Approximating Fixed Point of Weak Contraction Mapping Using General Picard-Mann Iteration Process

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Abstract. Fixed point theory plays a crucial role in solved real world problems in terms of nonlinear problems or differential equations for a suitable mapping. In numerous problems, it is difficult to find an exact fixed point for some contractions mapping. Hence, it is necessary to develop numerical algorithm to approximating the fixed point of contractions mapping. The aim of this article is to study the estimation of fixed point of weak contraction mapping via General Picard Mann (GPM) algorithm. This iteration process is combined Picard and Mann iterative scheme and generalized by making k iteration of that combination. In this research, we prove that the sequences that generated by GPM iterative scheme convergent strongly to unique fixed for weak contractions mapping in uniformly convex Banach spaces. We also prove that GPM algorithm is converge to fixed point faster than any other well-known iteration processes. We compare GPM iterative methods to six other iterative methods related to speed of convergence Finally, we give numerical results for find fixed point of weak contraction mapping using GMP iteration with k = 3 and k = 5 and compare with others under different number of parameters and different function.

Keywords: Fixed Point, Weak Contraction, General Picard-Mann

1 Introduction

Fixed point theory is a branch of mathematical analysis that focused on the existence and uniqueness of specific mapping in some abstract spaces. It has many applications for solving mathematical problems that related to differential equations, dynamical system, or engineering. Many real-world problems can be solved by fixed point for suitable mapping. This theory starts from Banach contraction principle that proves the uniqueness of fixed point of contraction mapping in complete metric spaces.

In the beginning of twentieth centuries, many researchers extended the notion of Banach contraction principle, in various abstract spaces and also formulated the new kind or type of contraction mapping. At first, focus of the theory of fixed point is just to find the fixed point. However, in some situations, it is difficult to find fixed points

N. Djam'an et al. (eds.), *Proceedings of the 5th International Conference on Statistics, Mathematics, Teaching, and Research 2023 (ICSMTR 2023)*, Advances in Computer Science Research 109,
https://doi.org/10.2991/978-94-6463-332-0_4

using direct methods especially for nonlinear mapping. Therefore, an iterative approximations is presented as an alternative to estimate the fixed point.

Charles Picard [1] proposed simple iterative algorithm to find fixed points of contraction mapping. However, Picard method fails to converge for nonexpansive mapping. This situation forces many researchers to find new iterative scheme for nonexpansive mapping such as Mann iteration process [2]. After that, many researchers proposed new type of iterative algorithm to approximate fixed point of other types contraction mapping with improvement in their rate of convergence [3 - 7].

Motivated by above, the aim of this study is to investigate a strong convergence theorem of iterative process for the class of weak contraction mappings in Banach spaces using General Picard-Mann iteration process and compare with well-known iteration process. We also give numerical example for supporting our analytical prove that shows General Picard-Mann iteration converges faster than other methods in estimate the fixed point of weak contraction mapping.

2 Basic Properties

In this part, we provide some definitions and basic properties related to weak contraction mapping, F-stable, rate convergence that used in main results.

Definition 1. [9] Let K be a nonempty subset of Banach spaces X. A self-map $F: K \to K$ is called weak contraction if there is a number $\delta \in (0,1)$ and some $L \geq 0$ that satisfied

$$\|Fx - Fy\| \leq \delta \|x - y\| + L\|y - Fx\| \tag{1}$$

for all $x, y \in K$.

Theorem 1. [9] Let K be a nonempty subset of Banach spaces X. If $F: K \to K$ be a mapping that satisfying (1) and

$$\|Fx - Fy\| \leq \delta \|x - y\| + L\|x - Fx\| \tag{2}$$

then the mapping F has a unique fixed point in X.

Weak contraction mapping is a generalization of contraction mapping that was proposed by Berinde. If we take $L = 0$ in inequalities (1), we will get the contraction mapping. In Theorem 1 we need to add condition (2) to ensure that weak contractions mapping has unique fixed point.

The following definitions proposed by Berinde that used to compare the rate of convergence of iterative methods.

