The Hilbert–Mumford criterion for representations of network quivers

Marco A. Armenta[1], Alexander H.W. Schmitt[2,*]

[1]*Institut quantique, Université de Sherbrooke, 2500 Boulevard de l'Université, Sherbrooke, QC, Canada*
[2]*Institute of Mathematics, Freie Universität Berlin, D-14195 Berlin, Germany*
**Correspondence author. Email: alexander.schmitt@fu-berlin.de, ORCID: 0000-0002-4454-1461*

ABSTRACT

The architecture of a neural network is given by a quiver, and the choice of weights (which can be numbers or even matrices) may be interpreted as a framed representation of that quiver. Armenta and Jodoin observed that the network function is invariant under certain rescaling operations. This led them to introduce moduli spaces of framed representations of network quivers as a new tool for investigating the theoretical foundations of machine learning. Geometric invariant theory tells us that we need to determine semistable and stable framed representations to get some basic information about the moduli spaces. This task was performed by Armenta, Brüstle, Hassoun, and Reineke. In this note, we will give an alternative proof of this basic result. We will work directly on the space of framed representations instead of taking a detour via some space of unframed representations. The main tool is the Hilbert–Mumford criterion from geometric invariant theory.

Keywords: *Quiver, representation, neural network, stability, geometric invariant theory, Hilbert–Mumford criterion*

INTRODUCTION

A (feed forward) neural network is composed of a network quiver Q, activation functions, and weights. The quiver Q consists of a finite set V of vertices and a finite set E of arrows between these vertices and describes the architecture of the network. Inside V, we have the disjoint sets I and O, consisting of the input and output vertices, respectively. The remaining vertices, i.e., the vertices of $V_h := V \setminus (I \sqcup O)$ are the hidden vertices. For each hidden vertex $v \in V_h$, there is an activation function f_v, such as, e.g., the rectifier.[1] Last but not least, a weight w_e is chosen for each arrow $e \in E$. In the setting of neural networks, the data f_v, $v \in V$, and w_e, $e \in E$, are real. In this note, we will work with their complex counterparts. A neural network comes with a network function

$$\Psi\colon \mathbb{C}^{\#I} \longrightarrow \mathbb{C}^{\#O}.$$

The training of a network usually just modifies the weights and yields a new network function. The weights form the complex vector space

$$R := \bigoplus_{a\in E} \mathbb{C}.$$

[1]One may keep track of the activation functions by adding a loop at each hidden vertex in the network quiver, but we will not do this, here.

The group

$$G := \prod_{v\in V} \mathbb{C}^\star$$

acts linearly on R via

$$\begin{aligned} G\times R &\longrightarrow R \\ \big((g_v, v\in V), (w_e, e\in E)\big) &\longmapsto \\ &\big(g_{t(e)}\cdot w_e \cdot g_{s(e)}^{-1}, e\in E\big). \end{aligned}$$

This action restricts to a linear action of the group

$$\widetilde{G} := \prod_{v\in V_h} \mathbb{C}^\star$$

on R. It is a basic observation of Armenta and Jodoin (Corollary 5 in [2]) that the network function is invariant under the action of $\widetilde{G}$. To make the statement of Armenta and Jodoin more precise, we need to introduce the moduli space

$$\mathcal{M} := R /\!\!/ \widetilde{G}.$$

It is the categorical quotient of R by the action of $\widetilde{G}$ in the category of (affine) algebraic varieties. (We will come back to this below.) Now, Armenta and Jodoin construct a map

$$\widehat{\Psi}\colon \mathcal{M} \longrightarrow \mathbb{C}^{\#O}$$

which does not depend on the weights and a map

$$\varphi\colon \mathbb{C}^{\#I} \xrightarrow{,} \mathcal{M},$$

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such that the network function factorizes as

$$\Psi = \widehat{\Psi} \circ \varphi.$$

Looking at the vector space R as an affine variety basically means that we endow R with the so-called regular functions that are given as polynomials in the coordinate functions x_f, $f \in E$, with

$$\begin{aligned} x_f \colon R &\longrightarrow \mathbb{C} \\ (w_e, e \in E) &\longmapsto w_f, \quad f \in E. \end{aligned}$$

The $\mathbb{C}$-algebra of regular functions of R is denoted by $\mathbb{C}[R]$. When trying to form the quotient, it makes sense to look at the regular functions which are constant on all $\widetilde{G}$-orbits, i.e., at the elements of the $\mathbb{C}$-algebra

$$\mathbb{C}[R]^{\widetilde{G}} := \big\{ F \in \mathbb{C}[R] \mid \forall g \in \widetilde{G},\ \forall S \in R : \\ F(g \cdot S) = F(S) \big\}.$$

By a famous result of Hilbert's, $\mathbb{C}[R]^{\widetilde{G}}$ is the algebra of regular functions of an affine algebraic variety that we shall denote by $R/\!\!/\widetilde{G}$. Furthermore, the inclusion $\mathbb{C}[R]^{\widetilde{G}} \subset \mathbb{C}[R]$ is realized by a unique map

$$\pi \colon R \longrightarrow R/\!\!/\widetilde{G}$$

of algebraic varieties. The pair $(R/\!\!/\widetilde{G}, \pi)$ is the categorical quotient. It is characterized by a certain universal property which makes it unique up to some canonical identifications. The map π separates closed orbits in R. However, there often do exist non-closed orbits. Each non-closed orbit Ω contains exactly one closed orbit $\Omega_\circ$ in its closure, and the orbit Ω is mapped to the same point as the closed orbit $\Omega_\circ$.

