHs with impaired unit volume. As effect of S-curve oscillations is transmitted in resultant altered duration UHs. Therefore, S-shaped lower incomplete gamma function (LIGF) and its analytically modified flexible version (modified S-curve function) are proposed for oscillation-free (hunting-free) fitting of *D-hr* S-curve against parent/conventional *D-hr* S-curve. The reliable function is selected on the basis of least deviation of analytical *D-hr* S-curve from the conventional *D-hr* S-curve, evaluated using Nash-Sutcliffe efficiency (η_{NS}). Herein, oscillation-free analytical *D-hr* S-curve and its lagged versions lead us to the smooth shaped altered duration UHs conserving unit volume. Proposed analytical S-curves outperform the conventional S-curve. However, the modified S-curve function (with $\eta_{NS} = 99.85\%$) was found comparatively more reliable than LIGF (with $\eta_{NS} = 99.62\%$) over 1st dataset (or UH) and hence used for analysis over 2nd dataset whereas; the LIGF was skipped. For 2nd dataset modified S-curve function rebuild the S-curve reliably with $\eta_{NS} = 99.78\%$. Overall, the complexity associated with classical method i.e. lengthy calculations; erroneous lagging and incompatible interpolation of parent UH or S-curve; tedious manual/graphical smoothing of derived S-curve or altered duration UHs for sustaining unit volume has been justifiably resolved by proposed analytical approaches.

Keywords: Conventional or Classical S-Curve · Lower Incomplete Gamma Function · Modified S-Curve Function · Altered Duration Unit Hydrograph

1 Introduction

The impact of spatial and temporal variability of rainfall its interactions with surface or groundwater flows and of variable basin attributes viz. type of soil, vegetation, land-uses and land-cover, hydrologic state and pre-existing soil moisture is reflected in the flow regimes [4, 7]. Hence, basin respond uniquely to each storm. Thus, irregular storm event UHs produced need to be brought to a preferred unique duration of effective precipitation. The conventional method of superposition is useful only when requisite UH duration

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exceeds the parent or original UH duration by an integer multiple. In short, if a *D-hr* UH is known and it is desired to develop an UH of *nD-hr* (where *n* is an integer), it can be easily derived by superposing *n* UHs lagged by *D-hr*. Alternatively, if it is desired to develop an *mD-hr* UH (where *m* is a fraction), then the method of superposition is unsuitable. In such cases, the conventional S-hydrograph approach which is still integral to linear systematic hydrology is applied [1, 2, 17, 19] but shortening the UH duration, invite more complexities [15] as well as desired accuracy about the response time is lost for broader time intervals [6]. The hypothetically smooth S- or mass-curve denotes the rising discharge up to its upper limit (equilibrium). Being a mass curve, it does not have negative slope at any point but, its slope become flat after the base period of parent UH, when the discharge is uniform. The S-hydrograph bears identical duration as of its parent UH. If modified UH duration is split multiple of original UH duration, interpolation of original UH is essential before estimating an S-curve. The linear interpolation results in stepped altered UHs and are difficult to explain in a physical sense. The incompatibility of linear interpolation to natural nonlinear S-curve results in erratic peak (long duration UH $>$ short duration UH). In practice, the S-curve plotted using field data is normally bears oscillations/hunting effect. There is doubt that, an oscillation-free finite period UH will yield a hunting-free S-curve. Therefore, in practice, before taking the difference of parent and lagged S-hydrograph the parent S-hydrograph smoothed manually so that it would attain the equilibrium at a time equal to base time (t_b) of known UH. The oscillations in S-curve represent that, the parent UH does not actually represent runoff at a uniform rate for *t-hr*. Such S-curve fluctuations may results in grossly inaccurate or negative tail end ordinates of a modified UH [18]. Such unexpected fluctuations may affect the constraints for hydrologic stability [16]. Hence, these UHs are also reshaped manually to smooth out oscillations and satisfying their volume as unity, which is the only time-consuming and laborious practice advised in many hydrologic textbooks [12].

