Invariance properties of time fractional linear diffusion-wave equations

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ABSTRACT

We study a class of time fractional diffusion-wave equations with variable coefficients using Lie symmetry analysis. We obtain not only infinitesimal symmetries but also a complete group classification and a classification of group invariant solutions of this class of equations. Group invariant solutions are given explicitly corresponding to every element in an optimal system of Lie algebras generated by infinitesimal symmetries of equations in the class. We express the solutions in terms of Mittag-Leffler functions, generalized Wright functions, and Fox H-functions. These solutions contain previously known solutions as particular cases.

Keywords: *Time fractional differential equation, diffusion wave equation, invariance properties*

1. INTRODUCTION

In 1987, Bluman and Kumei [1] gave a complete group classification and some invariant solutions to variable coefficient wave equation $u_{tt} = c^2(x)u_{xx}$ and its corresponding system $u_t = c^2(x)v_x,\ v_t = u_x$. In [2], some group invariant solutions to a time fractional generalization of the corresponding system of the wave equation were given. As a continuation of [2], in this work we consider a class of time fractional diffusion-wave equations with variable coefficients of the following form:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = c^2(x)\frac{\partial^2}{\partial x^2}u(x,t), \quad x>0,\ t>0,\ \alpha>0, \tag{1}$$

where $c(x)$ is a sufficiently differentiable, nonzero function. The equation (1) can be considered as a time fractional generalization of diffusion-wave equations with variable coefficients. Here, fractional differentiation is defined by the following Riemann-Liouville manner:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \text{for } \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, & \\ \quad \text{for } \alpha \in (n-1,n), \text{ with } n\in\mathbb{N}. \end{cases} \tag{2}$$

In recent years, the study of time fractional diffusion-wave equations gains increasing attention as they model anomalous diffusion processes. There are a variety of inhomogeneous media ranging from plasma to living cell where diffusion exhibits anomalous properties. So, studying invariance properties and presenting explicit invariant solutions of (1) is of great practical importance.

The particular case of the equation (1) with constant diffusion coefficient

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = C\frac{\partial^2}{\partial x^2}u(x,t) \tag{3}$$

has been studied extensively. In [3], [4], when $0<\alpha<2$, the solutions of (3) with appropriate initial conditions were expressed in terms of Fox H-functions using Mellin integral transformations. Also, in the sequential works of F. Mainardi et al. [5], [6], [7], the fundamental solutions of Cauchy and boundary value problems of (3) in means of Riemann-Liouville and Caputo fractional derivatives were obtained using Laplace transformations. In [8] the invariance of the equation (3) under scaling transformations was studied and the scale-invariant solutions were found in terms of generalized Wright functions when $\alpha>1$.

Solutions of (1) with $c(x) = x^m$ were given for $\alpha = 1$ in [9] and for $0<\alpha<1$ in [10] using Laplace transformations. To the best of our knowledge (see the references), the classification of infinitesimal symmetries and invariant solutions of (1) have not been studied when the diffusion coefficient is non-constant. Thus, the main purpose of this study is to obtain a complete group classification depending on the function $c(x)$ and to give explicitly invariant solutions that correspond to each infinitesimal symmetry in an optimal system of infinitesimal symmetries of (1) in terms of special

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functions: Mittag-Leffler functions, generalized Wright functions, and Fox H-functions. The solutions obtained in this work coincide with the previously known solutions for particular choice of $c(x)$ or for particular values of α. From the definition (2), we know that the equation (1) interpolates between the diffusion equations and the wave equations as α varies from 1 to 2. In Section 5, this interpolating behaviour can be seen from the plots of the solution obtained.

The structure of this work is as follows. In Section 2, we present a simple introduction to the Lie symmetry analysis of fractional partial differential equations (FPDEs) and provide formulas that are useful in studying (1). In Section 3, we carry out a complete group classification with respect to the function $c(x)$. In Section 4, we investigate the structure of the corresponding Lie algebras of infinitesimal symmetries and determine optimal systems. Then, we reduce (1) to fractional ordinary differential equations (FODEs) in accordance with these optimal systems. In Section 5, we explicitly present solutions to (1) using the results of preceding work [11] by the authors. Additionally, we include a brief introduction to the special functions and relevant formulas in order to make the current study self-contained. Finally, we show that for $\alpha = 1$ and $\alpha = 2$ the well known solutions can be derived from the solutions that we give in this work.

2. LIE SYMMETRY ANALYSIS FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

To study the equation given in (1) via Lie symmetry analysis, we present basic definitions and formulas that are needed to carry out the Lie symmetry analysis of FPDEs. The general form of time fractional PDEs with two independent variables x and t is as follows:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = F(x,t,u,u_x,u_{xx},\ldots), \tag{4}$$

where the subscripts denote partial derivatives and α is a positive real number. In the Lie symmetry analysis, the infinitesimal generator and the corresponding prolonged infinitesimal generator of (4) are given by

$$\begin{aligned} X &= \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial u} \text{ and} \\ \tilde{X} &= X + \mu^{(\alpha)}\frac{\partial}{\partial t^\alpha} + \mu^{(1)}\frac{\partial}{\partial u_x} + \cdots, \end{aligned}$$

respectively, where ξ, τ and μ are infinitesimals and $\mu^{(\alpha)}$, $\mu^{(n)}$ $(n = 1,2,\ldots)$ are extended infinitesimals. Explicitly, $\mu^{(n)}$ is given by