In view of the above result of Armenta and Jodoin and the general complications caused by non-closed orbits, it is necessary to study the map $\pi \colon R \longrightarrow \mathcal{M} = R/\!\!/\widetilde{G}$ more closely. In geometric invariant theory, one distinguishes the following points in R:

- A point $S \in R$ is a *nullform*, if 0 is contained in the closure of the $\widetilde{G}$-orbit of S. (As noted above, this means $\pi(S) = \pi(0)$.)
- A point $S \in R$ is *semistable*, if it is not a nullform, i.e., if 0 is not contained in the closure of the orbit of S.
- A point $S \in R$ is *stable*, if the orbit of S is closed and the stabilizer of S is finite. (If $V_h \neq \varnothing$, then S is semistable.)

There exists an open subset $V \subset \mathcal{M}$, such that $U := \pi^{-1}(V) \subset R$ is the set of stable points. Since π separates closed orbits, every fiber of the map

$$\pi_{|U} \colon U \longrightarrow V$$

consists exactly of one orbit (of a stable point). So, the open subset U is a $\widetilde{G}$-invariant open subset of R, such that one may endow the set of $\widetilde{G}$-orbits in U in a natural way with the structure of an algebraic variety. (The set $P \subset R$ of all points of R whose $\widetilde{G}$-orbit is closed contains U, but is, in general, not an open subset of R and, thus, not in a natural way an algebraic variety. For this reason, the set P plays only an auxiliary role.) These phenomena explain the significance of stable points.

The datum of weights $(w_e, e \in E)$ is a representation of the network quiver Q whose dimension vector has only ones as entries. More generally, one may allow arbitrary dimension vectors (compare [1]). This means that we fix, for each vertex $v \in V$, a positive integer d_v and look at tuples $(r_e, e \in E)$ where r_e is a $(d_{t(e)} \times d_{s(e)})$-matrix, $e \in E$. The characterization of the semistable points in this framework follows from a general result of Halic and Stupariu (Theorem 1.1 in [5]). Section 3 in [1] gives a characterization of the semistable as well as of the stable points.[2] To put this into perspective, the case when I and O are both empty is the classical case, and the semistable and stable points in this context are well-known (see [7], [6], and [10], p. 75ff, for different discussions). Now, both the paper [5] and the paper [1] contain techniques for modifying the quiver Q to a quiver Q', such that the problem with non-empty I and O for Q is equivalent to the problem for Q' with empty I and O. Though this so-called deframing procedure is quite elementary, it is not completely natural. For instance, a stable point S in R has, by definition, a finite stabilizer. Its associated representation S' in the corresponding space R' has a one-dimensional stabilizer in the corresponding symmetry group.

On the other hand, the Hilbert–Mumford criterion is the central tool of geometric invariant theory for finding the semistable and the stable points. In this note, we will directly apply it on R to reprove Theorem 2.9. The advantage of this direct approach is that we need only the most basic notions from the representation theory of quivers. This might make this fundamental result more accessible to people who are mainly interested in neural networks. Of course, we still need the tools of algebraic geometry which are necessary to study quotients. (These are also used in [1].)

1. REVIEW OF REPRESENTATIONS OF FRAMED QUIVERS

A *quiver* is a quadruple $Q = (V, E, s, t)$, consisting of finite sets V and E and maps $s, t \colon E \longrightarrow V$. We will call the elements of V *vertices* and the elements of E *arrows* or *oriented edges*. For an arrow $e \in E$, the vertex $s(e)$ is the *source* and the vertex $t(e)$ the *target* of e. So, we will think of e as an arrow pointing from $s(e)$ to $t(e)$. Let $n \geq 1$. Then, a *path* of *length* n is a tuple $p = (e_1, \ldots, e_n)$ of arrows, such that $t(e_i) = s(e_{i+1})$, $i = 1, \ldots, n-1$.[3] We call $s'(p) := s(e_1)$ the *source* of p and $t'(p) := t(e_n)$ the *target* of p. An *oriented cycle* is a path p with $s'(p) = t'(p)$ and a *loop* an oriented cycle of length one. We will suppose that we have a decomposition $V =$

[2] The words semistable and stable do not appear in Section 3 of [1]. So, one has to go back to the definitions and the constructions given in [1] to see this.

[3] So, a path of length one is just an arrow.

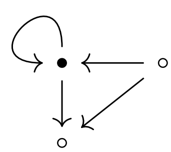
$V_u \sqcup V_h$ into non-empty subsets. The vertices $v \in V_u$ are said to be *unmarked* and the vertices $v \in V_h$ *hidden*. In pictures, we will draw unmarked vertices as circles and hidden vertices as dots. We call $Q = (V_u, V_h, E, s, t)$ a *framed quiver*. The *hidden subquiver* of Q is $\widetilde{Q} = (V_h, E_h, s_{|E_h}, t_{|E_h})$ with $E_h := \{ e \in E \mid s(e) \in V_h \wedge t(e) \in V_h \}$.

