Some Notes of Iteration Double Sequences and Double Series on Real Number

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Abstract. A development of single sequences and single series (with domain N) is double sequences and double series (with domain $N \times N$). The discussion in this research focuses specifically on the iterated series of a double series. The research method used is literature review. The study begins by providing a definition of an iterated series, then shows its properties related to the convergence of double series. This research concludes that if a double series converges to $s \in R$ then the following applies: (1) Iterated series $\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) = s$ if and only if $\sum_{m=1}^{\infty} x(n,m)$ exists for every $n \in N$, (2) Iterated series $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right) = s$ if and only if $\sum_{n=1}^{\infty} x(n,m)$ exists for every $m \in N$. On the other hand, a double series where the terms can be expressed as the product of two single series which converge respectively to $x \in R$ and $y \in R$, then the double series converges to xy and both iterated series also have the value xy. All double sequences and double series discussed are limited to the codomain of real numbers.

Keywords: Convergent, Double Sequences, Double Series, Iterated Series

1 Introduction

Mathematics is a science that studies and develops structures containing a number of axioms, definitions, postulates, propositions, theorems, lemmas, consequences and conjectures that support each other and do not conflict. Several branches of mathematics include analysis, algebra, statistics, applied mathematics, actuarial science, financial mathematics, computational mathematics, geometry, and others. One of the studies in the analysis is real analysis [1]. One of the topics in real analysis is sequences. In mathematics, a sequence is used to represent specific numbers within an infinite sequence. The individual numbers in a sequence are referred to as its terms [2]. According to [3], the definition of a sequence of real numbers (denoted as $\{x_n\}_{n=1}^{\infty}$, (x_n), or $(x_n : n \in N)$) is a function defined on the set of natural numbers $N = \{1,2,3,\ldots\}$ where the result area is contained in the set of real numbers R.. By adding the terms of the sequence, namely $x_1 + x_2 + x_3 + \cdots + x_n + \cdots$ an infinite series will be formed. So an infinite series is the sum of the series $x_1, x_2, x_3, \ldots.$ (referred to as series terms) and can be written as $\sum_{n=1}^{\infty} x_n$ [2].

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As part of mathematics, sequences have developed more specifically with the emergence of basic properties of real number sequences, one of which is sequence convergence [4]. According to [5], the sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is said to converge to $x \in R$ if given any number $\varepsilon > 0$, there exists a natural number $K \in N$ such that $|x_n - x| < \varepsilon$ when $n \geq K$. Likewise, a series of real numbers is considered convergent if the sequence of partial sums meets the definition of sequence convergence.

The expansion of sequences and series of real numbers occurs if the domain is formed from natural numbers with dimensions of two $(N \times N)$. In other words, the terms of a sequence or series are determined by using two natural number variables simultaneously. In previous research, the convergence of double sequences and double series has been discussed [6] as a generalization of sequences and series of real numbers. Some of the sub-topics contained regarding double sequences and double series are double limits, monotonicity, iterated limits, Cauchy criteria, sub-sequences, divergence, convergence of series, iterated series, series with non-negative terms, and several other series convergence tests. Previous studies also examined the definition of double series convergence and several applications of the double series convergence test [7]. Then [8] wrote in his article about a summary of certain iteration series, which proves that for an iteration series with certain conditions, its value can be calculated from the subtraction of two convergent single series. Meanwhile, in this paper, we will analyze the theorems around the iterated series of a double series of real numbers. All double sequences and double series discussed are limited to the codomain of real numbers.

2 Methodology

The research method used is a literature review, which involves collecting library sources in the form of articles, papers, and books related to the iterated series of a double series of real numbers. After that, proceed with the formulation of the problem and then, from the library sources that have been collected, those that are relevant are selected to be processed in the discussion so that conclusions can be drawn. The stages in studying iterated series are to first study sequences and series of real numbers along with their convergence and basic properties through several literatures, namely [9]–[12]. Next, examine the definitions of double series, double series, and iterated series, along with their convergence and properties, in the literature [6]–[8]. To support the construction process in showing the theorems around the iterated series of a double series, the following is a definition of a double sequence and a double series of real numbers.