$$\begin{aligned} &\mu^{(1)} = D_x(\mu) - u_x D_x(\xi) - u_t D_x(\tau), \\ &\mu^{(2)} = D_x(\mu^{(1)}) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau), \\ &\vdots \end{aligned}$$

where D_x is the total derivative operator defined as

$$D_x := \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \cdots.$$

The αth order extended infinitesimal $\mu^{(\alpha)}$ has the following form [12], [13]:

$$\begin{aligned} \mu^{(\alpha)} = &\frac{\partial^\alpha \mu}{\partial t^\alpha} + (\mu_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \mu_u}{\partial t^\alpha} \\ &+ \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \mu_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] D_t^{\alpha-n}(u) \\ &- \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) + \mu_1, \end{aligned}$$

where

$$\begin{aligned} \mu_1 = &\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!} \\ &\times \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^r \frac{\partial^m}{\partial t^m}\left(u^{k-r}\right)\frac{\partial^{n-m+k}\mu}{\partial t^{n-m}\partial u^k}. \end{aligned}$$

Here we denote the generalized binomial coefficient $\binom{\alpha}{n}$ and the total derivative operator D_t, respectively by

$$\begin{aligned} \binom{\alpha}{n} &= \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)} \text{ and} \\ D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + \cdots. \end{aligned}$$

It should be noted that $\mu_1 = 0$ when the infinitesimal μ is linear in u.

We have the following initial condition

$$\tau(x,t,u)|_{t=0} = 0 \tag{5}$$

because of the fixed lower limit in the integral of (2).

The infinitesimal invariance criterion in the Lie symmetry analysis for the equation (4) is

$$\tilde{X}(u_{t^\alpha} - F(t,x,u,u_x,u_{xx},\cdots))|_{(4)} = 0. \tag{6}$$

Now we are prepared for the investigation of the infinitesimal symmetries of the class of equations given in (1).

3. LIE SYMMETRY ANALYSIS OF THE DIFFUSION-WAVE EQUATION GIVEN IN (1)

In this section, we study (1) using the formulas obtained in the previous section. We show that there are six cases regarding the symmetry groups of (1), as determined by the form of the function $c(x)$, five in which $c(x)$ possesses the special forms (specified below), and one in which it does not. In the former five cases, the symmetry groups of (1) possess additional symmetries which do not exist in the latter case. For each of the six cases we obtain the infinitesimal symmetries.

The invariance criterion (6) for the equation given in (1) is

$$\tilde{X}(u_{t^\alpha} - c^2(x)u_{xx})|_{(1)} = 0,$$

which is in explicit form

$$\left(\mu^{(\alpha)} - 2c(x)c'(x)\xi u_{xx} - c^2(x)\mu^{(2)}\right)\Big|_{(1)} = 0. \quad (7)$$

From (7), we obtain the following (overdetermined) system of determining equations by setting the coefficients of the linearly independent partial derivatives $D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x$, u_{xx}, u_x^2, u_x, u_{xt}, $u_x u_{xt}$ and $u_x u_{xx}$ equal to zero:

$$\begin{aligned}
&\binom{\alpha}{n}\frac{\partial^n \mu_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}\tau = 0, \quad n = 1, 2, \dots,\\
&D_t^n(\xi) = 0, \quad n = 1, 2, \dots,\\
&2c^2(x)\xi_x - \alpha c^2(x)\tau_t - 2c(x)c'(x)\xi = 0,\\
&\mu_{uu} = 0,\\
&2\mu_{xu} - \xi_{xx} = 0,\\
&\frac{\partial^\alpha \mu}{\partial t^\alpha} - u\frac{\partial^\alpha \mu_u}{\partial t^\alpha} + c^2(x)\mu_{xx} + \mu_1 = 0,\\
&\tau_x = \tau_u = 0,\\
&\xi_u = 0.
\end{aligned}$$

Analyzing the above overdetermined system with the initial condition (5), we are able to deduce the following infinitesimal symmetries of (1):

Case 0. This is the generic situation, which applies to all forms of $c(x)$ except the following five cases. In this case, the infinitesimal symmetries are

$$X_1 = u\frac{\partial}{\partial u}, \quad X_g = g(x,t)\frac{\partial}{\partial u},$$

where $g(x,t)$ is a solution of the equation (1).

For the following five cases of function $c(x)$, we get additional symmetries along with X_1 and X_g given above.

Case 1. In the case $c(x) = c$ (here $c \in \mathbb{R}$), the additional symmetries are

$$X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial x} + \frac{2}{\alpha}t\frac{\partial}{\partial t}.$$

Case 2. In the case $c(x) = (k_1x + k_2)^2$ (here $k_1, k_2 \in \mathbb{R}$ and $k_1 \neq 0$), the additional symmetries are

$$X_4 = -\left(x + \frac{k_2}{k_1}\right)\frac{\partial}{\partial x} + \frac{2}{\alpha}t\frac{\partial}{\partial t},$$

$$X_5 = (k_1x + k_2)^2\frac{\partial}{\partial x} + k_1^2 xu\frac{\partial}{\partial u}$$