While deriving UH with ordinary least-square method the oscillation [20] appeared are need to be smooth out to sustain the stability of hydrologic system. The ridge least-square and other optimization techniques proposed to eliminate oscillations, are either complex or subjective [3]. Polynomial S-curve was not found entirely satisfactory. The sigmoid curve is not well suited for fitting natural asymmetrical S-hydrographs. The fluctuations-free S-hydrograph can be reliably mimicked analytically [12] by using Obha's Inflection S-Shaped growth model [10] conventionally used for software reliability check. Recently, a Savitzky-Golay (SG) smoothing-differentiation filter method to mitigate oscillations proposed by [16], which reduces ambiguity while deciding the filter parameters and allows the S-hydrograph's numerical differentiation with a function having finite number of breaks to derive an instantaneous UH using efficient matrix computation. The growth curve at a catchment outlet is its S-hydrograph. The mass curve of the Soil Conservation Service (SCS) dimensionless UH is often referred as S-hydrograph as it bear S-shape [5]. Researchers [14] claimed that, an incomplete gamma function represent a mass curve. Owing to this concept, the lower incomplete gamma function (LIGF) and its analytically modified version (Modified S-curve) are proposed for analytical reproduction of oscillation-free *D-hr* S-curve or its lagged versions, which led us to the smooth shaped altered duration UHs of unit volume.

2 Materials and Methods

2.1 Conventional or Classical-S-Curve or S-Hydrograph Approach

The parent unit duration (*D-hr*) UH was lagged continuously by *D-hr* till $t = t_b$ and resultant UHs are added together to derive the *D-hr* parent S-hydrograph (Subramanya 2013). A modified UH of the preferred integer or fractional (τ*-hr*) duration is obtained by shifting the known *D-hr* S-hydrograph by τ*-hr*, subtracting the shifted S-hydrograph from parent S-hydrograph, and then normalizing the volume i.e. dividing the resultant ordinates by the ratio (τ/D). The derived τ*-hr* UH replicates distribution of rainfall at the rate l/τ depth units/hr for τ*-hrs* and is termed as altered duration (τ*-hr*) conventional or classical UH.

2.2 Analytical LIGF Based S-Hydrograph Approach

Gamma cumulative distribution function (CDF) 'F(x)' is defined in terms of LIGF as [9] shown in Eq. (1). The LIGF (Eq. 2) is expressed by rearranging Eq. (1).

$$F[x] = \frac{\gamma\left(shape, x/\,scale\right)}{\Gamma(shape)} = \frac{\gamma\left(c, x/\,b\right)}{\Gamma(c)} \tag{1}$$

$$\gamma\left[c, x/\,b\right] = \Gamma(c) \times F[x] = \Gamma(c) \times \frac{1}{b^c\Gamma(c)} \int_0^x t^{c-1} e^{\frac{-t}{b}}\, dt \quad for x \geq 0;\, b, c > 0$$

$$when\ x = 0,\ \gamma\left(c, x/\,b\right) = 0;\ \ and\, x = t_b,\, \gamma\left(c, x/\,b\right) = Q_{eq} \tag{2}$$

where, t_b = base time of *D-hr* parent UH, Q_{eq} = equilibrium outflow of S-curve. Equation (2) defines the analytical *D-hr* parent S-curve and its parameters (b and c) needs calibration. This function is 'incomplete' because it integrates over a part of the region (i.e. 0 to positive value, x) defining the Gamma function. Analytical LIGF based *D-hr* parent S-curve $\gamma[c, x/b]$ (Eq. 2) can be shifted by desired τ*-hr* duration $\gamma[c, x - \tau/b]$ as:

$$\gamma\left[c, x - \tau/\,b\right] = \Gamma(c) \times F[x - \tau] = \Gamma(c) \times \frac{1}{b^c\Gamma(c)} \int_0^{x-\tau} t^{c-1} e^{\frac{-t}{b}}\, dt$$

$$for\ x \geq 0;\ \ b,\ c, \tau > 0 \tag{3}$$

The difference of *D-hr* S-curves $[\gamma(c, x/b) - \gamma(c, x - \tau/b)]$ shifted by (τ*-hr*) duration represents the ordinates of a modified duration (τ*-hr*) UH. The greatest deviation between shifted S-curves replicates peak corresponds to time to peak (t_p) of τ*-hr* UH.

Procedure for Fitting of Altered/Desired/Modified/Preferred Duration UH.

1. Fit the analytical (LIGF based) *D-hr* S-curve (Eq. 2) against the conventional *D-hr* S-curve to optimize b & c using generalized reduced gradient nonlinear programming (GRG-NLP) algorithm by maximizing η_{NS} (Eq. 6) as an objective function.
2. With known b & c, lag analytical *D-hr* S-curve with preferred duration (τ*-hr*) (Eq. 3).
3. Subtract shifted S-curve from original S-curve and divide the resulting ordinates by the ratio (τ/D) at each pulse to obtain the modified duration (τ*-hr*) UH.