Definition 1. *A double sequence of real numbers (double sequence in) is a function on the set* $N \times N$*with a range contained in* R*, namely* X*:* $N \times N \longrightarrow R$*. The double sequence of real numbers* $X: N \times N \longrightarrow R$ *can be denoted by*

$$X, \big(x(n,m)\big), (x_{n,m}) \text{ or } (x(n,m) : n,m \in N).$$

Definition 2. *Given a double sequence of real numbers* $(x(n,m))$, *define the double sequence* $(s(n,m))$ *by* $s(n,m) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x(i,j)\right) \forall n,m \in N$.

The pair (x,s) *is called a double series, denoted as* $\sum_{n,m=1}^{\infty} x(n,m)$ *or* $\sum x(n,m)$. *Each number* $x(n,m)$ *is called a term of the double series, and* $s(n,m)$ *is the partial sum of the double series.*

Example 1. $\sum_{n,m=1}^{\infty} \left(\frac{1}{n}+\frac{1}{m}\right)$ is a double series of real numbers where $x(n,m) = \frac{1}{n}+\frac{1}{m} \forall n,m \in N$ are the terms of the double series and $s(n,m) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \left(\frac{1}{i}+\frac{1}{j}\right)\right) \forall n,m \in N$ is the partial sum. For more details, each term of the double sequence $(x(n,m))$ and $(s(n,m))$ can be seen in **Table 1** and **Table 2**.

Table 1. Double sequence terms $x(n,m) = \frac{1}{n}+\frac{1}{m} \forall n,m \in N$

$x(n,m)$		m								
		1	2	3	4	5	6	7	8	⋯
n	1	2	$1\frac{1}{2}$	$1\frac{1}{3}$	$1\frac{1}{4}$	$1\frac{1}{5}$	$1\frac{1}{6}$	$1\frac{1}{7}$	$1\frac{1}{8}$	⋯
	2	$1\frac{1}{2}$	1	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{7}{10}$	$\frac{2}{3}$	$\frac{9}{14}$	$\frac{5}{8}$	⋯
	3	$1\frac{1}{3}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{8}{15}$	$\frac{1}{2}$	$\frac{10}{21}$	$\frac{11}{24}$	⋯
	4	$1\frac{1}{4}$	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{1}{2}$	$\frac{9}{20}$	$\frac{5}{12}$	$\frac{11}{28}$	$\frac{3}{8}$	⋯
	5	$1\frac{1}{5}$	$\frac{7}{10}$	$\frac{8}{15}$	$\frac{9}{20}$	$\frac{2}{5}$	$\frac{11}{30}$	$\frac{12}{35}$	$\frac{13}{40}$	⋯
	6	$1\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{11}{30}$	$\frac{1}{3}$	$\frac{13}{42}$	$\frac{7}{24}$	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 2. Terms of a double sequence of partial sums $s(n,m) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \left(\frac{1}{i}+\frac{1}{j}\right)\right)$

$x(n,m)$		m								
		1	2	3	4	5	6	7	8	⋯
n	1	2	$3\frac{1}{2}$	$4\frac{5}{6}$	$6\frac{1}{12}$	$7\frac{17}{60}$	$8\frac{9}{20}$	$9\frac{83}{140}$	$10\frac{201}{280}$	⋯
	2	$3\frac{1}{2}$	6	$8\frac{1}{6}$	$10\frac{1}{6}$	$12\frac{1}{15}$	$13\frac{9}{10}$	$15\frac{24}{35}$	$17\frac{61}{140}$	⋯