Definition 2. [13] Given sequences $\{a_n\}, \{b_n\} \subseteq \mathbf{R}^+$ that converges to a and b, respectively. Suppose that

$$L = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} \quad (3)$$

1. If $L = 0$, then $\{a_n\}$ converges to a faster than $\{b_n\}$ converges to b.
2. If $0 < L < \infty$, then $\{a_n\}$ converges with the same rate as $\{b_n\}$.

Definition 3. [13] Let $\{x_n\}$ and $\{y_n\}$ are two iterative schemes that converges to same point p with the error estimates $|x_n - x| \le a_n$ and $|y_n - y| \le b_n$. If $\lim_{n\to\theta} \frac{a_n}{b_n} = 0$, then $\{x_n\}$ converges faster than $\{y_n\}$.

3 Some Iterative Methods

For a mapping $F: K \to K$, some researcher defined iterative methods to estimate the fixed point of *F*. For given $x_1 \in K$ and sequences $\{a_n\},\{b_n\},\{c_n\} \subseteq (0,1)$, we have

1. Mann iteration [2]

$$x_{n+1} = (1 - a_n) x_n + a_n F x_n \quad (4)$$

2. Ishikawa iteration [13]

$$\begin{cases} y_n = (1 - b_n) x_n + b_n F x_n \\ x_{n+1} = (1 - a_n) x_n + a_n F y_n \end{cases} \quad (5)$$

3. Sintunavarat iteration [14]

$$\begin{cases} z_n = (1 - b_n) x_n + b_n F x_n \\ y_n = (1 - c_n) x_n + c_n z_n \\ x_{n+1} = (1 - a_n) F y_n + a_n F z_n \end{cases} \quad (6)$$

4. F^*-iteration [15]

$$\begin{cases} y_n = F\left((1 - a_n) x_n + a_n F x_n\right) \\ x_{n+1} = F y_n \end{cases} \quad (7)$$

5. Piri et al. iterative algorithm [16]

$$\begin{cases} z_n = F\left[(1 - b_n) x_n + b_n F x_n\right] \\ y_n = F z_n \\ x_{n+1} = (1 - a_n) y_n + a_n F y_n \end{cases} \quad (8)$$

6. Jubair et al. iterative algorithm [17]

$$\begin{cases} z_n = (1-c_n)x_n + c_n F x_n \\ y_n = (1-b_n)Fx_n + b_n F z_n \\ x_{n+1} = F\left((1-a_n)y_n + a_n F y_n\right) \end{cases} \tag{9}$$

Shukla, et.al. [18] defined a new type of iterative algorithm which combined Mann Iteration with k number of repetition for the Picard iteration. This iterative scheme named General Picard-Mann Iterative (GPM) with

$$x_{n+1} = F^k\left((1-a_n)x_n + a_n F x_n\right) \tag{10}$$

for some natural number k.

4 Results and Discussion

In this section we give some results related to converges of using GPM in convex Banach Spaces.

Theorem 2. Let K be a nonempty subset convex and closed of Banach space X. If $F: K \to K$ be weak contraction satisfying condition (2), then the sequence defined by GPM iterative scheme (10) converges strongly to a unique fixed point of F.

Proof.

Let $p \in K$ be a fixed point of F. Since F is a weak contraction and satisfying condition (2), so we have

$$\begin{aligned} \|Fx_n - p\| &= \|Fx_n - Fp\| \\ &\leq \delta\|x_n - p\| + L\|p - Tp\| \\ &= \delta\|x_n - p\| \end{aligned} \tag{11}$$

Using GPM iterative method (10), for $k = 0$

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-a_n)x_n + a_n F x_n - p\| \\ &\leq (1-a_n)\|x_n - p\| + a_n\|Fx_n - p\| \\ &\leq (1-a_n + \delta a_n)\|x_n - p\| \end{aligned} \tag{12}$$

For $k = 1$, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|F\{(1-a_n)x_n + a_n F x_n\} - p\| \\ &\leq \delta\{(1-a_n)\|x_n - p\| + a_n\|Fx_n - p\|\} \\ &= \delta(1-a_n + \delta a_n)\|x_n - p\| \end{aligned} \tag{13}$$