Case 3. In the case $c(x) = (k_1x+k_2)^{k_3}$ (here $k_1, k_2, k_3 \in \mathbb{R}$, $k_1 \neq 0$ and $k_3 \neq 0, 2$), the additional symmetry is

$$X_6 = \left(x + \frac{k_2}{k_1}\right)\frac{\partial}{\partial x} + \frac{2(1-k_3)}{\alpha}t\frac{\partial}{\partial t}.$$

Case 4. In the case $c(x) = k_1 e^{k_2 x}$ (here $k_1, k_2 \in \mathbb{R}$ and $k_1k_2 \neq 0$), the additional symmetry is

$$X_7 = -\frac{1}{2k_2}\frac{\partial}{\partial x} + \frac{1}{\alpha}t\frac{\partial}{\partial t}.$$

Case 5. In the case

$$\begin{aligned}
c(x) &= \left((k_1x + k_2)^2 + k_3\right)\\
&\quad\times \exp\left(k_4\int\frac{dx}{(k_1x+k_2)^2 + k_3}\right)
\end{aligned}$$

(here $k_1, k_2, k_3, k_4 \in \mathbb{R}$ and $k_1 \neq 0$, $(k_3, k_4) \neq (0,0)$), the additional symmetry is

$$X = ((k_1x+k_2)^2 + k_3)\frac{\partial}{\partial x} - \frac{2k_4}{\alpha}t\frac{\partial}{\partial t} + k_1^2 xu\frac{\partial}{\partial u}.$$

Since we derived a complete group classification of (1), we are ready to determine the one-dimensional optimal systems of Lie algebras of corresponding infinitesimal symmetries and the classification of group invariant solutions of (1). In subsequent calculations, we ignore the trivial infinitesimal symmetry X_g and in Case 0, the Lie algebra is generated by only X_1. Since there are no invariant solutions corresponding to X_1, we consider Cases 1 through 5.

4. THE CLASSIFICATIONS OF INVARIANT SOLUTIONS

In this section, we determine the group invariant solutions of (1) corresponding to infinitesimal symmetries obtained in the previous section. More explicitly, we express the invariant solutions as solutions of so-called reduced fractional ordinary differential equations. The invariant solutions of (1) corresponding to any infinitesimal symmetry can be obtained through continuous symmetry transformations, which are applied to the invariant solutions corresponding to the infinitesimal symmetries of any optimal system of one-dimensional subalgebras of infinitesimal symmetries [14], [15]. For this reason, we need only to describe the invariant solutions corresponding to the infinitesimal symmetries of an optimal system of one-dimensional subalgebras of infinitesimal symmetries. The optimal systems of low-dimensional Lie algebras are determined in [16]. We describe the structure of Lie algebras generated by the infinitesimal symmetries and choose the optimal systems by using the results of [16].

In the following subsections, we present the optimal systems and corresponding reduced equations for five cases specified above. Notice that the point transformation

$$\bar{x} = ax + b, \ \bar{t} = ht, \ \bar{u} = gu + \sum_{i=1}^{n} g_i t^{\alpha - i},$$
$$a, b, h, g, g_i \in \mathbb{R} \ \text{ and } \ a, h > 0 \quad (8)$$

is an equivalent transformation for (1). In other words, the equation (1) can be transformed to an equivalent one

$$\frac{\partial^\alpha}{\partial t^\alpha}\bar{u}(\bar{x},\bar{t}) = \bar{c}^2(\bar{x})\frac{\partial^2}{\partial x^2}\bar{u}(\bar{x},\bar{t}),$$
$$\text{where } \bar{c}^2(\bar{x},\bar{t}) = \frac{a^2}{h^\alpha}c^2\left(\frac{\bar{x}-b}{a}\right).$$

We apply the transformation (8) in order to simplify the coefficient $c(x)$ in (1). This means, without loss of generality, we assume that $c(x) = x^{k_3}$ instead of $c(x) = (k_1x+k_2)^{k_3}$, $k_1 \neq 0$ and $c(x) = e^{-\frac{x}{2}}$ instead of $c(x) = k_1e^{k_2x}$, $k_1k_2 \neq 0$, respectively.

4.1. Reduced equations of (1) with $c(x) = c$, $c \in \mathbb{R}$

We assume, without loss of generality, that $c(x) = 1$. In this case, the equation given in (1) possesses the following infinitesimal symmetries

$$X_1 = u\frac{\partial}{\partial u}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = x\frac{\partial}{\partial x} + \frac{2}{\alpha}t\frac{\partial}{\partial t}.$$

Except the Lie commutator of X_2, X_3, which is $[X_2, X_3] = X_2$, all the other Lie commutators are zero. Thus, the Lie algebra generated by X_1, X_2 and X_3 is identical to the Lie algebra $A_1 \oplus A_2$ given in [16]. The one-dimensional optimal system of this Lie algebra is that obtained in [16],

$$\begin{aligned} U_1 &= sX_1 + X_3 = x\frac{\partial}{\partial x} + \frac{2}{\alpha}t\frac{\partial}{\partial t} + su\frac{\partial}{\partial u}, \; s \in \mathbb{R}, \\ U_2 &= X_1 + X_2 = \frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \\ U_3 &= X_1 - X_2 = -\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \\ U_4 &= X_2 = \frac{\partial}{\partial x}, \\ U_5 &= X_1 = u\frac{\partial}{\partial u}. \end{aligned}$$

So, all we need to do is to present the invariant solutions corresponding to each infinitesimal symmetry in the above optimal system.