	3	$4\frac{5}{6}$	$8\frac{1}{6}$	11	$13\frac{7}{12}$	$16\frac{1}{60}$	$18\frac{7}{20}$	$20\frac{257}{420}$	$22\frac{689}{840}$	⋯
	4	$6\frac{1}{12}$	$10\frac{1}{6}$	$13\frac{7}{12}$	$16\frac{2}{3}$	$19\frac{11}{20}$	$22\frac{3}{10}$	$24\frac{401}{420}$	$27\frac{113}{210}$	⋯
	5	$7\frac{17}{60}$	$12\frac{1}{15}$	$16\frac{1}{60}$	$19\frac{11}{20}$	$22\frac{5}{6}$	$25\frac{19}{20}$	$28\frac{199}{210}$	$31\frac{719}{840}$	⋯
	6	$8\frac{9}{20}$	$13\frac{9}{10}$	$18\frac{7}{20}$	$22\frac{3}{10}$	$25\frac{19}{20}$	$29\frac{2}{5}$	$32\frac{99}{140}$	$35\frac{127}{140}$	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋰ ⋮

3 Results and Discussion

Before going into the discussion of iterated series, first we give a definition of the convergence of double sequences and double series of real numbers, which will later become a prerequisite for the iterated series theorems.

3.1 Convergence of Double Sequences and Double Series

Definition 3. *The double sequence $(x(n,m))$ is said to converge to $x \in R$ and is written $x(n,m) = x$, if it holds $\forall\, \varepsilon > 0, \exists\, K \in N$ is such that*

$$\forall\, n,m \geq K \text{ obtain } |x(n,m) - x| < \varepsilon.$$

The number x is called the double limit of the double sequence $(x(n,m))$. If there are no satisfying values for x, the sequence $(x(n,m))$ is called a divergent sequence.

Example 2. Given the following double sequence of real numbers $(x(n,m)) = \left\{\frac{1}{n+m} : n,m \in N\right\}$. Using **Definition 3**, prove that the double sequence $(x(n,m))$ converges to 0.

Solution. Let $x(n,m) = \frac{1}{n+m}\ \forall\, n,m \in N$.
It will be proven that $\frac{1}{n+m} = 0$.
For any $\varepsilon > 0$.
Choose $K = \lceil \frac{2}{\varepsilon} \rceil \in N$ so as $\forall\, n,m \geq K = \lceil \frac{2}{\varepsilon} \rceil$, we have

$$|x(n,m) - 0| = \left|\frac{1}{n+m} - 0\right| = \frac{1}{n+m} < \frac{1}{n} < \frac{1}{n} + \frac{1}{m} \leq \frac{1}{K} + \frac{1}{K} = \frac{2}{K} \leq \frac{2}{\frac{2}{\varepsilon}} = \varepsilon.$$

This implies that $\frac{1}{n+m} = 0$.

Theorem 1. Uniqueness of double sequence limits:
A double sequence $(x(n,m))$ can have at most one limit.

Proof. Suppose the double sequence $(x(n,m))$ converges to two different limits.
We say $x(n,m) = x' \in R$ and $x(n,m) = x'' \in R$ with $x' \neq x''$.

For any $\varepsilon > 0$.
Since $x(n,m) = x'$ and $x(n,m) = x''$, then from **Definition 3**, exists $K_1, K_2 \in N$ so that

$$\text{if } n,m \geq K_1 \text{ then } |x(n,m) - x'| < \frac{\varepsilon}{2}. \quad \text{... (1)}$$

and

$$\text{if } n,m \geq K_2 \text{ then } |x(n,m) - x''| < \frac{\varepsilon}{2}. \quad \text{... (2)}$$

Choose $K = maks\{K_1, K_2\}$ then from (1) and (2) if $n,m \geq K$, we have

$$\begin{aligned} 0 \leq |x' - x''| &= |x' - x(n,m) + x(n,m) - x''| \\ &\leq |x' - x(n,m)| + |x(n,m) - x''| \quad \text{... (triangle inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Obtained $0 \leq |x' - x''| < \varepsilon$, for all $\varepsilon > 0$. It can be concluded that $x' - x'' = 0$ that is $x' = x''$.
This contradicts the assumption that the double sequence $(x(n,m))$ converges to two different limits. So the double sequence $(x(n,m))$ must have at most one limit. ■