Using the similar process, we have

$$\|x_{n+1}-p\|\leq \delta^k\left(1-a_n+\delta a_n\right)\|x_n-p\| \tag{14}$$

Because $\delta\in(0,1)$ and the sequence $\{a_n\}\subseteq(0,1)$, then $(1-a_n+\delta a_n)<1$. Hence,

$$\|x_{n+1}-p\|\leq \delta^k\|x_n-p\| \tag{15}$$

Using inequality (15), we get

$$\|x_{n+1}-p\|\leq \delta^{k(n+1)}\|x_0-p\| \tag{16}$$

Since $\delta\in(0,1)$, then the sequence $\{x_n\}$ strongly converges to p.

Therem shows that GPM iteration scheme is faster than Mann, Ishikawa, Sintunavarat, F*-iteration, Piri, and Jubair iteration process.

Theorem 3. Suppose that K be a nonempty subset convex and closed of Banach space X. Let $\{x_{(1)n}\}$ is sequence defined by Mann, $\{x_{(2)n}\}$ by Ishikawa, $\{x_{(3)n}\}$ by Sintunavarat, $\{x_{(4)n}\}$ by Ali (F* iteration), $\{x_{(5)n}\}$ by Piri et al, $\{x_{(6)n}\}$ Jubair et al. and $\{x_n\}$ dined by Shukla et al (GPM). If $F:K\rightarrow K$ be weak contraction satisfying condition (2) and all iterative methods converge to similar fixed point, then GPM with $k\geq 3$ converges faster than any algorithms a fixed point.

Proof.

Let's look inequalities (16) in Theorem 2, then

$$\|x_{n+1}-p\|\leq \delta^{k(n+1)}\|x_0-p\| \tag{17}$$

Let $\mu_n=\delta^{k(n+1)}\|x_0-p\|$ for all $n\in\mathbf{N}$ and $k\geq 3$. From Ali and Ali [15] (Theorem 2.3), we have

$$\left\|x_{(4)n}-p\right\|\leq \delta^{2(n+1)}\left\|x_{(4)0}-p\right\|=\mu_{(4)n} \tag{18}$$

Therefore,

$$\frac{\mu_n}{\mu_{(4)n}}=\frac{\delta^{k(n+1)}\|x_0-p\|}{\delta^{2(n+1)}\left\|x_{(4)0}-p\right\|}=\frac{\delta^{(k-2)(n+1)}\|x_0-p\|}{\left\|x_{(4)0}-p\right\|} \tag{19}$$

Since $k\geq 3$ and $\delta<1$, so $\frac{\mu_n}{\mu_{(4)n}}\rightarrow 0$ as $n\rightarrow\infty$. According to definition, the sequence $\{x_n\}$ converges to fixed point faster than $\{x_{(4)n}\}$ or GPM algorithm with

$k \geq 3$ is faster than F* algorithm. Since the F* algorithm is faster than Mann, Ishikawa and Sintunavarat algorithm, according to Ali and Ali, then GPM algorithm with $k \geq 3$ is also converge faster than that algorithm.

For Piri Iterative methods (8), we have

$$\|Fx_n - p\| \leq \delta \|x_n - p\| \tag{20}$$

After that

$$\begin{aligned} \|z_n - p\| &= \left\|F\left[\left(1-b_n\right)x_n + b_n F x_n\right] - p\right\| \\ &\leq \delta \left\|\left(1-b_n\right)x_n + b_n F x_n - p\right\| \\ &\leq \delta\left(1-b_n+\delta b_n\right)\|x_n - p\| \end{aligned} \tag{21}$$

and

$$\|y_n - p\| = \|Fz_n - p\| \leq \delta^2 \left(1-b_n+\delta b_n \|x_n - p\|\right) \tag{22}$$

Therefore,

$$\begin{aligned} \|x_{n+1} - p\| &= \left\|\left(1-a_n\right)y_n + a_n F y_n - p\right\| \\ &\leq \left(1-a_n\right)\|y_n - p\| + \delta a_n \|y_n - p\| \\ &\leq \delta^2 \left(1-a_n+\delta a_n\right)\left(1-b_n+\delta b_n\right)\|x_n - p\| \end{aligned} \tag{23}$$