2.3 Analytical Modified S-Curve Function Based S-Hydrograph Approach

A minor modification in 'LIGF' leads us to another more accurate S-shaped function (modified S-curve function), to portray a smooth hunting-free S-curve with negligible deviation from actual/classical S-curve. Herein, the Gamma function '$\Gamma(c)$' of Eqs. (2 or 3) is replaced by exponential function pertaining a new parameter (d), whereas the associated Gamma CDF, 'F(x)' is kept unmodified.

The modified S-curve function based *D-hr* parent S-curve (Eq. 4) and its shifted form at preferred τ*-hr* duration (Eq. 5) expressed using parameters b, c, and d, are as:

$$S_{(x)} = e^{d\times\Gamma(\ln(c))} \times F(x) = e^{d\times\Gamma(\ln(c))} \times \frac{1}{b^c\Gamma(c)} \int\limits_0^x t^{c-1} e^{\frac{-t}{b}} dt$$
$$\text{for } x \geq 0;\ b,\ c, d > 0 \text{ when } x = 0,$$
$$S_{(x)} = 0;\ \text{and if } x = t_b,\ S_{(x)} = Q_{eq} \tag{4}$$

$$S_{(x-\tau)} = e^{d\times\Gamma(\ln(c))} \times F(x-\tau)$$
$$= e^{d\times\Gamma(\ln(c))} \times \frac{1}{b^c\Gamma(c)} \int\limits_0^{x-\tau} t^{c-1} e^{\frac{-t}{b}} dt \quad \text{for } x \geq 0;\ b,\ c, d, \tau > 0 \tag{5}$$

Repeat the procedure given in Sect. 2.2 by replacing 'Eq. (2) with Eq. (4)' and 'Eq. (3) with Eq. (5).' Here, the parameters calibrated primarily are b, c, and d.

3 Application and Performance Evaluation

Unit duration (4-*hr* and 6-*hr*) parent UHs from popular hydrology textbooks [11, 13] were employed to test the analytical approaches against conventional S-curve using η_{NS} [8] given as:

$$\underset{[0\ \cdots\ 100\ \%]}{\eta_{NS}} = \left[1 - \left(\sum_{i=1}^{No\ of\ Ordinates} (Observed - Computed)^2 \Big/ \sum_{i=1}^{No\ of\ Ordinates} \left(Observed - \overline{Observed}\right)^2\right)\right] \times 100 \tag{6}$$

The parameters of LIGF (b and c) and of modified S-curve function (b, c, and d) were calibrated instantly using maximization of η_{NS} as an objective function defined in solver routine of spreadsheet, to obtain close hunting-free analytical fitting of both the *D-hr* S-curves w.r.t. conventional *D-hr* S-curve.

4 Analysis, Results and Discussion

4.1 Conversion of 6-*hr* Parent UH to Preferred Duration (3-*hr* and 9-*hr*) UHs

Classical S-hydrograph (Table 1): For conversion of a 6-*hr* parent UH to a preferred duration (3-*hr* and 9-*hr*) UHs, initially 'conventional/classical 6-*hr* S-curve (Fig. 1)' is derived by adding 6-*hr* parent UHs lagged continuously by 6-*hr* till the time = t_b = 54-*hrs*. Further, classical 6-*hr* S-curve is smoothed manually such a that an equilibrium flow, $Q_{eq} = Ad/0.36D = [(1/0.36) \times (35100 km^2) \times (1 cm/6\text{-}hr)] = 16250 \text{m}^3/\text{s}$ is achieved at a time = t_b = 54-*hr* (Fig. 1). The 'manually refined classical 6-*hr* S-curve' is comparatively more stable/smoother than the 'classical 6-*hr* S-curve' and hence, adopted hereinafter. The difference of ordinates of manually refined classical 6-*hr* S-curve and its shifted forms at 3-*hr* and 9-*hr* gives '3-*hr* and 9-*hr* classical UHs' when divided by the ratio τ/D = 3/6 and 9/6, respectively at each ordinate (Figs. 2 and 3). Both these modified UHs conserves same volume $[\Sigma Q_i \times (\Delta t/D) = 16250 \text{ m}^3/\text{s}]$ and satisfies the volumetric equality (with Q_{eq}) perfectly but still bears undulations (Figs. 2 and 3) which infer that the undulations have not been fully obviated even by physically smoothing the classical 6-*hr* S-curve. Hence, the analytical approaches having efficacy in eliminating oscillations have been exemplified.