Definition 4. *Let $X : N \times N \longrightarrow R$ be a double sequence of real numbers and define a double sequence of partial sums $(s(n,m))$ as*

$$s(n,m) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x(i,j) \right) \forall\, n,m \in N.$$

The double series $\sum_{n,m=1}^{\infty} x(n,m)$ is said to converge to a real number if there exists $s \in R$ such that $s(n,m) = s$. If there is no such s, the double series $\sum_{n,m=1}^{\infty} x(n,m)$ is divergent.

Example 3. Given a double sequence $(x(n,m))$ with $x(n,m) = \frac{1}{2^{n+m}}\ \forall\, n,m \in N$. Prove that the double series of real numbers $\sum_{n,m=1}^{\infty} \left(\frac{1}{2^{n+m}}\right)$ converges to 1 using **Definition 4**.

Solution. For any $\varepsilon > 0$. Choose $K = \lceil \frac{2}{\varepsilon} \rceil \in N$ so that if $n,m \geq K$ we have

$$\begin{aligned} |s(n,m) - 1| &= \left| \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{m} \frac{1}{2^{i+j}} \right) \right) - 1 \right| \\ &= \left| \left(\sum_{i=1}^{n} \left(\frac{1}{2^i}\right) \right) \left(\sum_{j=1}^{m} \left(\frac{1}{2^j}\right) \right) - 1 \right| \\ &= \left| \left(1 - \left(\frac{1}{2}\right)^n\right) \left(1 - \left(\frac{1}{2}\right)^m\right) - 1 \right| \\ &= \left| \frac{1}{2^{n+m}} - \frac{1}{2^n} - \frac{1}{2^m} + 1 - 1 \right| \\ &= \frac{1}{2^n} + \frac{1}{2^m} - \frac{1}{2^{n+m}} < \frac{1}{2^n} + \frac{1}{2^m} < \frac{1}{n} + \frac{1}{m} < \frac{1}{K} + \frac{1}{K} = \frac{2}{K} = \frac{2}{\lceil \frac{2}{\varepsilon} \rceil} \leq \varepsilon. \end{aligned}$$

This proves that the double series $\sum_{n,m=1}^{\infty} \left(\frac{1}{2^{n+m}}\right)$ converges to 1.

Furthermore, the following theorem makes it easier to calculate the convergent value of a double series under certain conditions. This theorem will be connected to **Theorem 5** in building **Theorem 6**.

Theorem 2. *If the double series* $\sum_{n,m=1}^{\infty} x(n,m)$ *can be written as*

$$\sum_{n,m=1}^{\infty} x(n,m) = \sum_{n,m=1}^{\infty} x(n)y(m)$$

where $\sum_{n=1}^{\infty} x(n) = x$ *and* $\sum_{m=1}^{\infty} y(m) = y$ *then* $\sum_{n,m=1}^{\infty} x(n,m)$ *converges. Further*

$$\sum_{n,m=1}^{\infty} x(n,m) = (\sum_{n=1}^{\infty} x(n))(\sum_{m=1}^{\infty} y(m)) = xy.$$

Proof. Let $\sum_{n,m=1}^{\infty} x(n,m)$ be a double series such that $\sum_{n,m=1}^{\infty} x(n,m) = \sum_{n,m=1}^{\infty} x(n)y(m)$, for sequences $(x(n))$ and $(y(m))$ with $\sum_{n=1}^{\infty} x(n) = x$ and $\sum_{m=1}^{\infty} y(m) = y$.

