Theory of Minimum Dissipation Rate in Fluid Dynamics

Hao Wu*, Zhipei Zhang

Yellow River Engineering Consulting Co., Ltd., Zhengzhou, Henan 450003, China

*250320158@qq.com

Abstract. The incompressible, viscous and steady fluid is studied basing the energy dissipation equation, motion equations, continuity equation and boundary conditions. It is proved that the necessary and sufficient condition of the energy rate obtaining the extreme value is the vorticity field meeting the harmonic equations. The vorticity field meeting the harmonic equations is equivalent to the viscous items with potential. The relationship between the energy rate and the motion equations has also been studied. For the real motion, the necessary and sufficient condition of the energy rate obtaining the extreme value is equivalent to the inertia items with potential. Finally, an example about the theory is given.

Keywords: Theory of Minimum Rate of Energy Dissipation, Necessary and Sufficient Condition, Viscous Items, Inertia Items, Motion Equations

1 Introduction

Helmholtz in 1868 proposed the "theory of minimum rate of energy dissipation" for the slow viscous flow. The basic point is that in the gravity field, for incompressible, viscous fluid, if the inertia items of the motion equations can be ignored, the rate of energy dissipation by the real velocity distribution is less than any others (virtual) in this volume with the same velocity distribution in the volume's surface[1].

Some scholars had studied the theory of the minimum dissipation rate after Helmholtz. In 1970s, Chinese-American scholar Yang C.T. and Chang H.H. had got great progress in the theory[2~9], and their findings have some certain influence in later research. Current research in the theory with influence mainly are the literatures[10~13]. Some others[14~19] are mainly focused on the application of the theory of the minimum dissipation rate.

Helmholtz conclusion about the "theory of minimum rate of energy dissipation" gives a sufficient condition of fluid movements, but if the inertia items in the fluid motion equations are not zero, the "minimum rate of energy dissipation" is still a controversial issue. In literature[10], the necessary and sufficient condition of the minimum rate of energy dissipation for planar, incompressible, steady and viscous fluid is proposed by introducing the stream function, but this conclusion can not be extended to the three-dimensional flow, because the stream function may not exist.

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For three-dimensional flow, "the theory of minimum rate of energy dissipation" is still an unresolved issue. In this paper, the necessary and sufficient condition of the "theory of minimum rate of energy dissipation" for incompressible, steady and viscous fluid in three-dimensions is proposed and proved. As well, the relationship between the "theory of minimum rate of energy dissipation" and the motion equations will be analyzed.

2 Definitions

For narrative convenience, some definitions are given below.

If a group of velocity distribution meets the continuity equation in a closed region and meets the velocity boundary conditions on the closed region's border, this group of velocity distribution is called the possible velocity distribution.

If a group of velocity distribution meets the continuity equation and motion equations in the closed region and meets the velocity boundary conditions on the closed region's border, this group of velocity distribution is called the real velocity distribution.

The real velocity distribution must be the possible velocity distribution, while the possible velocity distribution maybe not the real velocity distribution, because it does not always satisfy the motion equations.

For a possible velocity distribution $u_x i + u_y j + u_z k$, if a function U exists, and it makes the inertia items of the possible velocity distribution meet the follow equations

$$\frac{\partial U}{\partial x} = u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \tag{1}$$

$$\frac{\partial U}{\partial y} = u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} \tag{2}$$

$$\frac{\partial U}{\partial z} = u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} \tag{3}$$

the inertia items are called the potential inertia items. Similarly, if a function V exists, and makes the viscous items of the possible velocity distribution meet the follow equations

$$\frac{\partial V}{\partial x} = \Delta u_x \tag{4}$$

$$\frac{\partial V}{\partial y} = \Delta u_y \tag{5}$$

$$\frac{\partial V}{\partial z} = \Delta u_z \tag{6}$$

the viscous items are called the potential viscous items. In equation(4), equation (5) and equation (6), $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$.

3 Theory Of Minimum Rate of Energy Dissipation

For incompressible, steady and viscous fluid, the theory of minimum rate of energy dissipation can be described as follow: in all the possible velocity distributions, the necessary and sufficient condition of the minimum rate of energy dissipation is that the viscous items are the potential items.

According to the total differential theorem, it is not difficult to prove that the viscous items with potential function are equivalent to a vorticity field satisfying the harmonic equations, therefore, the theory of minimum rate of energy dissipation can be expressed as follow: in all the possible velocity distributions, the necessary and sufficient condition of the minimum rate of energy dissipation is the vorticity field of the velocity distribution satisfying the harmonic equations.

4 Sufficiency

Sufficiency: in all the possible velocity distributions, if the viscous items of the velocity distribution are the potential items, the rate of energy dissipation is the minimum.

Proof: it is supposed that $u_x i + u_y j + u_z k$ is a possible velocity distribution with the potential viscous items, and $(u_x + \varepsilon_x)i + (u_y + \varepsilon_x)j + (u_z + \varepsilon_x)k$ is another possible velocity distribution.