Let k be a field. As usual, a *representation* of Q is a tuple $(W_v, v \in V, r_e, e \in E)$, consisting of k-vector spaces W_v, $v \in V$, and k-linear maps $r_e \colon W_{s(e)} \longrightarrow W_{t(e)}$, $e \in E$. Given representations $(W_v^1, v \in V, r_e^1, e \in E)$ and $(W_v^2, v \in V, r_e^2, e \in E)$ with $W_v^1 = W_v^2$, $v \in V_u$, a *homomorphism* is a collection $(\varphi_v, v \in V_h)$ of k-linear maps $\varphi_v \colon W_v^1 \longrightarrow W_v^2$, $v \in V_h$, such that, setting $\varphi_v := \mathrm{id}_{W_v^1}$, $v \in V_u$, the relation

$$\varphi_{t(e)} \circ r_e^1 = r_e^2 \circ \varphi_{s(e)}$$

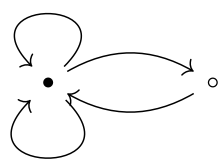
holds, for all arrows $e \in E$. A *subrepresentation* of a representation $(W_v, v \in V, r_e, e \in E)$ is a collection $A = (A_v, v \in V)$ of k-linear subspaces $A_v \subseteq W_v$, $v \in V$, such that $r_e(A_{s(e)}) \subseteq A_{t(e)}$ is satisfied, for all arrows $e \in E$. Finally, the *dimension vector* of a representation $(W_v, v \in V, r_e, e \in E)$ is the tuple $d = (\dim_k(W_v), v \in V)$, and the *hidden dimension vector* the tuple $d_h = (\dim_k(W_v), v \in V_h)$.

Example 1.1. a) Representations of the framed quiver



are called *linear dynamical systems* and appear in control theory. We refer to the paper [4] for a discussion and references.

b) The quiver



is known as the *ADHM-quiver*, named after the paper [3]. It is discussed with references to the literature in [10], p. 349ff.

c) Armenta and Jodoin [2] introduced a large class of framed quivers, called *network quivers*, to the theory of neural networks. The toy example which will be useful in illustrating some basic properties is $\circ \longrightarrow \bullet \longrightarrow \circ$.

We fix a vector $d = (d_v, v \in V)$. Here, d_v is a natural number, $v \in V$. Set

$$\mathrm{Rep}_d(Q) := \bigoplus_{e \in E} \mathrm{Hom}_k(k^{d_{s(e)}}, k^{d_{t(e)}}).$$

This is a parameter space for representations of Q with dimension vector d. In fact, we identify $r = (r_e, e \in V)$ with the representation $(k^{d_v}, v \in V, r_e, e \in E)$.

The *change of basis group* is

$$\widetilde{G}_d(Q) := \prod_{v \in V_h} \mathrm{GL}_{d_v}(k).$$

We also introduce

$$\mathrm{GL}_d(Q) := \prod_{v \in V} \mathrm{GL}_{d_v}(k).$$

Then,

$$\begin{aligned} \widetilde{\alpha} \colon \mathrm{GL}_d(Q) \times \mathrm{Rep}_d(Q) &\longrightarrow \mathrm{Rep}_d(Q) \\ (g, r) &\longmapsto (g_{t(e)} \cdot r_e \cdot g_{s(e)}^{-1}, e \in E), \end{aligned}$$

$$(g, r) := \big((g_v, v \in V), (r_e, e \in E)\big),$$

is a group action. We will use the restricted action

$$\widetilde{\alpha}_{|\widetilde{G}_d(Q) \times \mathrm{Rep}_d(Q)} \colon \widetilde{G}_d(Q) \times \mathrm{Rep}_d(Q) \longrightarrow \mathrm{Rep}_d(Q),$$

denoted by α, to describe and investigate the classification problem of framed quiver representations. Indeed, $r^1 = (r_e^1, e \in E)$ and $r^2 = (r_e^2, e \in E)$ lie in the same $\widetilde{G}_d(Q)$-orbit if and only if the representations $(k^{d_v}, v \in V, r_e^1, e \in E)$ and $(k^{d_v}, v \in V, r_e^2, e \in E)$ are isomorphic.

Since the classification of all representations of a given dimension vector by hand is usually not feasible, the tool of choice for studying the classification problem is a moduli space.

2. SEMISTABLE AND STABLE REPRESENTATIONS

We will now work over the algebraically closed field $\mathbb{C}$. Note that all assertions concerning semistability and the construction of quotients hold also true over non-algebraically closed fields of characteristic zero. Only the central notion of stability does not behave well under changing the base field.

In view of our description of the classification problem in terms of the group action, the first moduli space to consider is the categorical quotient

$$\mathcal{M}_d^\circ(Q) := \mathrm{Rep}_d(Q) /\!\!/ \widetilde{G}_d(Q).$$

This is an affine algebraic variety whose algebra of regular functions is the algebra of invariant functions

$$\begin{aligned} &\mathbb{C}[\mathrm{Rep}_d(Q)]^{\widetilde{G}_d(Q)} := \\ &\Big\{ F \in \mathbb{C}[\mathrm{Rep}_d(Q)] \,\Big|\, \forall g \in \widetilde{G}_d(Q),\ \forall S \in \mathrm{Rep}_d(Q) : \\ &\qquad F\big(\alpha(g, S)\big) = F(S) \Big\}. \end{aligned}$$

Generators for this algebra were determined by Halic and Stupariu (Theorem 1.1 in [5]). They may be obtained as follows:

- For a hidden vertex $v \in V_h$ and an oriented cycle c in the hidden quiver $\widetilde{Q}$, starting and ending at v, we may assign to any representation $(r_e, e \in E) \in$

$\mathrm{Rep}_d(Q)$ an endomorphism $r_c\colon \mathbb{C}^{d_v} \longrightarrow \mathbb{C}^{d_v}$. Clearly,

$$\begin{aligned} F_c\colon \mathrm{Rep}_d(Q) &\longrightarrow \mathbb{C} \\ (r_e, e \in E) &\longmapsto \mathrm{Trace}(r_c) \end{aligned}$$

is an invariant function.