Define $(s_x(n))$, $(s_y(m))$, and $(s(n,m))$ which respectively are sequences of partial sums of the series $\sum_{n=1}^{\infty} x(n), \sum_{m=1}^{\infty} y(m)$ and the double series $\sum_{n,m=1}^{\infty} x(n,m)$.
We will show $\sum_{n,m=1}^{\infty} x(n,m) = (\sum_{n=1}^{\infty} x(n))(\sum_{m=1}^{\infty} y(m)) = xy$.
Note that

$$s(n,m) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x(i)y(j)\right) = \left(\sum_{i=1}^{n} x(i)\right)\left(\sum_{j=1}^{m} y(j)\right)$$
$$= s_x(n)s_y(m).$$

Since $\sum_{n=1}^{\infty} x(n) = x$ and $\sum_{m=1}^{\infty} y(m) = y$, it means

$$s_x(n) = x \text{ and} s_y(m) = y. \qquad \text{... (3)}$$

Referring to one of the arithmetic theorems for calculating the double limit value in [6], that is, if the terms of the double sequence $(x(n,m))$ can be written as $x(n,m) = x_n y_m$ where (x_n) and $(y_m)$is a sequence with limit values $x_n = x$ and $y_m = y$, Then

$$(x(n,m)\,) = (x(n,m)\,) = x(n,m) = xy.$$

So based on the theorem and (3), it is obtained

$$s(n,m) = \left(s_x(n)s_y(m)\right) = (s_x(n)\,)\left(s_y(m)\,\right) = xy.$$

This Show that $\sum_{n,m=1}^{\infty} x(n,m) = (\sum_{n=1}^{\infty} x(n))(\sum_{m=1}^{\infty} y(m)) = xy$.

■

Example 4. In the previous example, the double series $\sum_{n,m=1}^{\infty} x(n,m)$ with $x(n,m) = \frac{1}{2^{n+m}}\ \forall\, n,m \in N$ converging to 1 can be calculated using **Theorem 2** as follows.

Solution. Define the sequence $x(n) = \frac{1}{2^n}$ and $y(m) = \frac{1}{2^m}\ \forall\, n,m \in N$
It is easy to show that the series $\sum_{n=1}^{\infty} x(n)$ and $\sum_{m=1}^{\infty} y(m)$ both converge to 1.
Note that

$$x(n,m) = \frac{1}{2^{n+m}} = \frac{1}{2^n \times 2^m} = \frac{1}{2^n} \times \frac{1}{2^m} = x(n)y(m)$$

Since $\sum_{n,m=1}^{\infty} x(n,m) = \sum_{n,m=1}^{\infty} x(n)y(m)$, where $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{2^m}$ both converge to 1, So according to **Theorem 2** we obtain

$$\sum_{n,m=1}^{\infty} \frac{1}{2^{n+m}} = \left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right)\left(\sum_{m=1}^{\infty} \frac{1}{2^m}\right) = (1)(1) = 1.$$

3.2 Iterated Series of a Double Series

A topic that we don't find in single series is iterated series. As in **Definition 5**, the iterated series uses two sigmas with different index orders, the convergence value may also exist, or vice versa.

Definition 5. *Double series*

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) \text{ and } \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right)$$

is called an ***iterated series.***

Example 5. Define the double sequence $\left(x(n,m)\right)$ as follows

$$x(n,m) = \{\ 1;\ if\ m-n=1, -1;\ if\ n-m=1\ , 0;\ n,m\ else\}.$$

Show that the iterated series

$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right)$ and $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right)$ exists and both are not the same and the double series $\sum_{n,m=1}^{\infty} x(n,m)$ is divergent in real numbers.

Solution. To make it easier to understand the form of the double sequence, **Table 3** below presents the terms of the double sequence $\left(x(n,m)\right)$.