These two possible velocity distributions should satisfy the continuity equation as follow in the closed region D

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \tag{7}$$

$$\frac{\partial (u_x + \varepsilon_x)}{\partial x} + \frac{\partial (u_y + \varepsilon_y)}{\partial y} + \frac{\partial (u_z + \varepsilon_z)}{\partial z} = 0 \tag{8}$$

According to equation (7) and equation(8), thus

$$\frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_y}{\partial y} + \frac{\partial \varepsilon_z}{\partial z} = 0 \tag{9}$$

The possible velocity distributions should satisfy the velocity boundary conditions as follow on the closed region D's border ∂D

$$u_x\big|_{\partial D} = f(x,y,z),\ u_y\big|_{\partial D} = g(x,y,z),\ u_z\big|_{\partial D} = h(x,y,z) \tag{10}$$

$$u_x+\varepsilon_x\big|_{\partial D}=f(x,y,z),\ u_y+\varepsilon_y\big|_{\partial D}=g(x,y,z),\ u_z+\varepsilon_z\big|_{\partial D}=h(x,y,z) \quad (11)$$

According to equation (10) and equation(11), thus

$$\varepsilon_x\big|_{\partial D}=0,\ \varepsilon_y\big|_{\partial D}=0,\ \varepsilon_z\big|_{\partial D}=0 \quad (12)$$

According to the expression of the rate of energy dissipation, thus

$$\Phi(\vec{u}+\vec{\varepsilon})=\mu\iiint_D 2\left[\left(\frac{\partial(u_x+\varepsilon_x)}{\partial x}\right)^2+\left(\frac{\partial(u_y+\varepsilon_y)}{\partial y}\right)^2+\left(\frac{\partial(u_z+\varepsilon_z)}{\partial z}\right)^2\right]$$

$$+\left[\left(\frac{\partial(u_x+\varepsilon_x)}{\partial y}+\frac{\partial(u_y+\varepsilon_y)}{\partial x}\right)^2+\left(\frac{\partial(u_y+\varepsilon_y)}{\partial z}+\frac{\partial(u_z+\varepsilon_z)}{\partial y}\right)^2\right.$$

$$\left.+\left(\frac{\partial(u_z+\varepsilon_z)}{\partial x}+\frac{\partial(u_x+\varepsilon_x)}{\partial z}\right)^2\right]dxdydz \quad (13)$$

$$\Phi(\vec{u})=\mu\iiint_D 2\left[\left(\frac{\partial u_x}{\partial x}\right)^2+\left(\frac{\partial u_y}{\partial y}\right)^2+\left(\frac{\partial u_z}{\partial z}\right)^2\right]$$

$$+\left[\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)^2+\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)^2+\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right)^2\right]dxdydz \quad (14)$$

$$\Phi(\vec{\varepsilon})=\mu\iiint_D 2\left[\left(\frac{\partial \varepsilon_x}{\partial x}\right)^2+\left(\frac{\partial \varepsilon_y}{\partial y}\right)^2+\left(\frac{\partial \varepsilon_z}{\partial z}\right)^2\right]$$

$$+\left[\left(\frac{\partial \varepsilon_x}{\partial y}+\frac{\partial \varepsilon_y}{\partial x}\right)^2+\left(\frac{\partial \varepsilon_y}{\partial z}+\frac{\partial \varepsilon_z}{\partial y}\right)^2+\left(\frac{\partial \varepsilon_z}{\partial x}+\frac{\partial \varepsilon_x}{\partial z}\right)^2\right]dxdydz \quad (15)$$

According to equation(13), equation (14) and equation(15), thus

$$\Phi'=\Phi(\vec{u}+\vec{\xi})-\Phi(\vec{u})-\Phi(\vec{\xi})=\mu\iiint_D 4\left(\frac{\partial u_x}{\partial x}\frac{\partial \xi_x}{\partial x}+\frac{\partial u_y}{\partial y}\frac{\partial \xi_y}{\partial y}+\frac{\partial u_z}{\partial z}\frac{\partial \xi_z}{\partial z}\right)$$

$$+2\left[\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)\left(\frac{\partial \xi_y}{\partial x}+\frac{\partial \xi_x}{\partial y}\right)+\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right)\left(\frac{\partial \xi_z}{\partial y}+\frac{\partial \xi_y}{\partial z}\right)\right.$$

$$\left.+\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)\left(\frac{\partial \xi_x}{\partial z}+\frac{\partial \xi_z}{\partial x}\right)\right]dxdydz$$

$$=2\mu\iiint_D\left(\frac{\partial u_x}{\partial x}\frac{\partial \xi_x}{\partial x}+\frac{\partial u_y}{\partial y}\frac{\partial \xi_y}{\partial y}+\frac{\partial u_z}{\partial z}\frac{\partial \xi_z}{\partial z}\right)$$