Analytical LIGF based S-hydrograph (Tables 1 and 2): The manually refined conventional 6-*hr* S-curve has been replicated (Fig. 4) with η_{NS} = 99.62% using analytical LIGF (Eq. 2) by optimizing *b* and *c* (Table 2), where Q_{eq} (= 16250 m^3/s) is achieved at

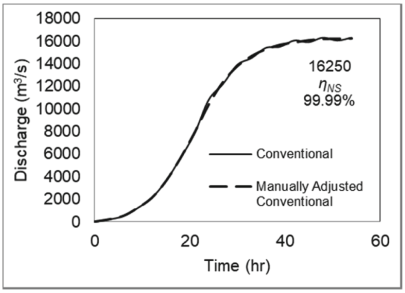

Fig. 1. Classical and manually refined classical 6-*hr* S-curves.

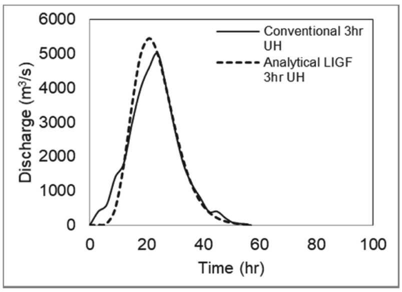

Fig. 2. 3-hr UH by classical and analytical LIGF approaches.

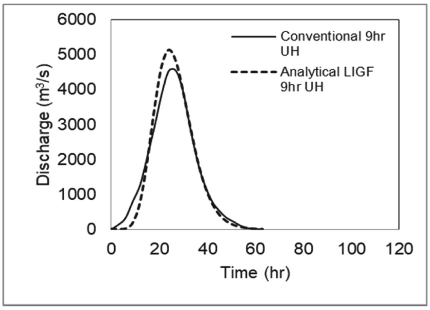

Fig. 3. 9-hr UH by classical and analytical LIGF approaches.

time = t_b = 54-*hrs*. The difference of analytical LIGF based 6-*hr* S-hydrograph (Eq. 2) and its lagged forms at 3-*hr* and 9-*hr* (Eq. 3) gives the 'analytical LIGF based 3-*hr* and 9-*hr* UHs' when divided by τ/D = 3/6 and 9/6, respectively (Table 1; Figs. 2 and 3). The oscillations in the classical UHs are wiped out absolutely by the analytical LIGF based 3-*hr* and 9-*hr* UHs (Figs. 2 and 3) reflecting symmetric shapes with higher peaks over shortened time base and are justified by the prefect mass conservation as of Q_{eq} in terms of their identical runoff volumes [$\Sigma Q_i \times (\Delta t/D) = 16250$ m^3/s] (Table 1).

Analytical Modified S-curve Function based S-hydrograph (Tables 1 and 2): In order to rebuild 'manually refined conventional 6-*hr* S-curve' the modified S-curve function (Eq. 4) is utilized with optimized *b*, *c*, & *d* (Table 2) and 'analytical modified 6-*hr* S-curve' is achieved (Table 1; Fig. 5). Here, Q_{eq} (=16250 m^3/s) is achieved at the same time = t_b = 54-*hrs* as of analytical LIGF based 6-*hr* S-curve. Higher $\eta_{NS} = 99.85\%$ achieved in this case reflects the precision of modified S-curve function (Fig. 5). The 'analytical modified 3-*hr* and 9-*hr* UH (Table 1, Figs. 6 and 7)' were derived by dividing the difference of analytical modified 6-*hr* S-curve (Eq. 4) and its shifted versions (Eq. 5) at 3-*hr* and 9-*hr* with the ratio τ/D = 3/6 and 9/6, respectively. Oscillations free analytical UHs derived is the best alternative over classical UHs as justified by the minimum deviation in the peaks and exact mass conservation as of Q_{eq} attained through analogous runoff volumes [$\Sigma Q_i \times (\Delta t/D) = 16250$ m^3/s]. Figures 2, 3, 4, 5, 6, and 7 reveals that

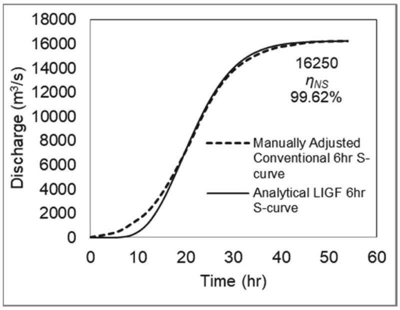

Fig. 4. Manually refined classical and analytical LIGF based 6-*hr* S-curves.