- For unmarked vertices $v_1, v_2 \in V_u$ and a path inside Q, beginning at v_1 and ending at v_2, we may associate with any representation $(r_e, e \in E) \in \mathrm{Rep}_d(Q)$ a $\mathbb{C}$-linear map $r_p\colon \mathbb{C}^{d_{v_1}} \longrightarrow \mathbb{C}^{d_{v_2}}$. This can be seen as a matrix $(r_p(i,j))_{\substack{i=1,\dots,m_p\\ j=1,\dots,n_p}}$, $m_p := d_{v_2}$, $n_p := d_{v_1}$. Since r_p is invariant under the action of $\widetilde{G}_p(Q)$, the functions

$$\begin{aligned} \Xi_p(i,j)\colon \mathrm{Rep}_d(Q) &\longrightarrow \mathbb{C} \\ (r_e, e \in E) &\longmapsto r_p(i,j) \end{aligned}$$

are invariant, too, $i = 1, \dots, m_p$, $j = 1, \dots, n_p$.

Theorem 2.1 (Halic/Stupariu)**.** *The ring of invariants $\mathbb{C}[\mathrm{Rep}_d(Q)]^{\widetilde{G}_d(Q)}$ is generated by the functions F_c, c an oriented cycle in $\widetilde{Q}$, and $\Xi_p(i,j)$, $i = 1, \dots, m_p$, $j = 1, \dots, n_p$, p a path in Q, starting and ending at unmarked vertices.*

Example 2.2. We look at the quiver $\circ \longrightarrow \bullet \longrightarrow \circ$ and the dimension vector $(1,1,1)$. So, we are looking at the action

$$\begin{aligned} \mathbb{C}^\star \times (\mathbb{C} \oplus \mathbb{C}) &\longrightarrow \mathbb{C} \oplus \mathbb{C} \\ \big(z, (u,v)\big) &\longrightarrow (z^{-1} \cdot u, z \cdot v). \end{aligned}$$

The ring of invariant functions is $\mathbb{C}[u \cdot v]$.

Recall that, given a reductive linear algebraic group G, a vector space R, and an action $G \times R \longrightarrow R$ of G on R by linear transformations, a point $x \in R$ is *semistable*, if there exists a homogeneous invariant function F of positive degree with $F(x) \neq 0$. A *polystable* point is a semistable point whose orbit is closed inside R, and a *stable* point is a polystable point whose stabilizer is finite.[4]

A vertex $v \in V$ in a connected quiver is a *sink*, if there is no arrow $e \in E$ with $s(e) = v$, and a *source*, if there is no arrow $e \in E$ with $t(e) = v$.[5]

Assumptions 2.3. We assume that the quiver Q is connected, does not have oriented cycles nor multiple arrows, that there are no arrows between unmarked vertices, and that every unmarked vertex is either a sink or a source. Since the case that all unmarked vertices are sources has already been covered by Nakajima [8] and Reineke [9] and this includes, by dualizing, the case that all unmarked vertices are sinks, we will assume that there are both sinks and sources among the unmarked vertices.

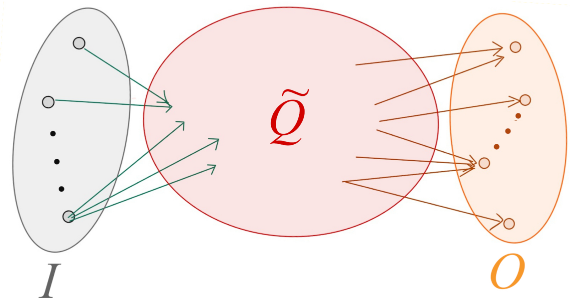
[4]These definitions are equivalent to those stated in the introduction.

[5]Since we assume the quiver to be connected, there is, for every vertex $v \in V$, an arrow starting or ending at v, unless, of course, Q is the one point quiver without arrows.

We write

$$V_u = I \sqcup O$$

with I the set of sources and O the set of sinks. Schematically, we depict our quiver in the following way:



Let us introduce

$$\begin{aligned} H_- &:= \bigoplus_{\substack{e\in E:\\ s(e)\in I}} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{s(e)}, \mathbb{C}^{t(e)}), \\ H^+ &:= \bigoplus_{\substack{e\in E:\\ t(e)\in O}} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{s(e)}, \mathbb{C}^{t(e)}), \end{aligned}$$

so that

$$\mathrm{Rep}_d(Q) = H_- \times \mathrm{Rep}_{d_h}(\widetilde{Q}) \times H^+.$$

Accordingly, we write an element of $\mathrm{Rep}_d(Q)$ in the form $S = (h_-, r, h^+)$ with $h_- \in H_-$, $r = (r_v, v \in V_h) \in \mathrm{Rep}_{d_h}(\widetilde{Q})$, and $h^+ \in H^+$.

Remark 2.4. This formalism reflects the fact that we view our representations as framed representations of the hidden quiver $\widetilde{Q}$. It is also convenient, because the notions of stability for a framed representation (h_-, r, h^+) that we will encounter are phrased in terms of subrepresentations of the representation r of the hidden quiver $\widetilde{Q}$.