The characteristic equation of U_1 reads as

$$\frac{dx}{x} = \frac{\alpha dt}{2t} = \frac{du}{su},$$

which gives the similarity variable $z = x^{-\frac{2}{\alpha}}t$. Thus, the similarity transformation (ansatz) or the group invariant solution is

$$u = x^s\varphi(z). \tag{9}$$

Substituting (9) into (1) with $c(x) = 1$, we obtain the reduced equation

$$\frac{d^\alpha\varphi}{dz^\alpha} = s(s-1)\varphi + \frac{2}{\alpha}\left(\frac{2}{\alpha} - 2s + 1\right) z\varphi_z + \frac{4}{\alpha^2}z^2\varphi_{zz}. \tag{10}$$

The invariant solutions $u = e^x\varphi(t)$ and $u = e^{-x}\varphi(t)$ are found corresponding to U_2 and U_3, respectively. For both cases, we get the reduced equation

$$\frac{d^\alpha\varphi}{dt^\alpha} = \varphi. \tag{11}$$

The invariant solution corresponding to U_4 is found as $u(x,t) = \varphi(t)$. Thus, the reduced equation is

$$\frac{d^\alpha\varphi}{dt^\alpha} = 0. \tag{12}$$

The infinitesimal symmetry U_5 appears in each optimal systems of the following subsections and it does not yield any invariant solutions.

4.2. Reduced equations of (1) with $c(x) = (k_1x + k_2)^2$ (here $k_1, k_2 \in \mathbb{R}$ and $k_1 \neq 0$)

We assume, without loss of generality, that $c(x) = x^2$. Then, the infinitesimal symmetries become

$$X_1 = u\frac{\partial}{\partial u},\; X_4 = -x\frac{\partial}{\partial x} + \frac{2}{\alpha}t\frac{\partial}{\partial t},\; X_5 = x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}.$$

Except the Lie commutator of X_4, X_5, which is $[X_5, X_4] = X_5$, all the other Lie commutators are zero. As in the previous case, the optimal system is

$$\begin{aligned} U_5 &= X_1 = u\frac{\partial}{\partial u}, \\ U_6 &= sX_1 - X_4 = x\frac{\partial}{\partial x} - \frac{2}{\alpha}t\frac{\partial}{\partial t} + su\frac{\partial}{\partial u}, \; s \in \mathbb{R}, \\ U_7 &= X_1 + X_5 = x^2\frac{\partial}{\partial x} + (x+1)u\frac{\partial}{\partial u}, \\ U_8 &= X_1 - X_5 = -x^2\frac{\partial}{\partial x} - (x-1)u\frac{\partial}{\partial u}, \\ U_9 &= X_5 = x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}. \end{aligned}$$

The invariant solution $u = x^s\varphi(z)$, where $z = x^{\frac{2}{\alpha}}t$, is found corresponding to U_6. The reduced equation is

$$\frac{d^\alpha\varphi}{dz^\alpha} = s(s-1)\varphi + \frac{2}{\alpha}\left(\frac{2}{\alpha} + 2s - 1\right) z\varphi_z + \frac{4}{\alpha^2}z^2\varphi_{zz}. \tag{13}$$

The invariant solutions $u = xe^{-\frac{1}{x}}\varphi(t)$ and $u = xe^{\frac{1}{x}}\varphi(t)$ are found corresponding to U_7 and U_8, respectively. For both cases, the reduced equation is same, given as

$$\frac{d^\alpha\varphi}{dt^\alpha} = \varphi. \tag{14}$$

The invariant solution corresponding to U_9 is found as $u(x,t) = x\varphi(t)$. Thus, the reduced equation is

$$\frac{d^\alpha\varphi}{dt^\alpha} = 0. \tag{15}$$

4.3. Reduced equations of (1) with $c(x) = (k_1x + k_2)^{k_3}$ (here $k_1, k_2, k_3 \in \mathbb{R}$, $k_1 \neq 0$ and $k_3 \neq \{0, 2\}$)

We may assume, without loss of generality, that $c(x) = x^m$ (here $m \in \mathbb{R}$ and $m \neq 0, 2$). Thus, the Lie symmetries become

$$X_1 = u\frac{\partial}{\partial u} \text{ and } X_6 = x\frac{\partial}{\partial x} + \frac{2(1-m)}{\alpha}t\frac{\partial}{\partial t}.$$

The commutator of the Lie symmetries is zero, i.e., $[X_1, X_6] = 0$, and thus, the one-dimensional optimal system consists of

$$\begin{aligned} U_5 &= X_1 = \frac{\partial}{\partial u}, \\ U_{10} &= sX_1 + X_6 = x\frac{\partial}{\partial x} + \frac{2(1-m)}{\alpha}t\frac{\partial}{\partial t} + su\frac{\partial}{\partial u}, \end{aligned}$$

with $s \in \mathbb{R}$. The characteristic equation for U_{10} reads as

$$\frac{dx}{x} = \frac{\alpha dt}{2(1-m)t} = \frac{du}{au},$$

which gives the similarity variable $z = x^{\frac{2(m-1)}{\alpha}} t$ and the invariant solution

$$u = x^s \varphi(z). \tag{16}$$

Substituting (16) into (1) with $c(x) = x^m$, we obtain the following reduced FODE

$$\frac{d^\alpha \varphi}{dz^\alpha} = s(s-1)\varphi + \frac{2(m-1)}{\alpha}\left(\frac{2(m-1)}{\alpha} + 2s - 1\right) \times z\varphi_z + \frac{4(m-1)^2}{\alpha^2} z^2 \varphi_{zz}. \tag{17}$$