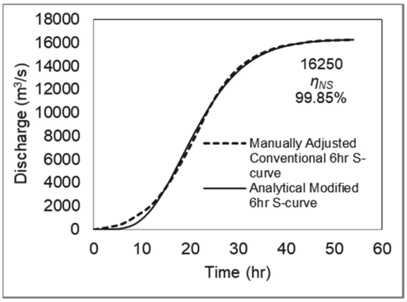

Fig. 5. Manually refined classical and analytical modified function based 6-*hr* S-curves.

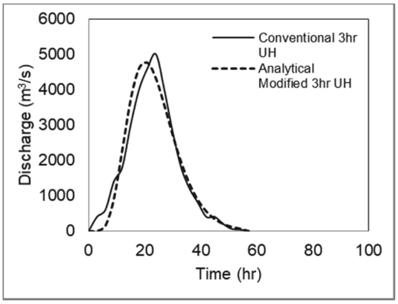

Fig. 6. 3-hr UH by classical & analytical modified S-curve based approaches.

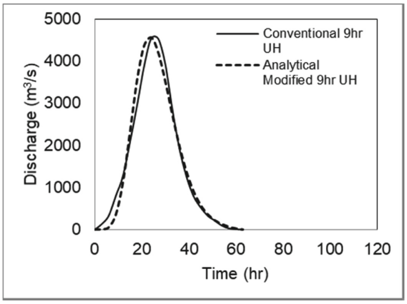

Fig. 7. 9-hr UH by classical & analytical modified S-curve based approaches.

the modified S-curve function has performed wisely than the LIGF in altered duration UH derivation and therefore further applied on second dataset or parent UH.

4.2 Altered UH Derivation from a Given 4-hr Parent UH [13]

With a given 4-*hr* parent UH, analytical modified 4-*hr* S-curve (Fig. 8) is derived with $\eta_{NS} = 99.78\%$ against the conventional 4-*hr* S-curve. Further, the preferred duration

Table 1. S-curves and modified duration UHs derived from 6-*hr* parent UH [11] using by classical and analytical approaches

Time (*hr*)	6-*hr* Parent UH (m^3/s)	6-*hr* S-curve (m^3/s)				3-*hr* UH (m^3/s)			9-*hr* UH (m^3/s)		
		Classical	Manually Refined Classical	Analytical LIGF	Analytical Modified S-curve	Classical	Analytical LIGF	Analytical Modified S-curve	Classical	Analytical LIGF	Analytical Modified S-curve
0	0	0	0	0	0	0	0	0	0	0	0
3	200	200	200	0	2	400	0	4	133	0	1
6	500	500	500	22	86	600	44	167	333	15	57
9	1000	1200	1200	261	557	1400	477	941	800	174	371
12	1600	2100	2100	1135	1726	1800	1749	2338	1267	757	1149
15	2400	3600	3600	2908	3604	3000	3546	3758	2067	1924	2346
18	3500	5600	5600	5396	5918	4000	4977	4626	2933	3424	3574
21	4200	7800	7900	8118	8297	4600	5443	4759	3867	4655	4381
24	5200	10800	10400	10605	10450	5000	4975	4305	4533	5132	4564
27	4400	12200	12400	12591	12217	4000	3971	3535	4533	4797	4200
30	3100	13900	13800	14017	13563	2800	2853	2692	3933	3933	3511
33	2300	14500	14700	14959	14528	1800	1883	1930	2867	2902	2719
36	1500	15400	15300	15539	15186	1200	1160	1317	1933	1965	1979
39	1000	15500	15700	15876	15618	800	674	862	1267	1239	1370
42	650	16050	15900	16063	15890	400	373	545	800	736	908

(*continued*)