$$+\left(\frac{\partial u_z}{\partial x}\frac{\partial \xi_z}{\partial x}+\frac{\partial u_z}{\partial y}\frac{\partial \xi_z}{\partial y}+\frac{\partial u_z}{\partial z}\frac{\partial \xi_z}{\partial z}\right)+\left(\frac{\partial u_y}{\partial x}\frac{\partial \xi_x}{\partial y}+\frac{\partial u_x}{\partial y}\frac{\partial \xi_y}{\partial x}\right)$$

$$+\left(\frac{\partial u_z}{\partial y}\frac{\partial \xi_y}{\partial z}+\frac{\partial u_y}{\partial z}\frac{\partial \xi_z}{\partial y}\right)+\left(\frac{\partial u_x}{\partial z}\frac{\partial \xi_z}{\partial x}+\frac{\partial u_z}{\partial x}\frac{\partial \xi_x}{\partial z}\right)dxdydz \quad (16)$$

According to Green's first formula and the equation(12), thus

$$\iiint_D(\frac{\partial u_x}{\partial x}\frac{\partial \xi_x}{\partial x}+\frac{\partial u_x}{\partial y}\frac{\partial \xi_x}{\partial y}+\frac{\partial u_x}{\partial z}\frac{\partial \xi_x}{\partial z})dxdydz$$

$$=\oiint_{\partial D}\xi_x\left(\frac{\partial u_x}{\partial x}\cos\alpha+\frac{\partial u_x}{\partial y}\cos\beta+\frac{\partial u_x}{\partial z}\cos\gamma\right)ds-\iiint_D\xi_x\Delta u_x dxdydz$$

$$=-\iiint_D\xi_x\Delta u_x dxdydz \quad (17)$$

Similarly,

$$\iiint_D(\frac{\partial u_y}{\partial x}\frac{\partial \xi_y}{\partial x}+\frac{\partial u_y}{\partial y}\frac{\partial \xi_y}{\partial y}+\frac{\partial u_y}{\partial z}\frac{\partial \xi_y}{\partial z})dxdydz=-\iiint_D\xi_y\Delta u_y dxdydz \quad (18)$$

$$\iiint_D(\frac{\partial u_z}{\partial x}\frac{\partial \xi_z}{\partial x}+\frac{\partial u_z}{\partial y}\frac{\partial \xi_z}{\partial y}+\frac{\partial u_z}{\partial z}\frac{\partial \xi_z}{\partial z})dxdydz=-\iiint_D\xi_z\Delta u_z dxdydz \quad (19)$$

Thus

$$\Phi'=2\mu\iiint_D\left(\frac{\partial u_x}{\partial x}\frac{\partial \xi_x}{\partial x}+\frac{\partial u_y}{\partial y}\frac{\partial \xi_y}{\partial y}+\frac{\partial u_z}{\partial z}\frac{\partial \xi_z}{\partial z}\right)$$

$$+\left(\frac{\partial u_y}{\partial x}\frac{\partial \xi_x}{\partial y}+\frac{\partial u_x}{\partial y}\frac{\partial \xi_y}{\partial x}\right)+\left(\frac{\partial u_z}{\partial y}\frac{\partial \xi_y}{\partial z}+\frac{\partial u_y}{\partial z}\frac{\partial \xi_z}{\partial y}\right)+\left(\frac{\partial u_x}{\partial z}\frac{\partial \xi_z}{\partial x}+\frac{\partial u_z}{\partial x}\frac{\partial \xi_x}{\partial z}\right)$$

$$-(\xi_x\Delta u_x+\xi_y\Delta u_y+\xi_z\Delta u_z)dxdydz$$

$$=2\mu\iiint_D\left(\frac{\partial u_x}{\partial x}\frac{\partial \xi_x}{\partial x}+\frac{\partial u_y}{\partial x}\frac{\partial \xi_x}{\partial y}+\frac{\partial u_z}{\partial x}\frac{\partial \xi_x}{\partial z}\right)$$

$$+\left(\frac{\partial u_y}{\partial y}\frac{\partial \xi_y}{\partial y}+\frac{\partial u_x}{\partial y}\frac{\partial \xi_y}{\partial x}+\frac{\partial u_z}{\partial y}\frac{\partial \xi_y}{\partial z}\right)+\left(\frac{\partial u_z}{\partial z}\frac{\partial \xi_z}{\partial z}+\frac{\partial u_y}{\partial z}\frac{\partial \xi_z}{\partial y}+\frac{\partial u_x}{\partial z}\frac{\partial \xi_z}{\partial x}\right)$$

$$-(\xi_x\Delta u_x+\xi_y\Delta u_y+\xi_z\Delta u_z)dxdydz \quad (20)$$

According to Green's first formula, equation(7) and equation(12), thus

$$\iiint_D \left(\frac{\partial u_x}{\partial x}\frac{\partial \xi_x}{\partial x}+\frac{\partial u_y}{\partial x}\frac{\partial \xi_x}{\partial y}+\frac{\partial u_z}{\partial x}\frac{\partial \xi_x}{\partial z}\right)dxdydz$$

$$=-\iiint_D \xi_x \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)dxdydz=0 \quad (21)$$