Associated with Q, there is the set $\mathscr{P}$ of paths starting at a vertex in I and ending at a vertex in O. We let $s', t'\colon \mathscr{P} \longrightarrow I \sqcup O$ be the maps that assign to a path its starting point and its target point, respectively. Using these paths, we define

$$H_\pm := \bigoplus_{p\in\mathscr{P}} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{s'(p)}}, \mathbb{C}^{d_{t'(p)}})$$

and a morphism

$$\begin{aligned} X_\pm\colon \mathrm{Rep}_d(Q) &\longrightarrow H_\pm \\ (h_-, r, h^+) &\longmapsto h_\pm. \end{aligned}$$

The space $H_\pm$ is zero, if $\mathscr{P}$ is empty. Theorem 2.1 implies that (h_-, r, h^+) is semistable if and only if $h_\pm \neq 0$.[6]

Remark 2.5. The morphism $X_\pm$ is clearly invariant under the action of $\widetilde{G}_d(Q)$, so that it induces a morphism

$$\overline{X}_\pm\colon \mathscr{M}_d^\circ(Q) \longrightarrow H_\pm.$$

[6]Recall that we are supposing that there are no oriented cycles.

Theorem 2.1 actually states that this morphism is a closed embedding.

Next, let us verify this observation with the Hilbert–Mumford criterion and also determine the stable points. It states that a point $S = (h_-, r, h^+) \in \mathrm{Rep}_d(Q) \setminus \{0\}$ is semistable if and only if

$$\mu(\lambda, S) \geq 0$$

holds for every one parameter subgroup $\lambda\colon \mathbb{C}^\star \longrightarrow \widetilde{G}_d(Q)$, and stable if and only if

$$\mu(\lambda, S) > 0$$

holds for every non-constant one parameter subgroup $\lambda\colon \mathbb{C}^\star \longrightarrow \widetilde{G}_d(Q)$. We refer to [10], Section 1.5.1, for the definition of μ that we use.

We introduce the subgroup

$$\mathrm{S}\widetilde{G}_d(Q) := \Big\{ (g_v, v \in V_h) \in \widetilde{G}_d(Q) \,\Big|\, \prod_{v \in V_h} \det(g_v) = 1 \Big\}.$$

The homomorphism

$$\begin{aligned} \mathbb{C}^\star \times \mathrm{S}\widetilde{G}_d(Q) &\longrightarrow \widetilde{G}_d(Q) \\ \big(z, (g_v, v \in V_h)\big) &\longmapsto (z \cdot g_v, v \in V_h) \end{aligned}$$

is surjective and has finite kernel. For constructing the quotient and determining semistable and stable points, we may replace the group $\widetilde{G}_d(Q)$ by the group $\mathbb{C}^\star \times \mathrm{S}\widetilde{G}_d(Q)$.

Let us first look at one parameter subgroups of $\mathrm{S}\widetilde{G}_d(Q)$. In order to study these, it will be convenient to use the embedding

$$\iota\colon \mathrm{S}\widetilde{G}_d(Q) \hookrightarrow \mathrm{SL}_D(\mathbb{C}), \quad D := \sum_{v \in V_h} d_v.$$

A one parameter subgroup $\lambda\colon \mathbb{C}^\star \longrightarrow \mathrm{S}\widetilde{G}_d(Q)$ then induces the one parameter subgroup $\iota \circ \lambda$ of $\mathrm{SL}_D(\mathbb{C})$.

In order to specify $\lambda\colon \mathbb{C}^\star \longrightarrow \mathrm{S}\widetilde{G}_d(Q)$, we need to specify bases $(w(v,1), \dots, w(v, d_v))$ of $\mathbb{C}^{d_v}$ and integral weights $\gamma(v,i)$, $i = 1, \dots, d_v$, $v \in V_h$, such that

$$\sum_{v \in V_h} \sum_{i=1}^{d_v} \gamma(v,i) = 0.$$

Next, we choose a bijection

$$\beta\colon \{1, \dots, D\} \longrightarrow \{ (v,i) \mid v \in V_h,\ i = 1, \dots, d_v \},$$

such that

$$\gamma\big(\beta(1)\big) \leq \cdots \leq \gamma\big(\beta(D)\big),$$

and define $\gamma_1 < \cdots < \gamma_{s+1}$ by the condition

$$\{ \gamma_1, \dots, \gamma_{s+1} \} = \Big\{ \gamma\big(\beta(1)\big), \dots, \gamma\big(\beta(D)\big) \Big\}$$

as well as

$$\delta_j := \max\big\{ i \mid 1 \leq i \leq D \wedge \gamma\big(\beta(i)\big) \leq \gamma_j \big\},$$

$j = 1, \dots, s+1$, Now, set

$$\mathbb{W}_0 := 0 \quad \text{and} \quad \mathbb{W}_j := \Big\langle w\big(\beta(1)\big), \dots, w\big(\beta(\delta_j)\big) \Big\rangle,$$

$j = 1, \dots, s+1$. Then,

$$\mathbb{W}_\bullet: \quad \mathbb{W}_0 \subsetneq \mathbb{W}_1 \subsetneq \cdots \subsetneq \mathbb{W}_s \subsetneq \mathbb{W}_{s+1} = \mathbb{C}^D$$

is a (partial) flag in $\mathbb{C}^D$. We also define

$$\varepsilon_j := \frac{\gamma_{j+1} - \gamma_j}{D}, \ j = 1, \dots, s, \quad \text{and} \quad \varepsilon_\bullet := (\varepsilon_1, \dots, \varepsilon_s).$$

The pair $(\mathbb{W}_\bullet, \varepsilon_\bullet)$ is the *weighted flag* of λ. This is the weighted flag of $\iota \circ \lambda$ as defined in [10], Example 1.5.1.36.