4.4. Reduced equations of (1) with $c(x) = k_1 e^{k_2 x}$ (here $k_1, k_2 \in \mathbb{R}$ and $k_1 k_2 \neq 0$)

We assume, without loss of generality, that $c(x) = e^{-\frac{x}{2}}$. Thus, the Lie symmetries become

$$X_1 = u\frac{\partial}{\partial u}, \qquad X_7 = \frac{\partial}{\partial x} + \frac{1}{\alpha} t \frac{\partial}{\partial t}.$$

The commutator of the symmetries is zero, i.e., $[X_1, X_7] = 0$, as in the previous case, the optimal system is

$$\begin{aligned} U_5 &= X_1 = u\frac{\partial}{\partial u} \\ U_{11} &= sX_1 + X_7 = \frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} + su\frac{\partial}{\partial u}, \quad s \in \mathbb{R}. \end{aligned}$$

We obtain the invariant solution $u = e^{sx}\varphi(z)$ with the similarity variable $z = e^{-\frac{1}{\alpha}x} t$ corresponding to U_{11}. Consequently, the reduced FODE is

$$\frac{d^\alpha \varphi}{dz^\alpha} = s^2 \varphi + \frac{1}{\alpha}\left(\frac{1}{\alpha} - 2s\right) z\varphi_z + \frac{1}{\alpha^2} z^2 \varphi_{zz}. \tag{18}$$

4.5. Reduced equations of (1) with $c(x) = ((k_1 x + k_2)^2 + k_3)\exp\left(\int \frac{k_4 dx}{(k_1 x + k_2)^2 + k_3}\right)$ (here $k_1, k_2, k_3, k_4 \in \mathbb{R}$, and $k_1 \neq 0$, $(k_3, k_4) \neq (0, 0)$)

In the case

$$c(x) = \left[(k_1 x + k_2)^2 + k_3\right] \exp\left(\int \frac{k_4 dx}{(k_1 x + k_2)^2 + k_3}\right),$$

the infinitesimal symmetries are

$$\begin{aligned} X_1 &= u\frac{\partial}{\partial u} \text{ and} \\ X &= ((k_1 x + k_2)^2 + k_3)\frac{\partial}{\partial x} - \frac{2k_4}{\alpha} t \frac{\partial}{\partial t} + k_1^2 x u \frac{\partial}{\partial u}. \end{aligned}$$

We obtain the following three subcases of function $c(x)$ through the equivalent transformation (8).

Case A

If $c(x) = (x^2+1)e^{2m \arctan x}$ (here $m \in \mathbb{R}$), then the infinitesimal symmetries become

$$X_1 = u\frac{\partial}{\partial u}, \qquad X_8 = (x^2+1)\frac{\partial}{\partial x} - \frac{4m}{\alpha} t \frac{\partial}{\partial t} + xu\frac{\partial}{\partial u}.$$

The Lie algebra generated by the infinitesimal symmetries in this case is Abelian. Thus, the optimal system is

$$\begin{aligned} U_5 &= X_1 = u\frac{\partial}{\partial u}, \\ U_{12} &= 2sX_1 + X_8 \\ &= (x^2+1)\frac{\partial}{\partial x} - \frac{4m}{\alpha} t \frac{\partial}{\partial t} + (x+2s)u\frac{\partial}{\partial u} \end{aligned}$$

with $s \in \mathbb{R}$. The invariant solution corresponding to U_{12} is

$$u = \sqrt{1+x^2} e^{2s \arctan x} \varphi(z),$$

where $z = e^{\frac{4m}{\alpha} \arctan x} t$ and the reduced equation is

$$\frac{d^\alpha \varphi}{dz^\alpha} = (4s^2+1)\varphi + \frac{16m}{\alpha}\left(\frac{m}{\alpha} + s\right) z\varphi_z + \frac{16m^2}{\alpha^2} z^2 \varphi_{zz}. \tag{19}$$

Case B

If $c(x) = \frac{(1-x)^{m+1}}{(1+x)^{m-1}}$ (here $m \in \mathbb{R}$ and $m \neq \pm 1$), then the infinitesimal symmetries become

$$X_1 = u\frac{\partial}{\partial u}, \qquad X_9 = (x^2-1)\frac{\partial}{\partial x} - \frac{4m}{\alpha} t \frac{\partial}{\partial t} + xu\frac{\partial}{\partial u}.$$

The Lie algebra generated by X_1, X_9 is also Abelian. Thus, the optimal system is

$$\begin{aligned} U_5 &= X_1 = u\frac{\partial}{\partial u}, \\ U_{13} &= 2sX_1 + X_2 \\ &= (x^2-1)\frac{\partial}{\partial x} - \frac{4m}{\alpha} t \frac{\partial}{\partial t} + (x+2s)u\frac{\partial}{\partial u} \end{aligned}$$

with $s \in \mathbb{R}$. Following the characteristic method, we obtain the similarity variable $z = \left(\frac{1-x}{1+x}\right)^{\frac{2m}{\alpha}} t$ and the invariant solution

$$u = \sqrt{1-x^2}\left(\frac{1-x}{1+x}\right)^s \varphi(z).$$

Thus, the reduced equation is

$$\frac{d^\alpha \varphi}{dz^\alpha} = (4s^2-1)\varphi + \frac{16m}{\alpha}\left(\frac{m}{\alpha} + s\right) z\varphi_z + \frac{16m^2}{\alpha^2} z^2 \varphi_{zz}. \tag{20}$$