Similarly,

$$\iiint_D \left(\frac{\partial u_y}{\partial y}\frac{\partial \xi_y}{\partial y}+\frac{\partial u_x}{\partial y}\frac{\partial \xi_y}{\partial x}+\frac{\partial u_z}{\partial y}\frac{\partial \xi_y}{\partial z}\right)dxdydz=0 \quad (22)$$

$$\iiint_D \left(\frac{\partial u_z}{\partial z}\frac{\partial \xi_z}{\partial z}+\frac{\partial u_y}{\partial z}\frac{\partial \xi_z}{\partial y}+\frac{\partial u_x}{\partial z}\frac{\partial \xi_z}{\partial x}\right)dxdydz=0 \quad (23)$$

According to equation(20), equation(21), equation(22) and equation(23), thus

$$\Phi' = 2\mu\iiint_D -(\xi_x\Delta u_x+\xi_y\Delta u_y+\xi_z\Delta u_z)dxdydz \quad (24)$$

According to equation(4), equation(5) and equation(6), equation(24) can be written as follows

$$\Phi' = 2\mu\iiint_D -\left(\xi_x\frac{\partial V}{\partial x}+\xi_y\frac{\partial V}{\partial y}+\xi_z\frac{\partial V}{\partial z}\right)dxdydz \quad (25)$$

According to Green's first formula and the equation(12), thus

$$\iiint_D \xi_x\frac{\partial V}{\partial x}dxdydz=\oiint_{\partial D}\xi_x V\cos\alpha ds-\iiint_D V\frac{\partial \xi_x}{\partial x}dxdydz=-\iiint_D V\frac{\partial \xi_x}{\partial x}dxdydz \quad (26)$$

Similarly,

$$\iiint_D \xi_y\frac{\partial V}{\partial y}dxdydz=-\iiint_D V\frac{\partial \xi_y}{\partial y}dxdydz \quad (27)$$

$$\iiint_D \xi_z\frac{\partial V}{\partial z}dxdydz=-\iiint_D V\frac{\partial \xi_z}{\partial z}dxdydz \quad (28)$$

According to equation(9), equation(25), equation(26), equation(27) and equation(28), thus

$$\Phi' = 2\mu\iiint_D V\left(\frac{\partial \xi_x}{\partial x}+\frac{\partial \xi_y}{\partial y}+\frac{\partial \xi_z}{\partial z}\right)dxdydz=0 \quad (29)$$

According to equation(16) and equation(29), thus

$$\Phi(\vec{u}+\vec{\varepsilon})-\Phi(\vec{u})-\Phi(\vec{\varepsilon})=0 \tag{30}$$

For any $\vec{u}$ and $\vec{\varepsilon}$, $\vec{\Phi}$ is not negative, thus

$$\Phi(\vec{u}+\vec{\varepsilon})-\Phi(\vec{u})=\Phi(\vec{\varepsilon})\geq 0 \tag{31}$$

Equation(31) shows that in all the possible velocity distributions, if the viscous items of the velocity distribution are the potential items, the rate of energy dissipation is the minimum. Proven.

5 Necessity

Necessity: in all the possible velocity distributions, if the rate of energy dissipation is the minimum, the viscous items of the velocity distribution are the potential items.

Proof of the necessity involves a variational problem with differential constraint. Functional as follow

$$J[y]=\int_{x_0}^{x} F(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n')dx \tag{32}$$

its constraint conditions are

$$\phi_i(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n')=0 \quad (i=1,2,\cdots,m, m<n) \tag{33}$$

its boundary conditions are

$$y_j(x_0)=y_{j0}, y_j(x_1)=y_{j1} \qquad (j=1,2,\cdots,n) \tag{34}$$

Boundary conditions equation (33) are claimed differential constraints, where $\phi_i(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n')=0 \ (i=1,2,\cdots,m, m<n)$ are independent from each other. Differential constraints are characterized by containing the partial derivative of y. Equation (32) is called the cost functional of the differential constraints. The extreme value problem of the functional (32) under the differential constraints (33) and boundary conditions (34) is known as the Lagrange problem.

Theorem 1: If the cost functional equation (32) obtains the extreme value under differential constraints (33) and boundary conditions (34), the selected functions $\lambda_i(x)$ exist, and make functions $y_1, y_2, \cdots, y_n$ satisfy the Euler equations of auxiliary functional

$$J^*[y]=\int_{x_0}^{x_1}\left[F+\sum_{i=1}^{m}\lambda_i(x)\phi_i\right]dx=\int_{x_0}^{x_1} Hdx \tag{35}$$

Euler equations as follow

$$H_{y_j} - \frac{d}{dx} H_{y_j'} = 0 (j = 1, 2, \cdots, n) \tag{36}$$

In equation(35), $H = F + \sum_{i=1}^{m} \lambda_i(x)\phi_i$.