Remark 2.6. Setting

$$\gamma_D^{(\delta)} := \big(\underbrace{\delta - D, \dots, \delta - D}_{\delta \times}, \underbrace{\delta, \dots, \delta}_{(D-\delta)\times} \big), \quad \delta = 1, \dots, D-1,$$

we have the identity

$$\Big(\gamma\big(\beta(1)\big), \dots, \gamma\big(\beta(D)\big)\Big) = \sum_{j=1}^{s} \varepsilon_j \cdot \gamma_D^{(j)}. \qquad (\star)$$

Let us first investigate $\mu(\lambda, r)$, for a one parameter subgroup $\lambda\colon \mathbb{C}^\star \longrightarrow \mathrm{S}\widetilde{G}_d(Q)$ and a (non-zero) representation $r = (r_v, v \in V_h) \in \mathrm{Rep}_d(\widetilde{Q})$ of the hidden quiver. The representation r can be viewed as an endomorphism of $\mathbb{C}^D$,[7] and

$$\mu(\lambda, r) = \mu(\iota \circ \lambda, r).$$

On the right hand side, we are dealing with the standard action of $\mathrm{SL}_D(\mathbb{C})$ on $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^D)$, and one readily checks:

$$\begin{aligned} \mu(\lambda, r) > 0 &\iff \exists j \in \{1, \dots, s\} : r(\mathbb{W}_j) \not\subseteq \mathbb{W}_j, \\ \mu(\lambda, r) < 0 &\iff \forall j \in \{1, \dots, s+1\} : r(\mathbb{W}_j) \subseteq \mathbb{W}_{j-1}. \end{aligned}$$

Now, there are uniquely defined subspaces $A_v^j \subseteq \mathbb{C}^{d_v}$, $v \in V_h$, $j = 0, \dots, s+1$, such that

$$\mathbb{W}_j = \bigoplus_{v \in V_h} A_v^j, \quad j = 0, \dots, s+1.$$

The fact that, for $j = 0, \dots, s+1$, the condition $r(\mathbb{W}_j) \subseteq \mathbb{W}_j$ holds if and only if $A^j := (A_v^j, v \in V_h)$ is a subrepresentation of r follows readily from the definitions and is crucial in order to express all the notions of semistability and stability for framed quiver representations in terms of subrepresentations of the representation of the hidden quiver.

Next, let us evaluate the quantity $\mu(\lambda, h_-)$, for a one parameter subgroup $\lambda\colon \mathbb{C}^\star \longrightarrow \mathrm{S}\widetilde{G}_d(Q)$, and $h_- \in H_- \subseteq \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}_-, \mathbb{C}^D)$, $\mathbb{H}_- := \bigoplus_{v \in I} \mathbb{C}^{d_v}$. Then, with

$$\delta_- := \min\Big\{ i \,\Big|\, 1 \leq i \leq D \quad \wedge \quad \mathrm{Im}(h_-) \subseteq \Big\langle w\big(\beta(1)\big), \dots, w\big(\beta(i)\big) \Big\rangle \Big\},$$

we get that

$$\mu(\lambda, h_-) = \gamma\big(\beta(\delta_-)\big).$$

[7] Here, we use the assumption that we do not have multiple arrows.

Remark 2.7. Let $j_m \in \{1, ..., s+1\}$ be minimal among the indices j, such that $\mathrm{Im}(h_-) \subseteq \mathbb{W}_j$. Then, we obviously have

$$\mu(\lambda, r) = \gamma_{j_m}.$$

On the other hand, by $(\star)$, we have that

$$\mu(\lambda, r) = \sum_{i=1}^{j_m-1} \varepsilon_i \cdot \delta_i + \sum_{i=j_m}^{s} \varepsilon_i \cdot (\delta_i - D).$$

We also have to determine $\mu(\lambda, h^+)$, for a one parameter subgroup $\lambda\colon \mathbb{C}^\star \longrightarrow S\widetilde{G}_d(Q)$, and $h^+ \in H^+ \subseteq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^D, \mathbb{H}^+)$, $\mathbb{H}^+ := \bigoplus_{v\in O} \mathbb{C}^{d_v}$. For

$$\delta^+ := \min\left\{ i \,\middle|\, 1 \le i \le D \wedge \left\langle w\big(\beta(1)\big), ..., w\big(\beta(i)\big) \right\rangle \not\subseteq \mathrm{Ker}(h^+) \right\},$$

we find that

$$\mu(\lambda, h^+) = -\gamma\big(\beta(\delta^+)\big).$$

Remark 2.8. Let $j_M \in \{1, ..., s+1\}$ be minimal among the indices j, such that $\mathbb{W}_j \not\subseteq \mathrm{Ker}(h^+)$. Then, we obviously have

$$\mu(\lambda, r) = -\gamma_{j_M}.$$

Moreover, $(\star)$ shows that

$$\mu(\lambda, r) = -\sum_{i=1}^{j_M-1} \varepsilon_i \cdot \delta_i + \sum_{i=j_M}^{s} \varepsilon_i \cdot (D - \delta_i).$$

Finally, we study one parameter subgroups of the form

$$\begin{aligned} \lambda_\gamma\colon \mathbb{C}^\star &\longrightarrow \mathbb{C}^\star \\ z &\longmapsto z^\gamma, \end{aligned}$$

$\gamma \in \mathbb{Z}$. This one parameter subgroup acts trivially on $\mathrm{Rep}_d(\widetilde{Q})$. For $h_- \neq 0$, one has

$$\mu(\lambda_\gamma, h_-) = \gamma, \quad \gamma \in \mathbb{Z},$$

and, for $h^+ \neq 0$, one sees

$$\mu(\lambda_\gamma, h^+) = -\gamma, \quad \gamma \in \mathbb{Z}.$$

We are now ready to give the characterization of stable and semistable points.