Case C

If $c(x) = x^2 e^{\frac{1}{x}}$, then infinitesimal symmetries become

$$X_1 = u\frac{\partial}{\partial u}, \qquad X_{10} = x^2\frac{\partial}{\partial x} + \frac{2}{\alpha} t \frac{\partial}{\partial t} + xu\frac{\partial}{\partial u}$$

and $[X_1, X_{10}] = 0$. Thus, the optimal system is

$$\begin{aligned} U_5 &= X_1 = u\frac{\partial}{\partial u}, \\ U_{14} &= -sX_1 + X_{10} \\ &= x^2\frac{\partial}{\partial x} + \frac{2}{\alpha} t \frac{\partial}{\partial t} + (x-s)u\frac{\partial}{\partial u}, \quad s \in \mathbb{R}. \end{aligned}$$

The invariant solution corresponding to U_{14} is

$$u = xe^{\frac{s}{x}}\varphi(z)$$

with similarity variable $z = e^{\frac{2}{\alpha x}}t$. The reduced equation is

$$\frac{d^\alpha \varphi}{dz^\alpha} = s^2\varphi + \frac{4}{\alpha}\left(\frac{1}{\alpha} + s\right)z\varphi_z + \frac{4}{\alpha^2}z^2\varphi_{zz}. \quad (21)$$

We have described the invariant solutions as solutions to the reduced equations corresponding to the infinitesimal symmetries of the optimal systems for different types of functions $c(x)$. The invariant solutions corresponding to any other infinitesimal symmetries can be obtained by symmetry transformations on the invariant solutions that we obtained in this section. To complete the work, we need to give solutions of reduced equations, which lead us to the explicit expressions of the invariant solutions.

5. EXPLICIT EXPRESSIONS OF INVARIANT SOLUTIONS

In this section, we derive solutions to the reduced equations (10)-(15) and (17)-(21). Then using these solutions, we give invariant solutions of (1) explicitly. Notice that all of the reduced equations we obtained in the previous section have the following general form:

$$\frac{d^\alpha \varphi}{dz^\alpha} = a\varphi + \frac{b}{\alpha}z\varphi' + \frac{c}{\alpha^2}z^2\varphi'', \quad (22)$$

where a, b and c are constants. Thus, the problem of finding invariant solutions of (1) is reduced into the problem of finding solutions of the equation given in (22).

5.1. Solutions of the reduced equation (22)

We derive solutions of (22) in terms of generalized Wright functions and Fox H-functions. Therefore, we give definitions of these two special functions in the following manner.

Definition 5.1. *The Fox H-function is defined by means of the Mellin-Barnes type contour integral*

$$H^{m,l}_{p,q}\left[z\left|\begin{array}{c}(A_i,\alpha_i)_{1,p}\\(B_j,\beta_j)_{1,q}\end{array}\right.\right] = \frac{1}{2\pi i}\int_L \mathcal{H}^{m,l}_{p,q}(s)z^s ds \quad (23)$$

with

$$\mathcal{H}^{m,l}_{p,q}(s) = \frac{\prod_{j=1}^{m}\Gamma(B_j-\beta_j s)\prod_{i=1}^{l}\Gamma(1-A_i+\alpha_i s)}{\prod_{i=l+1}^{p}\Gamma(A_i-\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-B_j+\beta_j s)}$$

for $z \in \mathbb{C}\setminus\{0\}$, where $m,l,p,q \in \mathbb{N}_0$, $(m,l) \neq (0,0)$, $\alpha_i,\beta_j \in \mathbb{R}_+$, $A_i,B_j \in \mathbb{R}$ ($i = 1,\ldots,p; j = 1,\ldots,q$). Here L is a suitable contour from $\gamma - i\infty$ to $\gamma + i\infty$, where γ is a real number.

The integral in (23) converges if the following conditions are satisfied [17]

$$\mu = \sum_{i=1}^{l}\alpha_i - \sum_{i=l+1}^{p}\alpha_i + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j > 0$$

and $|\arg z| < \frac{\pi\mu}{2}$. For large z the Fox H-function vanishes as [18]

$$H^{m,0}_{p,q}[z] \approx O\left(\exp\left(-\nu z^{\frac{1}{\nu}}\epsilon^{\frac{1}{\nu}}\right)z^{\frac{2\delta+1}{2\nu}}\right), \quad (24)$$

where $\epsilon = \prod_{i=1}^{p}(\alpha_i)^{\alpha_i}\prod_{j=1}^{q}(\beta_j)^{-\beta_j}$, $\delta = \sum_{j=1}^{q}B_j - \sum_{i=1}^{p}A_i + \frac{p-q}{2}$ and $\nu = \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > 0$.