Table 3. Double sequence terms $\left(x(n,m)\right)$

$x(n,m)$		m								
		1	2	3	4	5	6	7	8	⋯
n	1	0	1	0	0	0	0	0	0	⋯
	2	−1	0	1	0	0	0	0	0	⋯
	3	0	−1	0	1	0	0	0	0	⋯
	4	0	0	−1	0	1	0	0	0	⋯
	5	0	0	0	−1	0	1	0	0	⋯
	6	0	0	0	0	−1	0	1	0	⋯
	7	0	0	0	0	0	−1	0	1	⋯
	8	0	0	0	0	0	0	−1	0	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Let the terms of the double sequence of partial sums of the double series $\sum_{n,m=1}^{\infty} x(n,m)$ be $s(n,m) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x(i,j)\right)\ \forall\, n,m \in N$.

Note that $\forall\, n,m \in N$ pretend

$$s(1,m) = \sum_{j=1}^{m} x(1,j) = 1. \qquad \text{... (4)}$$

$$s(n,1) = \sum_{i=1}^{n} x(i,1) = -1. \qquad \text{... (5)}$$

Next, $\forall\, n,m \in N$ and $\forall\, k \geq 2$ pretend

$$s(k,m) = \sum_{j=1}^{m} x(k,j) = 0. \qquad \text{... (6)}$$

$$s(n,k) = \sum_{i=1}^{n} x(i,k) = 0. \qquad \text{... (7)}$$

Based on (4), (5), (6), and (7), we obtained

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) = \left(\sum_{m=1}^{\infty} x(1,m)\right) + \sum_{n=2}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right)$$
$$= 1 + 0$$
$$= 1$$

and

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right) = \left(\sum_{n=1}^{\infty} x(n,1)\right) + \sum_{n=2}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right)$$
$$= (-1) + 0$$
$$= -1$$

Therefore, $\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) = 1 \neq -1 = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right)$.
On the other hand, it is clear that $\forall\, n \in N$ holds

$$s(n,n) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x(i,j)\right) = 0 \qquad \text{... (8)}$$

and

$$s(n,n-1) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x(i,j)\right) = -1. \qquad \text{... (9)}$$

Based on (8) and (9), according to **Theorem 1**, it is impossible for a double sequence to converge to two different limits. As a result, $\left(s(n,m)\right)$ diverges.
Therefore, $\sum_{n,m=1}^{\infty} x(n,m)$ is divergent in real numbers.

It can be seen in the above example that the iterated series have different results, i.e. $\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) = 1 \; and \; \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right) = -1$.

This result shows that if the convergence value of the two iterated series exists (in R), it does not indicate that the two values must be the same. However, by adding the condition that the double series $\sum_{n,m=1}^{\infty} x(n,m)$ converges to $s \in R$ as well as certain conditions as in **Theorem 5** which implies the convergence value of the two iterated series is the same.

Previously, it was necessary to prove the following two theorems and combine them to form **Theorem 5**.

Theorem 3. *Suppose the double series* $\sum_{n,m=1}^{\infty} x(n,m)$ *converges to* $s \in R$. *Then* $\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) = s$ *if and only if* $\sum_{m=1}^{\infty} x(n,m)$ *exist for every* $n \in N$.

Proof. Let $\left(x(n,m)\right)$ is a double sequence with $\sum_{n,m=1}^{\infty} x(n,m)$ as its double sequence that converges to $s \in R$.

Define the double sequence $\left(s(n,m)\right)$ as the partial sum of the double sequence $\sum_{n,m=1}^{\infty} x(n,m)$ i.e. $s(n,m) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x(i,j)\right) \; \forall\, n,m \in N$.
According to **Definition 4**, $\sum_{n,m=1}^{\infty} x(n,m)$ converges to s if and only if

$$s(n,m) = s\,.$$

Note that

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x(i,j)\right)$$
$$= \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{m} x(i,j)\right)\right)$$
$$= \left(s(n,m)\right). \qquad \text{... (10)}$$

On the other hand,

$$\sum_{m=1}^{\infty} x(n,m) \text{ exist } \forall\, n \in N \;\Leftrightarrow\; s(n,m) \text{ exist } \forall\, n \in N. \qquad \text{... (11)}$$