Theorem 1 and its proof see literature[20], page from 181-188.

According to theorem 1, if under differential constraints

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \tag{37}$$

and boundary conditions

$$u_x\big|_{\partial D} = f(x, y, z), u_y\big|_{\partial D} = g(x, y, z), u_z\big|_{\partial D} = h(x, y, z) \tag{38}$$

the cost functional

$$\Phi(\vec{u}) = \mu \iiint_D 2\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] + \left[\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)^2\right] dxdydz \tag{39}$$

obtains the extreme value, a selected function $\lambda(x, y, z)$ exists, and makes functions $u_x(x, y, z)$, $u_y(x, y, z)$ and $u_z(x, y, z)$ satisfy the Euler equations

$$4\frac{\partial^2 u_x}{\partial x^2} + 2\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y}\right) + 2\left(\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial x \partial z}\right) + \frac{\partial}{\partial x}[\lambda(x, y, z)] = 0 \tag{40}$$

$$4\frac{\partial^2 u_y}{\partial y^2} + 2\left(\frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial x^2}\right) + 2\left(\frac{\partial^2 u_y}{\partial z^2} + \frac{\partial^2 u_z}{\partial y \partial z}\right) + \frac{\partial}{\partial y}[\lambda(x, y, z)] = 0 \tag{41}$$

$$4\frac{\partial^2 u_z}{\partial z^2} + 2\left(\frac{\partial^2 u_y}{\partial y \partial z} + \frac{\partial^2 u_z}{\partial y^2}\right) + 2\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_x}{\partial x \partial z}\right) + \frac{\partial}{\partial z}[\lambda(x, y, z)] = 0 \tag{42}$$

Equation(40), equation (41) and equation (42) can be written as

$$\Delta u_x + \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + \frac{1}{2}\frac{\partial \lambda}{\partial x} = 0 \tag{43}$$

$$\Delta u_y + \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + \frac{1}{2}\frac{\partial \lambda}{\partial y} = 0 \tag{44}$$

$$\Delta u_z + \frac{\partial}{\partial z}(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}) + \frac{1}{2}\frac{\partial \lambda}{\partial z} = 0 \tag{45}$$

According to equation(37), equation(43), equation (44) and equation(45), thus

$$\Delta u_x + \frac{1}{2}\frac{\partial \lambda}{\partial x} = 0 \tag{46}$$

$$\Delta u_y + \frac{1}{2}\frac{\partial \lambda}{\partial y} = 0 \tag{47}$$

$$\Delta u_z + \frac{1}{2}\frac{\partial \lambda}{\partial z} = 0 \tag{48}$$

The equation (46) partial derivative of y subtracts the equation (47) partial derivative of x, thus

$$\frac{\partial(\Delta u_y)}{\partial x} - \frac{\partial(\Delta u_x)}{\partial y} = 0 \tag{49}$$

Similarly,

$$\frac{\partial(\Delta u_z)}{\partial y} - \frac{\partial(\Delta u_y)}{\partial z} = 0 \tag{50}$$

$$\frac{\partial(\Delta u_x)}{\partial z} - \frac{\partial(\Delta u_z)}{\partial x} = 0 \tag{51}$$

According to equation (49), equation (50), equation (51) and total differential theorem, Δu_x, Δu_y and Δu_z must be the total differential of a function, namely, the viscous items of the velocity distribution are the potential viscous items. Proven.

6 Relationship Between Theory Of Minimum Rate Of Energy Dissipation And Motion Equations

As can be seen from the above results, the theory of minimum rate of energy dissipation is equivalent to the viscous items with potential or vorticity field satisfying the harmonic equations. There is no inevitable relationship between theory of minimum rate of energy dissipation and the motion equations.

It is not difficult to prove that if the possible velocity distribution meets the motion equations under a gravity field, the inertia items with potential are equivalent to the viscous items with potential. Therefore, the follow inference is available: in all the possible velocity distributions, the necessary and sufficient condition of the minimum rate of energy dissipation of the real velocity distribution is that the inertia items of the velocity distribution are the potential inertia items. This inference can be considered as the promotion of the Helmholtz conclusion about the theory of minimum rate of energy dissipation.