Since $\left(1-a_n+\delta a_n\right)$ and $\left(1-b_n+\delta b_n\right)$ is less than 1, then $\|x_{n+1} - p\| \leq \delta^2 \|x_n - p\|$ or

$$\|x_{n+1} - p\| \leq \delta^{2(n+1)} \|x_0 - p\| \tag{24}$$

So

$$\frac{\mu_n}{\mu_{(5)n}} = \frac{\delta^{k(n+1)} \|x_0 - p\|}{\delta^{2(n+1)} \left\|x_{(5)0} - p\right\|} = \frac{\delta^{(k-2)(n+1)} \|x_0 - p\|}{\left\|x_{(5)0} - p\right\|} \tag{25}$$

Since $k \geq 3$ and $\delta < 1$, so $\frac{\mu_n}{\mu_{(5)n}} \to 0$ as $n \to \infty$. Therefore, the sequence $\{x_n\}$ converges to fixed point faster than $\left\{x_{(5)n}\right\}$ or GPM algorithm with $k \geq 3$ is faster than Piri algorithm

Furthermore, by Jubair algorithm, we get

$$\left\|x_{(5)n} - p\right\| \leq \delta^{2(n+1)} \left\|x_{(5)0} - p\right\| = \mu_{(5)n} \tag{26}$$

So,

$$\frac{\mu_n}{\mu_{(6)n}} = \frac{\delta^{k(n+1)}\left\|x_0 - p\right\|}{\delta^{2(n+1)}\left\|x_{(6)0} - p\right\|} = \frac{\delta^{(k-2)(n+1)}\left\|x_0 - p\right\|}{\left\|x_{(6)0} - p\right\|} \tag{27}$$

Since $k \geq 3$ and $\delta < 1$, so $\frac{\mu_n}{\mu_{(6)n}} \to 0$ as $n \to \infty$. Therefore, the sequence $\{x_n\}$ converges to fixed point faster than $\{x_{(6)n}\}$ or GPM algorithm with $k \geq 3$ is faster than Jubair algorithm. Hence, GPM algorithm with $k \geq 3$ is faster than all iterative methods that mentioned before.

5 Numerical Results

Let $X = \mathbf{R}$ be a Banach Space with usual norm and $K = [1,50]$. Suppose that $F: K \to K$ defined by $Fx = x + 1 - \ln x$. For all $x \in K$. It can be verified that Fx is weak contraction mapping that satisfied condition (2) so that F has unique fixed point. Let defined sequence $\{a_n\} = \left\{\frac{3}{5n+7}\right\}, \{b_n\} = \{0.15\}, \{c_n\} = \left\{\frac{1}{\sqrt{n}}\right\}$ subset of $(0,1)$ and choose initial value $x_1 = 20$. It is clear that all conditions of Theorem 3 and Theorem 5 are satisfied. Using Matlab, we show that GPM algorithm with k = 3 and k = 5 converges faster to fixed point than any others iterative methods discussed earlier.

Table 1. A comparison a rate of convergence among different iteration methods

Iteration	Mann	Ishikawa	Sintunavarat	F*	Piri	Jubair	GPM k=3	GPM k=5
1	20.00000	20.00000	20.00000	20.00000	20.00000	20.00000	20.00000	20.00000
2	19.50110	17.93320	17.78240	15.66660	15.40460	15.48110	13.91510	10.774
3	19.00840	15.97980	15.72270	11.90250	11.46190	11.61820	9.08244	5.12112
⋮								
8	16.64300	8.14737	7.72759	3.09398	2.97018	3.02858	2.73540	2.71831
9	16.19000	7.01362	6.61464	2.86335	2.80852	2.83480	2.72222	2.71828
⋮								
14	14.03020	3.61301	3.44537	2.71923	2.71872	2.71898	2.71828	2.71828
⋮								
19	12.05370	2.82264	2.79585	2.71829	2.71828	2.71829	2.71828	2.71828
20	11.68140	2.78405	2.76658	2.71828	2.71828	2.71828	2.71828	2.71828
39	6.17156	2.71829	2.71829	2.71828	2.71828	2.71828	2.71828	2.71828
⋮								
41	5.77002	2.71829	2.71828	2.71828	2.71828	2.71828	2.71828	2.71828
42	5.58186	2.71828	2.71828	2.71828	2.71828	2.71828	2.71828	2.71828