Referring to one of the iterated limit theorems in [6] namely if $s(n,m) = s$ then $(s(n,m)) = s$ if and only if $s(n,m)$ exists $\forall\, n \in N$. Because $s(n,m) = s$, as a result, equivalence between (10) and (11) is obtained using the iterated limit theorem. This shows that if $\sum_{n,m=1}^{\infty} x(n,m) = s$ then $\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) = s$ if and only if $\sum_{m=1}^{\infty} x(n,m)$ exists for every $n \in N$. ■

Theorem 4. *Suppose the double series* $\sum_{n,m=1}^{\infty} x(n,m)$ *converges to* $s \in R$. *Then* $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right) = s$ *if and only if* $\sum_{n=1}^{\infty} x(n,m)$ *exists for every* $m \in N$.

Proof. Because the forms of **Theorem 3** and **Theorem 4** are symmetrical, **Theorem 4** can be proven in the same way. ■

So if you combine **Theorem 3** and **Theorem 4**, you get the following results.

Theorem 5. *Suppose the double series* $\sum_{n,m=1}^{\infty} x(n,m)$ *converges to* $s \in R$. *Iterated series* $\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right)$ *and* $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right)$ *exists and is both equal to* s *if and only if the following two conditions hold:*

(i) $\forall\, n \in N$, *the series* $\sum_{m=1}^{\infty} x(n,m)$ *convergent, and*

(ii) $\forall\, m \in N$, *the series* $\sum_{n=1}^{\infty} x(n,m)$ *convergent.*

Example 6. Define the double sequences $(x(n,m))$ as follows.

$x(n,m) = \{ \quad 1;\ if\ n = 1\ and\ m = 1\ -3;\ if\ n = 2\ and\ m = 2\ -1;\ if\ n \geq 3\ and\ m = 2\ -1;\ if\ n = 2\ and\ m \geq 3\quad 0;\ if\ n \geq 3\ and\ m \geq 3\,.$

To make it easier to understand the form of the double sequence, **Table 4** below presents the terms of the double sequence $(x(n,m))$.

Table 4. Double sequence terms $(x(n,m))$

$x(n,m)$		m								
		1	2	3	4	5	6	7	8	⋯
n	1	1	1	1	1	1	1	1	1	⋯
	2	1	−3	−1	−1	−1	−1	−1	−1	⋯
	3	1	−1	0	0	0	0	0	0	⋯
	4	1	−1	0	0	0	0	0	0	⋯
	5	1	−1	0	0	0	0	0	0	⋯
	6	1	−1	0	0	0	0	0	0	⋯
	7	1	−1	0	0	0	0	0	0	⋯
	8	1	−1	0	0	0	0	0	0	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Let the terms of the double series of partial sums of the double series $\sum_{n,m=1}^{\infty} x(n,m)$ be $s(n,m) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x(i,j)\right) \forall\, n,m \in N$.
Note that $\forall\, n,m \geq 2$, we have

$$\begin{aligned} s(n,m) &= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x(i,j)\right) \\ &= \sum_{i=1}^{2} \left(\sum_{j=1}^{2} x(i,j)\right) + \sum_{i=3}^{n} \left(\sum_{j=1}^{2} x(i,j)\right) \\ &\quad + \sum_{j=3}^{m} \left(\sum_{i=1}^{2} x(i,j)\right) + \sum_{i=3}^{n} \left(\sum_{j=3}^{m} x(i,j)\right) \\ &= \left(1+1+1+(-3)\right) + (0) + (0) + (0) \\ &= 0. \end{aligned}$$