According to the inference, if the rate of energy dissipation of the real velocity distribution is the minimum, the inertia items of the velocity distribution are the potential items. Therefore, the motion equations can be written as

$$\frac{\partial}{\partial x}\left(U-G+\frac{p}{\rho}-\frac{\mu}{\rho}V\right)=0 \tag{52}$$

$$\frac{\partial}{\partial y}\left(U-G+\frac{p}{\rho}-\frac{\mu}{\rho}V\right)=0 \tag{53}$$

$$\frac{\partial}{\partial z}\left(U-G+\frac{p}{\rho}-\frac{\mu}{\rho}V\right)=0 \tag{54}$$

In equation(52), equation (53) and equation(54), U stands for the inertia items potential, G for the gravity items potential, p for the pressure potential, V for the viscous items potential. According to equation(52), equation (53) and equation(54), thus

$$U-G+\frac{p}{\rho}-\frac{\mu}{\rho}V=const \tag{55}$$

7 Examples For Theory of Minimum Dissipation Rate

7.1 Steady, Incompressible and Axisymmetric Planar Flow

Consider the steady, incompressible and axisymmetric planar flow described by the following equations

$$\begin{cases} V_r=\dfrac{K}{r} \\ V_\theta=f(r) \end{cases} \tag{56}$$

It can be confirmed that the equation (56) satisfy the following continuity equations

$$\frac{1}{r}\frac{\partial(rV_r)}{\partial r}+\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}=0 \tag{57}$$

The motion equations for steady, incompressible, viscous fluid in polar coordinates as follow

$$\begin{cases} V_r \dfrac{\partial V_r}{\partial r} - \dfrac{V_\theta^{\ 2}}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} + f_r + \nu\left(\nabla^2 V_r - \dfrac{V_r}{r^2}\right) \\ V_r \dfrac{\partial V_\theta}{\partial r} + \dfrac{V_r V_\theta}{r} = \nu\left(\nabla^2 V_\theta - \dfrac{V_\theta}{r^2}\right) \end{cases} \tag{58}$$

The following equations can be got by substitution of the equations (56) into the second formula of the equations (58)

$$r^2 \frac{\partial^2 V_\theta}{\partial r^2} - \frac{K-\nu}{\nu} r \frac{\partial V_\theta}{\partial r} - \frac{K+\nu}{\nu} V_\theta = 0 \tag{59}$$

Equation (59) is the second order homogeneous Euler equation, taking the boundary conditions

$$V_r\Big|_{r=r_1} = \frac{K}{r_1}, V_r\Big|_{r=r_2} = \frac{K}{r_2}, V_\theta\Big|_{r=r_1} = V_1, V_\theta\Big|_{r=r_2} = V_2 \tag{60}$$

with $r_2 > r_1 > 0$. The solution is as follows

(1) While $\dfrac{K}{\nu} = -2$

$$\begin{cases} V_r = \dfrac{K}{r} \\ V_\theta = C_1 \dfrac{1}{r} + C_2 \dfrac{\ln r}{r} = \dfrac{r_1 V_1 \ln r_2 - r_2 V_2 \ln r_1}{\ln \dfrac{r_2}{r_1}} \dfrac{1}{r} + \dfrac{r_2 V_2 - r_1 V_1}{\ln \dfrac{r_2}{r_1}} \dfrac{\ln r}{r} \end{cases} \tag{61}$$

(2) While $\dfrac{K}{\nu} \neq -2$

$$\begin{cases} V_r = \dfrac{K}{r} \\ V_\theta = C_3 \dfrac{1}{r} + C_4 r^{\left(\frac{K}{\nu}+1\right)} = \dfrac{r_1 V_1 r_2^{\left(\frac{K}{\nu}+2\right)} - r_2 V_2 r_1^{\left(\frac{K}{\nu}+2\right)}}{r_2^{\left(\frac{K}{\nu}+2\right)} - r_1^{\left(\frac{K}{\nu}+2\right)}} \dfrac{1}{r} + \dfrac{r_2 V_2 - r_1 V_1}{r_2^{\left(\frac{K}{\nu}+2\right)} - r_1^{\left(\frac{K}{\nu}+2\right)}} r^{\left(\frac{K}{\nu}+1\right)} \end{cases} \tag{62}$$

The limit type of the equation (62) is $\dfrac{0}{0}$ while $\dfrac{K}{\nu} = -2$. According to L'Hospital rules, it is can be got that the limit of equation (62) is the equation (61) while $\dfrac{K}{\nu} \to -2$.

7.2 Energy Rate Of Plane Axisymmetric Flow

While $\frac{K}{v}=-2$, the strains of equation (61) are as follow

$$A_{rr}=\frac{\partial V_r}{\partial r}=-\frac{K}{r^2} \tag{63}$$

$$A_{\theta\theta}=\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{V_r}{r}=\frac{K}{r^2} \tag{64}$$

$$2A_{r\theta}=\frac{1}{r}\frac{\partial V_r}{\partial \theta}+r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)=\frac{-2C_1+C_2-2C_2\ln r}{r^2} \tag{65}$$

The energy rate of equation (61) is as follow

$$\begin{aligned}\Phi_1&=2\mu\int_0^{2\pi}\int_{r_1}^{r_2}\left(A_{rr}^2+A_{\theta\theta}^2+2A_{r\theta}^2\right)rdrd\theta\\&=4\mu\pi\left[K^2\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)+\frac{\left(r_2V_2-r_1V_1\right)^2}{4\left(\ln r_2-\ln r_1\right)^2}\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)+\left(V_1^2-V_2^2\right)\right]\end{aligned} \tag{66}$$