Definition 5.2. *The generalized Wright function is defined as*

$${}_p\Psi_q\left[z\left|\begin{array}{c}(A_i,\alpha_i)_{1,p}\\(B_j,\beta_j)_{1,q}\end{array}\right.\right] = \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(A_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(B_j+\beta_j k)}\frac{z^k}{k!} \quad (25)$$

for $z \in \mathbb{C}$, $p,q \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, $A_i,B_j \in \mathbb{C}$ and $\alpha_i,\beta_j \in \mathbb{R}\setminus\{0\}$ ($i = 1,\ldots,p; j = 1,\ldots,q$).

If $\Delta = \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > -1$ or $\Delta = -1$, then the series in (25) is absolutely convergent for $z \in \mathbb{C}$ or $|z| < \prod_{i=1}^{p}|\alpha_i|^{-\alpha_i}\prod_{j=1}^{q}|\beta_j|^{\beta_j}$, respectively [19]. Moreover, the Mittag-Leffler, Wright and Gauss hypergeometric functions can be expressed in terms of the generalized Wright functions, respectively, as

$$E_{\alpha,\beta}(z) = {}_1\Psi_1\left[z\left|\begin{array}{c}(1,1)\\(\beta,\alpha)\end{array}\right.\right], \quad (26)$$

$$\Psi(z;\alpha,\beta) = {}_0\Psi_1\left[z\left|\begin{array}{c}-\\(\beta,\alpha)\end{array}\right.\right], \quad (27)$$

$${}_2F_1\left(\begin{array}{c}\alpha,\beta\\\gamma\end{array};z\right) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}{}_2\Psi_1\left[z\left|\begin{array}{c}(\alpha,1),(\beta,1)\\(\gamma,1)\end{array}\right.\right]. \quad (28)$$

In [11], we studied the solutions of (22) in detail. Thus, we deduce the following two lemmas from the results of [11].

Lemma 5.1. *The equation given in (22) has the following solutions:*

1) If $a = 0$, $b = 0$ and $c = 0$ in (22) then

$$\varphi(z) = \sum_{k=1}^{n}c_k z^{\alpha-k} \text{ for } z \in \mathbb{R}.$$

2) If $b = 0$ and $c = 0$ in (22) then

$$\varphi(z) = \sum_{k=1}^{n}c_k z^{\alpha-k}E_{\alpha,1+\alpha-k}(az^\alpha) \text{ for } z \in \mathbb{R}.$$

3) If $b > 0$, $c = 0$ and $0 < \alpha < 1$ in (22) then

$$\varphi(z) = c_1 H^{1,0}_{1,1}\left[\frac{z^{-\alpha}}{b}\left|\begin{array}{c}(1,\alpha)\\(\frac{a}{b},1)\end{array}\right.\right] \text{ for } z \in \mathbb{R}_+.$$

4) If $b \neq 0,\ c = 0$ in (22) then

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k} {}_2\Psi_1 \left[bz^{\alpha} \left| \begin{array}{c} \left(1 - \frac{k}{\alpha} + \frac{a}{b}, 1\right), (1,1) \\ (1+\alpha-k, \alpha) \end{array} \right. \right]$$

for $z \in \mathbb{R}$ when $\alpha > 1$, or for $|z| < \frac{1}{|b|}$ when $\alpha = 1$.

5) If $D = \frac{1}{\alpha^2} - \frac{2b}{\alpha c} + \frac{b^2}{c^2} - \frac{4a}{c} \geq 0,\ c > 0$ and $0 < \alpha < 2$ in (22) then

$$\varphi(z) = c_1 H^{2,0}_{1,2} \left[\frac{z^{-\alpha}}{c} \left| \begin{array}{c} (1,\alpha) \\ (\kappa_1, 1), (\kappa_2, 1) \end{array} \right. \right] \text{ for } z \in \mathbb{R}_+.$$

6) If $D = \frac{1}{\alpha^2} - \frac{2b}{\alpha c} + \frac{b^2}{c^2} - \frac{4a}{c} \geq 0,\ c \neq 0$ in (22) then

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k}$$
$$\times {}_3\Psi_1 \left[cz^{\alpha} \left| \begin{array}{c} \left(1 - \frac{k}{\alpha} + \kappa_1, 1\right), \left(1 - \frac{k}{\alpha} + \kappa_2, 1\right), (1,1) \\ (1+\alpha-k, \alpha) \end{array} \right. \right]$$

for $z \in \mathbb{R}$ when $\alpha > 2$, or for $|z| < \frac{2}{\sqrt{|c|}}$ when $\alpha = 2$. Where n is a natural number satisfying $0 \leq n-1 < \alpha \leq n$, $\kappa_{1,2} = \frac{1}{2}\left(\frac{b}{c} - \frac{1}{\alpha} \pm \sqrt{D}\right)$ and c_k $(k = 1, \ldots, n)$ are constants.

Lemma 5.2. *Let $D = \frac{1}{\alpha^2} - \frac{2b}{\alpha c} + \frac{b^2}{c^2} - \frac{4a}{c} = \frac{1}{4}$ and $c \neq 0$ in (22). The equation given in (22) has the following solutions:*

1) If $c > 0$ and $0 < \alpha < 2$ then

$$\varphi(z) = c_1 z^{\frac{\alpha}{2}\left(\frac{1}{\alpha} - \frac{b}{c} + \frac{1}{2}\right)}$$
$$\times \Psi \left[-\frac{2z^{-\frac{\alpha}{2}}}{\sqrt{c}} \left| -\frac{\alpha}{2}, \frac{1}{2}\left(\frac{3}{\alpha} - \frac{b}{c} + \frac{1}{2}\right) \right. \right]$$

for $z \in \mathbb{R}_+$.