Table 1. *(continued)*

Time (*hr*)	6-*hr* Parent UH (m^3/s)	6-*hr* S-curve (m^3/s)				3-*hr* UH (m^3/s)			9-*hr* UH (m^3/s)		
		Classical	Manually Refined Classical	Analytical LIGF	Analytical Modified S-curve	Classical	Analytical LIGF	Analytical Modified S-curve	Classical	Analytical LIGF	Analytical Modified S-curve
45	400	15900	16100	16162	16058	400	198	335	533	415	581
48	250	16300	16200	16213	16158	200	102	200	333	224	360
51	150	16050	16230	16238	16216	60	50	117	220	117	217
54	0	16300	16250	16250	16250	40	24	67	100	59	128
57			16250	16250	16250	0	0	0	33	25	61
60			16250	16250	16250				13	8	22
63			16250	16250	16250				0	0	0
Volume of UH = $\Sigma Qi \times (\Delta t / D)$ m^3/s						16250	16250	16250	16250	16250	16250

Table 2. LIGF & modified S-curve function based S-curve and UHs parameters

Parent UH Duration (D-hr)	Altered UH Duration (τ-hr)	LIGF based Parameter Estimates					Modified S-curve Function based Parameter Estimates					
		b	c	t_p (hr)	z_{max} (m^3/s)	Q_p (m^3/s)	b	c	t_p (hr)	d	z_{max} (m^3/s)	Q_p (m^3/s)
6	3	2.55	8.57	21	2721.63	5443.26	3.43	6.40	21	1.77	2379.64	4759.48
	9			24	7697.36	5131.57			24		6845.37	4563.58

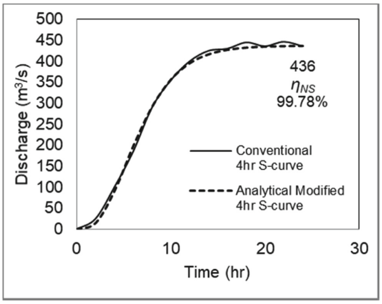

Fig. 8. Classical and analytical modified function based 4-*hr* S-curves.

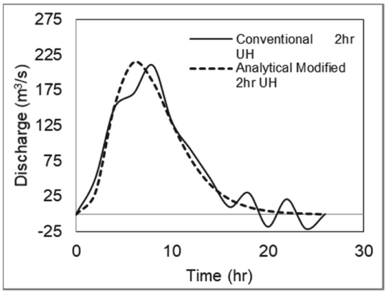

Fig. 9. 2-hr UH by classical & analytical modified S-curve based approaches.

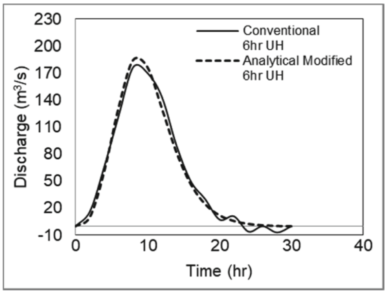

Fig. 10. 6-hr UH by classical & analytical modified S-curve based approaches.

(I.E. 2-*hr* and 6-*hr*) oscillation-free analytical modified UHs are compared with the 2-*hr* and 6-*hr* classical UHs (Figs. 9 and 10). The derived preferred duration analytical modified UHs satisfies volumetric equality reliably.

5 Conclusions

1. The LIGF and modified S-curve function proposed to build smooth *D-hr* S-curves and derive altered duration UHs are proved to be valuable while testing with field data in hydrologic analysis.

2. The proposed time saving analytical S-shaped approaches are capable enough to exactly replicate hunting-effect free *D-hr* parent S-curve or its lagged version at desired τ*-hr* duration. The proposed approaches outperforming the conventional S-curve approach and are ranked on the basis of their performance as modified S-curve function followed by LIGF.
3. The two parameters (b & c) LIGF has been analytically improved including one more parameter and it becomes three parameters (b, c, and d) modified S-curve function.
4. The modified S-curve function ($\eta_{NS} = 99.85\%$) was found comparatively more reliable or flexible than LIGF ($\eta_{NS} = 99.62\%$) in terms of rebuilding of hunting-free *D-hr* parent S-curve as well as reproduction of peak of altered duration UHs.
5. The analytical LIGF or modified S-curve based altered duration UHs of same duration are of approximately similar shape, as both these approaches are relied on same mathematical perception i.e. Gamma distribution and exhibits greater reliability.

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