Similarly,

(1) While $\frac{K}{v}=-1$, the energy rate of equation (62) is as follow

$$\Phi_2=4\mu\pi\left[K^2\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)+\frac{\left(r_2V_2-r_1V_1\right)^2}{2\left(r_2-r_1\right)^2}\ln\frac{r_2}{r_1}+\left(V_1^2-V_2^2\right)\right] \tag{67}$$

(2) While $\frac{K}{v}\neq-2$ and $\frac{K}{v}\neq-1$, the energy rate of equation (62) is as follow

$$\begin{aligned}\Phi_3=4\mu\pi\Bigg[&K^2\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)\\&+\left(r_2V_2-r_1V_1\right)^2\left(\frac{\frac{K}{v}+2}{r_2^{(\frac{K}{v}+2)}-r_1^{(\frac{K}{v}+2)}}\right)^2\frac{r_2^{2\left(\frac{K}{v}+1\right)}-r_1^{2\left(\frac{K}{v}+1\right)}}{4\left(\frac{K}{v}+1\right)}+\left(V_1^2-V_2^2\right)\Bigg]\end{aligned} \tag{68}$$

According to L'Hospital rules, it is can be got that the limit of equation (68) is the equation (66) while $\frac{K}{v}\to-2$, and the limit of equation (68) is the equation (67) while $\frac{K}{v}\to-1$.

7.3 The Possible Velocity Distribution With The Minimal Energy Rate

Consider the steady, incompressible and axisymmetric planar flow described by the following equations

$$\begin{cases} V_r = \dfrac{K}{r} \\ V_\theta = h(r) \end{cases} \tag{69}$$

It can be confirmed that the equation (69) satisfy the continuity equation (57) in the closed region. If it is assumed that the equation (69) satisfy the boundary conditions(60), the equation (69) are the possible velocity distribution, and its energy rate is as follow

$$\Phi = 4\mu\pi\left[K^2\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)+\frac{1}{2}\int_{r_1}^{r_2}\left(h'-\frac{h}{r}\right)^2 rdr\right] \tag{70}$$

According to the variational theory[3], if the equation (70) get the extreme value, h(r) meets the following equation

$$r^2h''+rh'-h=0 \tag{71}$$

The equation (71) is second order homogeneous Euler equation. According to the boundary conditions(60), the possible velocity distribution with the extreme energy rate can be got as follow

$$\begin{cases} V_r = \dfrac{K}{r} \\ V_\theta = \dfrac{C_5}{r}+C_6 r = \dfrac{r_1V_1r_2^2-r_2V_2r_1^2}{r_2^2-r_1^2}\dfrac{1}{r}+\dfrac{r_2V_2-r_1V_1}{r_2^2-r_1^2}r \end{cases} \tag{72}$$

The energy rate of the equation (72) is as follow

$$\Phi_4 = 4\mu\pi\left[K^2\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)+\frac{\left(r_2V_1-r_1V_2\right)^2}{r_2^2-r_1^2}\right] \tag{73}$$

In fact, the energy rate of the equation (72) is the minimal in all the possible velocity distribution. It is proved with the reduction to absurdity as follow.

It is assumed that the following velocity distribution is a possible velocity distribution with $\varepsilon(r)\neq 0$, and its energy rate is less than the equation(72).

$$V_r = \frac{K}{r}$$
$$V_\theta = \frac{C_5}{r} + C_6 r + \varepsilon(r) = \frac{r_1 V_1 r_2^2 - r_2 V_2 r_1^2}{r_2^2 - r_1^2}\frac{1}{r} + \frac{r_2 V_2 - r_1 V_1}{r_2^2 - r_1^2} r + \varepsilon(r) \tag{74}$$

It can be confirmed that the equation (74) satisfy the continuity equation (57) in the closed region. The equation (74) is a possible velocity distribution, so it satisfies the boundary conditions(60). Thus

$$\varepsilon(r)|_{r=r_1} = 0, \varepsilon(r)|_{r=r_2} = 0 \tag{75}$$

The energy rate of the equation(74) is as follow

$$\Phi = 4\mu\pi\left[K^2\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) + \frac{\left(r_2 V_1 - r_1 V_2\right)^2}{r_2^2 - r_1^2}\right] + 2\mu\pi\int_{r_1}^{r_2}\left[r^{\frac{3}{2}}\frac{\partial}{\partial r}\left(\frac{\varepsilon}{r}\right)\right]^2 dr \tag{76}$$

Contrasting the equation(76) and the equation(73), and taking into account the boundary conditions of the equation(75), if $\varepsilon(r) \neq 0$, the energy rate of the equation(76) is larger than the equation (73). It is a contradiction. Therefore, in all the virtual, planar, axisymmetric flow, the energy of the equation (72) is the minimum.