⋮								
78	2.86554	2.71828	2.71828	2.71828	2.71828	2.71828	2.71828	2.71828
79	2.85235	2.71828	2.71828	2.71828	2.71828	2.71828	2.71828	2.71828
80	2.84031	2.71828	2.71828	2.71828	2.71828	2.71828	2.71828	2.71828

From Table 1, we can see that General Picard-Iteration with $k = 5$ converges faster than others iterative algorithm. The GPM process with $k = 5$ converges to fixed point 2.71828 (with accuracy 5 decimal point) just need nine iterations. It is clear that GPM methods is faster than other methods because GPM iteration with $k = 5$ is appropriate with five step iterative schemes. However, for GPM with $k = 3$ is still faster than others three step iteration process (Jubair, Piri, and Sintunavarat iteration) which is need fourteen iterations compare with nineteen iterations for Piri, twenty for Jubair, and forty-one iterations for Sintunavarat iteration process. Interestingly for F^*-iteration, although it is classified as two step iterative schemes, it just needs 39 iteration process to estimate the fixed point. Ishikawa iteration process have to 78 iteration process to reach fixed point with five decimal place accuracy, while Mann iteration need more than eighty iterations to reach the fixed point.

The following figures show the behavior of every iterative method to reach the fixed point.

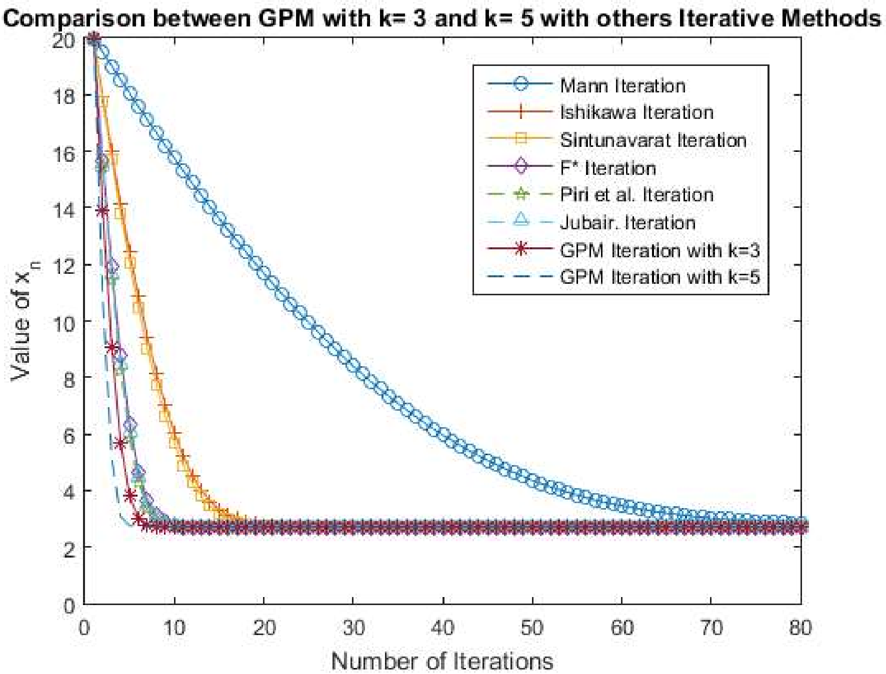

Fig. 1. Convergence behavior of GPM with $k = 3$ and $k = 5$ compare with others iteration process.

6 Conclusion

In this article, we study about convergence and stability of General Picard-Mann iteration process for weak contraction mapping in Banach spaces. We also show that General Picard-Mann algorithm is faster than iterative scheme defined in (4 – 9). Furthermore, we give numerical example to support our analytical results.

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