Theorem 2.9. a) *A representation is semistable if and only if the induced map* $h_\pm\colon \mathbb{H}_- \longrightarrow \mathbb{H}^+$ *is non-zero.*

b) *A representation* $r = (h_-, r_v, v \in V_h, h^+) \in \mathrm{Rep}_d(Q)$ *is stable if and only if the following two properties hold true:*

1. *There is no subrepresentation* $A = (A_v, v \in V_h) \neq 0$ *of the representation* $(r_v, v \in V_h)$ *of the hidden quiver with* $\bigoplus_{v\in V_h} A_v \subseteq \mathrm{Ker}(h^+)$.
2. *There is no subrepresentation* $A = (A_v, v \in V_h) \neq (\mathbb{C}^{d_v}, v \in V_h)$ *of the representation* $(r_v, v \in V_h)$ *of the hidden quiver with* $\mathrm{Im}(h_-) \subseteq \bigoplus_{v\in V_h} A_v$.

Note that the first condition, applied to $(\mathbb{C}^{d_v}, v \in V_h)$, shows $h^+ \neq 0$, and the second condition, applied to 0, yields $h_- \neq 0$.

Proof. a) As noted above, this result follows from Theorem 2.1. Nevertheless, we will explain how to see it with the Hilbert–Mumford criterion.[8] Suppose $S = (h_-, r, h^+)$ is a (non-zero) point of $\mathrm{Rep}_d(Q)$. First, we study the case that there is a one parameter subgroup $(\lambda_\gamma, \lambda)$ of $\mathbb{C}^\star \times S\widetilde{G}_d(Q)$, such that

$$\mu\big((\lambda_\gamma, \lambda), S\big) < 0.$$

If h_-, $r = (r_v, v \in V_h)$, and h^+ are all non-zero, then

$$\begin{aligned} &\mu\big((\lambda_\gamma, \lambda), S\big) \qquad (\star\star) \\ &= \max\big\{ \mu(\lambda, h_-) + \gamma, \mu(\lambda, r), \mu(\lambda, h^+) - \gamma \big\}. \end{aligned}$$

In general, only the non-zero components of S participate in forming the maximum.

i) If $\lambda = 0$, then necessarily $r = 0$. Furthermore, $h_- = 0$, if $\gamma > 0$, and $h^+ = 0$, if $\gamma < 0$. In both cases, $h_\pm = 0$.

ii) For $\lambda \neq 0$, we look at the weighted flag $(\mathbb{W}_\bullet, \varepsilon_\bullet)$ that this one parameter subgroup of $S\widetilde{G}_d(Q)$ defines in $\mathbb{C}^D$. If $h_- = 0$ or $h^+ = 0$, then also $h_\pm = 0$. In the remaining case $h_- \neq 0$ and $h^+ \neq 0$, we note that, by $(\star\star)$, we must have

$$0 > \mu(\lambda, h_-) + \gamma + \mu(\lambda, h_-) - \gamma = \gamma_{j_m} + \gamma - \gamma_{j_M} - \gamma,$$

i.e.,

$$\gamma_{j_m} < \gamma_{j_M},$$

or, equivalently,

$$j_m < j_M.$$

This shows that $\mathrm{Im}(h_-) \subseteq \mathbb{W}_{j_m} \subseteq \mathrm{Ker}(h_+)$ and, so, $h_\pm = 0$.

Now, we will show that S is unstable, if $h_\pm = 0$.

i) In the case that $S = (h_-, 0, h^+)$ and $h_- = 0$, we have $\mu((\lambda_1, 0), S) = -1 < 0$. If $h^+ = 0$, then $\mu((\lambda_{-1}, 0), S) = -1 < 0$.

ii) Suppose first that $h_+ = 0$ and $r \neq 0$.[9] Since we assume that Q has no oriented cycles, the representation r, viewed as an endomorphism of $\mathbb{C}^D$, is nilpotent. Let s be the largest natural number with $r^s \neq 0$. Then, we form the flag

$$\mathbb{W}_\bullet: \quad 0 \subsetneq r^s(\mathbb{C}^D) \subsetneq \cdots \subsetneq r(\mathbb{C}^D) \subsetneq \mathbb{C}^D.$$

Pick a one parameter subgroup $\lambda\colon \mathbb{C}^\star \longrightarrow S\widetilde{G}_d(Q)$ whose weighted flag is of the form $(\mathbb{W}_\bullet, \varepsilon_\bullet)$. By a previous remark, $\mu(\lambda, r) < 0$. If $h_- = 0$, we are done. Otherwise, choose an integer $\gamma > \mu(\lambda, h_-)$. Then, $\mu(\lambda, h_-) - \gamma < 0$. In view of $(\star\star)$, the pair $(\lambda_\gamma, \lambda)$ constitutes a destabilizing one parameter subgroup. We proceed in a similar manner, if $h_- = 0$.

[8]The determination of the one parameter subgroups that act with negative and zero weight is also necessary in order to understand the notion of semistability with respect to a linearization obtained by modifying the standard representation by a character of $\widetilde{G}_d(Q)$.

[9]Otherwise we may apply Part i) of the argument.