7.4 Confirmation Of The Necessary And Sufficient Condition (The Vorticity Field Meets The Harmonic Equation)

The vorticity field in cylindrical coordinates is as follow

$$\Omega = \text{rot}\vec{V} = \left[\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right] i_r + \left[\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right] i_\theta + \frac{1}{r}\left[\frac{\partial\left(r V_\theta\right)}{\partial r} - \frac{\partial V_r}{\partial \theta}\right] i_z \tag{77}$$

The vorticity field of the axisymmetric, planar flow is as follow

$$\Omega = \text{rot}\vec{V} = \frac{1}{r}\frac{\partial\left(r V_\theta\right)}{\partial r} i_z \tag{78}$$

The harmonic equation in cylindrical coordinates is as follow

$$\Delta\vec{u} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial u}{\partial z}\right)\right] \tag{79}$$

The harmonic equation of the axisymmetric, planar flow is as follow

$$\Delta\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{80}$$

The harmonic equation of the axisymmetric, planar flow's vorticity field is as follow

$$\Delta\vec{u}=\Delta\left(\text{rot}\vec{V}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\left(rV_\theta\right)}{\partial r}\right)\right)i_z \tag{81}$$

Two cases are confirmed according to the different velocity field expressions of the equation (61) and the equation(62).

(1) While $\frac{K}{v}=-2$

The vorticity field of the equation(61) is as follow

$$\begin{aligned}\Delta\vec{u}&=\Delta\left(\text{rot}\vec{V}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\left(C_1+C_2\ln r\right)}{\partial r}\right)\right)i_z\\&=C_2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\left(\ln r\right)}{\partial r}\right)\right)i_z\\&=C_2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial r^{-2}}{\partial r}\right)i_z=-2C_2\frac{1}{r}\frac{\partial r^{-2}}{\partial r}i_z=\frac{4C_2}{r^4}i_z\end{aligned} \tag{82}$$

Confirm 1: While $C_2=0$, then $\Delta\Omega=0$, the energy rate of the equation (61) is the minimal.

According to the equation(61), while $C_2=0$, then $r_2V_2-r_1V_1=0$, the velocity field of the equation (61) is the same as the velocity field of the equation(72) as follow

$$\begin{cases}V_r=\dfrac{K}{r}\\V_\theta=\dfrac{r_1V_1}{r}\end{cases} \tag{83}$$

So, the energy rate of the equation (61) is the same as the equation(72). It is proved that the energy rate of the equation (72) is the minimal in all the possible velocity distribution, so the energy rate of the equation (61) is the minimal.

Confirm 2: While $C_2\neq 0$, then $\Delta\Omega\neq 0$, the energy rate of the equation (61) is not the minimal.

According to the equation(61), while $C_2\neq 0$, then $r_2V_2-r_1V_1\neq 0$, the velocity field of the equation (61) is not the same as the velocity field of the equation(72). So, the energy rate of the equation (61) is not the same as the equation(72). It is proved that the energy rate of the equation (72) is the minimal in all the possible velocity distribution, so the energy rate of the equation (61) is not the minimal.

(2) While $\frac{K}{v}\neq -2$

The vorticity field of the equation(62) is as follow

$$\Delta\vec{u}=\Delta\left(\mathrm{rot}\vec{V}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\left(C_3+C_4r^{\left(\frac{K}{\nu}+2\right)}\right)}{\partial r}\right)\right)i_z$$

$$=C_4\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r^{\left(\frac{K}{\nu}+2\right)}}{\partial r}\right)\right)i_z \tag{84}$$

$$=C_4\left(\frac{K}{\nu}+2\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial r^{\frac{K}{\nu}}}{\partial r}\right)i_z=\left[C_4\left(\frac{K}{\nu}+2\right)\left(\frac{K}{\nu}\right)^2r^{\left(\frac{K}{\nu}-2\right)}\right]i_z$$

Confirm 3: While $C_4 = 0$ or $K = 0$, then $\Delta\Omega = 0$, the energy rate of the equation (62) is the minimal.

Slimily with the confirm 1.

Confirm 4: While $C_4 \neq 0$ or $K \neq 0$, then $\Delta\Omega \neq 0$, the energy rate of the equation (62) is not the minimal.

Slimily with the confirm 2.

In fact, it can also be confirmed by contrasting the energy rate expressions, the equation(66), the equation(67), the equation(68) and the equation(73), directly. The value of the equation(66), the equation(67) and the equation(68) is equivalent to the value of the equation(73) in case $\Delta\Omega = 0$, and the value of the equation(66), the equation(67) and the equation(68) is larger than the value of the equation(73) in case $\Delta\Omega \neq 0$. But the contrasting process is quit tedious, it is no longer listed.

In all above the cases, it is confirmed that the necessary and sufficient condition of the energy rate obtaining the extreme value is the vorticity field meeting the harmonic equations.

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