So the double limit value of $\left(s(n,m)\right)$ is $s(n,m) = 0$.
According to **Definition 4**, we obtain

$$\sum_{n,m=1}^{\infty} x(n,m) = 0. \qquad \text{... (12)}$$

On the other hand, $\forall\, n,m \in N$ hold

$$s(1,m) = \sum_{j=1}^{m} x(1,j) = m. \qquad \text{... (13)}$$

$$s(n,1) = \sum_{i=1}^{n} x(i,1) = n. \qquad \text{... (14)}$$

and $\forall\, n,m \in N$ with $n,m \geq 2$ hold

$$s(2,m) = \sum_{j=1}^{m} x(2,j) = -m. \qquad \text{... (15)}$$

$$s(n,2) = \sum_{i=1}^{n} x(i,2) = -n.. \qquad \text{... (16)}$$

From $(13), (14), (15)$, and (16) obtained
for the value $n \in \{1,2\}$ then the series $\sum_{m=1}^{\infty} x(n,m)$ diverges and
for the value $m \in \{1,2\}$ then the series $\sum_{n=1}^{\infty} x(n,m)$ diverges.
As a result, based on (12) and referring to the Double Series Cauchy Criterion (contrapositive form) in [6], it can be concluded that the iterated series

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) \text{ and } \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right)$$

is divergent in real numbers.

In the end, this last theorem is obtained by combining **Theorem 2** and **Theorem 5**, which state that for a double series with special conditions, the convergence value of the two iterated series is the same.

Theorem 6. *If the double series* $\sum_{n,m=1}^{\infty} x(n,m)$ *can be written as*

$$\sum_{n,m=1}^{\infty} x(n,m) = \sum_{n,m=1}^{\infty} x(n)y(m)$$

where $\sum_{n=1}^{\infty} x(n) = x$ *and* $\sum_{m=1}^{\infty} y(m) = y$ *then*

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right) = xy.$$

Proof. Let the double series $\sum_{n,m=1}^{\infty} x(n,m)$, the terms can be written as the product of two convergent sequences, i.e.

$$x(n,m) = x(n)y(m)\ \forall\, n,m \in N$$

where $\sum_{n=1}^{\infty} x(n) = x$ and $\sum_{m=1}^{\infty} y(m) = y$.
According to **Theorem 2**, the double series converges to xy that is $\sum_{n,m=1}^{\infty} x(n,m) = xy$. Next, For any $i,j \in N$.
Note that

$$\sum_{m=1}^{\infty} x(i,m) = \sum_{m=1}^{\infty} x(i)y(m) = x(i)\sum_{m=1}^{\infty} y(m) = x(i)y \in R,$$

$$\sum_{n=1}^{\infty} x(n,j) = \sum_{n=1}^{\infty} x(n)y(j) = y(j)\sum_{n=1}^{\infty} x(n) = y(j)x \in R.$$

This means that $\sum_{m=1}^{\infty} x(i,m)$ and $\sum_{n=1}^{\infty} x(n,j)$ exist $\forall\, i,j \in N$. Because the double series $\sum_{n,m=1}^{\infty} x(n,m) = xy$ then based on **Theorem 5**, we obtain

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right) = xy.$$

■

4 Conclusion

Based on the results and discussion, several theorems are obtained around the iterated series in a double series of real numbers, namely

1) Suppose the double series $\sum_{n,m=1}^{\infty} x(n,m)$ converges to $\in R$. So it applies
   (i) $\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right) = s$ if and only if $\sum_{n=1}^{\infty} x(n,m)$ exists $\forall\, m \in N$.
   (ii) $\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) = s$ if and only if $\sum_{m=1}^{\infty} x(n,m)$ exists $\forall\, n \in N$.
2) If the double series $\sum_{n,m=1}^{\infty} x(n,m)$ can be written as

$$\sum_{n,m=1}^{\infty} x(n,m) = \sum_{n,m=1}^{\infty} x(n)y(m)$$

   where $\sum_{n=1}^{\infty} x(n) = x$ and $\sum_{m=1}^{\infty} y(m) = y$ then

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} x(n,m)\right) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n,m)\right) = xy.$$

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