Finally, if $h_- \neq 0$ and $h_+ \neq 0$, let s be the maximal natural number with $r^s(\mathrm{Im}(h_-)) \neq 0$ and $\mathbb{W}_s$ the subrepresentation generated by $\mathbb{I} := \mathrm{Im}(h_-)$, i.e.,

$$\mathbb{W}_s := \sum_{i=0}^{s} r^i(\mathbb{I}).$$

We have $\mathbb{W}_s \neq \mathbb{C}^D$, because $h^+ \neq 0$ and $h_\pm = 0$. We further set

$$\mathbb{W}_{s-j} := r^j(\mathbb{W}_s) = \sum_{i=j}^{s} r^i(\mathbb{I}), \quad j = 1, ..., s.$$

Then,

$$\mathbb{W}_\bullet : \quad 0 =: \mathbb{W}_0 \subsetneq \mathbb{W}_1 \subsetneq \cdots \subsetneq \mathbb{W}_s \subsetneq \mathbb{W}_{s+1} := \mathbb{C}^D$$

is a flag in $\mathbb{C}^D$, such that

- $r(\mathbb{W}_j) \subseteq \mathbb{W}_{j-1}$, $j = 1, ..., s+1$,
- $\mathbb{W}_s \subseteq \mathrm{Ker}(h^+)$, because $h_\pm = 0$, and
- $\mathrm{Im}(h_-) \subseteq \mathbb{W}_s$, by construction.

We are still free to choose the vector $\varepsilon_\bullet$ of weights in $\mathbb{Z}[1/D]$. Once we have decided for such a weight vector and fixed a one parameter subgroup λ of $\mathrm{S}\widetilde{G}_d(Q)$ whose weighted flag is $(W_\bullet, \varepsilon_\bullet)$, then $\mu(\lambda, r) < 0$, by the first property of $\mathbb{W}_\bullet$. The s-th summand in the formula for $\mu(\lambda, h_+)$ in Remark 2.8 is the negative number $-\varepsilon_s \cdot \dim_k(\mathbb{W}_s)$, because of the second property of $\mathbb{W}_\bullet$, and the s-th summand in the formula for $\mu(\lambda, h_-)$ in Remark 2.7 is the negative number $\varepsilon_s \cdot (\dim_k(\mathbb{W}_s) - D)$, by the third property of $\mathbb{W}_\bullet$. So, if we choose ε_s large with respect to $\varepsilon_1, ..., \varepsilon_{s-1}$, both $\mu(\lambda, h_-)$ and $\mu(\lambda, h_+)$ will be negative, too, so that $\mu((0, \lambda), S) < 0$, by ($\star\star$), and we have verified that S is unstable.

b) If $S = (h_-, r, h^+)$ is stable and, so, semistable, then there is no subrepresentation A with $\mathrm{Im}(h_-) \subseteq A \subseteq \mathrm{Ker}(h^+)$, because this would imply $h_\pm = 0$. Now, suppose that $A = (A_v, v \in V_h)$ is a subrepresentation with $\mathbb{W}(A) \subseteq \mathrm{Ker}(r)$, $\mathbb{W}(A) := \bigoplus_{v \in V_h} A_v$. Then, we choose a one parameter subgroup $\lambda\colon \mathbb{C}^\star \longrightarrow \mathrm{S}\widetilde{G}_d(Q)$, such that its weighted flag is $((0 \subsetneq \mathbb{W}(A) \subsetneq \mathbb{C}^D), (1))$. By Remark 2.8, we have

$$\mu(\lambda, h^+) = -\delta, \quad \delta := \dim_k\big(\mathbb{W}(A)\big).$$

Since $\mathrm{Im}(h_-) \nsubseteq \mathbb{W}(A)$, it follows that

$$\mu(\lambda, h_-) = \delta,$$

using Remark 2.7. So, we infer that

$$\mu\big((\lambda_{-\delta}, \lambda), S\big) = 0.$$

In the same way, we may rule out the existence of a subrepresentation A with $\mathrm{Im}(r) \subseteq \mathbb{W}(A)$. So, a stable representation satisfies Conditions 1. and 2.

Now, let us assume that $S = (h_-, r, h^+)$ satisfies Conditions 1. and 2. Applying the first condition to the trivial subrepresentation and the second one to the whole representation clearly implies $h_- \neq 0$ and $h^+ \neq 0$, respectively. Let $(\lambda_\gamma, \lambda)$ be a one parameter subgroup of $\mathbb{C}^\star \times \mathrm{S}\widetilde{G}_d(Q)$. Then, in our previous conventions, Condition 1. and Remark 2.8 show

$$\mu(\lambda, h^+) = -\gamma_1 \geq 0,$$

and Condition 2. and Remark 2.8 imply

$$\mu(\lambda, h_-) = \gamma_{s+1} \geq 0.$$

So, if $\gamma < 0$, then

$$\mu(\lambda, h^+) - \gamma = -\gamma_1 - \gamma > 0,$$

and, if $\gamma \geq 0$, then

$$\mu(\lambda, h_-) = \gamma_{s+1} + \gamma > 0.$$

Here, we use that, if $\gamma = 0$, then $\lambda \neq 0$ and, so, $\gamma_{s+1} > 0$. In view of ($\star\star$), we infer that S is stable. □

CONCLUSION

We evaluated the Hilbert–Mumford criterion directly on the space of framed representations of a network quiver in order to recover a characterization of semistable and stable framed representations of such quivers found by Armenta, Brüstle, Hassoun, and Reineke. This criterion can be used to establish the existence of stable representations and to compute the dimension of the resulting moduli space. As we will explain in forthcoming work, the dimension of the moduli space of framed representations whose dimension vector consists only of ones agrees with a certain invariant for measuring the complexity of a neural network. This observation illustrates the potential of the investigation of moduli spaces in the study of the theoretical foundations of neural networks and machine learning.

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