2) If $\alpha \geq 2$ then

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k}{}_2$$
$$\times \Psi_1 \left[\frac{cz^{\alpha}}{4} \left| \begin{array}{c} \left(\frac{3}{2} - \frac{2k+1}{\alpha} + \frac{b}{c}, 2\right), (1,1) \\ (1+\alpha-k, \alpha) \end{array} \right. \right]$$

for $z \in \mathbb{R}$ when $\alpha > 2$, or for $|z| < \frac{2}{\sqrt{|c|}}$ when $\alpha = 2$. Here n is a natural number satisfying $0 \leq n-1 < \alpha \leq n$ and c_k $(k = 1, \ldots, n)$ are constants.

Knowing the solutions of (22), we give explicit invariant solutions of (1) for different types of function $c(x)$ and show that these solutions coincide with the known solutions for particular cases of $\alpha = 1$ and $\alpha = 2$.

5.2. *Solutions of (1) with $c(x) = 1$*

We give invariant solutions of (1) with $c(x) = 1$ as follows:

1) Since $D = \frac{1}{4}$ in (10), using Lemma 5.2 we have the following solutions to (10)

$$\varphi(z) = c_1 z^{\frac{s\alpha}{2}} \Psi \left(-z^{-\frac{\alpha}{2}}; -\frac{\alpha}{2}, 1 + \frac{s\alpha}{2} \right)$$

for $0 < \alpha < 2$ and

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k} {}_2\Psi_1 \left[z^{\alpha} \left| \begin{array}{c} \left(2 - \frac{2k}{\alpha} - s, 2\right), (1,1) \\ (1+\alpha-k, \alpha) \end{array} \right. \right]$$

for $\alpha \geq 2$. We obtain through (9) the following invariant solutions of (1) with $c(x) = 1$ correspond to U_1

$$u(x,t) = c_1 t^{\frac{s\alpha}{2}} \Psi \left(-xt^{-\frac{\alpha}{2}}; -\frac{\alpha}{2}, 1 + \frac{s\alpha}{2} \right) \quad (29)$$

for $0 < \alpha < 2$ and

$$u(x,t) = \sum_{k=1}^{n} c_k x^{s-2+\frac{2k}{\alpha}} t^{\alpha-k}$$
$$\times {}_2\Psi_1 \left[\frac{t^{\alpha}}{x^2} \left| \begin{array}{c} \left(2 - \frac{2k}{\alpha} - s, 2\right), (1,1) \\ (1+\alpha-k, \alpha) \end{array} \right. \right] \quad (30)$$

for $\alpha \geq 2$. E. Buckwar and Yu. Luchko [8] studied the diffusion-wave equation (1) with $c(x) = 1$ and found solutions via scaling transformations. The solution (29) and (30) coincide with those solutions obtained in [8].

Moreover, if we set $\alpha = 1$, $c_1 = \frac{1}{2}$ and $s = -1$ in (29), then it becomes

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{t}} \sum_{k=0}^{\infty} \frac{(-xt^{-\frac{1}{2}})^k}{k!\Gamma\left(\frac{1}{2} - \frac{k}{2}\right)} \\ &= \frac{1}{2\sqrt{t}} \sum_{k=0}^{\infty} \frac{\left(-xt^{-\frac{1}{2}}\right)^{2k}}{(2k)!\Gamma\left(\frac{1}{2} - k\right)}. \end{aligned}$$

Substituting $\Gamma\left(\frac{1}{2} - k\right) = \frac{(-4)^k k! \sqrt{\pi}}{(2k)!}$ into the above solution, we arrive to the fundamental solution of the diffusion equation

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$$

If we set $\alpha = 1$, $c_1 = \frac{1}{2}$ and $s = 0$ in (29), then it becomes

$$u(x,t) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-xt^{-\frac{1}{2}})^k}{k!\Gamma\left(1 - \frac{k}{2}\right)}.$$

Analogously to the previous computation, the above solution equals to

$$u(x,t) = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right),$$

where $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \frac{z^{2k+1}}{k!}$.

If we set $\alpha = 2$ in (30), then it becomes

$$u(x,t) = c_1 x^{s-1} t {}_2\Psi_1 \left[x^{-2}t^2 \left| \begin{array}{c} (1-s, 2), (1,1) \\ (2,2) \end{array} \right. \right]$$
$$+ c_2 x^s {}_2\Psi_1 \left[x^{-2}t^2 \left| \begin{array}{c} (-s, 2), (1,1) \\ (1,2) \end{array} \right. \right].$$

Applying the formulas

$$z {}_2\Psi_1 \left[z^2 \left| \begin{array}{c} (A,2), (1,1) \\ (2,2) \end{array} \right. \right] = \frac{\Gamma(A-1)}{2}$$
$$\times \left[(1-z)^{1-A} - (1+z)^{1-A} \right], \quad (31)$$

$$ {}_2\Psi_1\left[z^2\left|\begin{matrix}(A,2),(1,1)\\(1,2)\end{matrix}\right.\right]=\frac{\Gamma(A)}{2}\times\left[(1-z)^{-A}+(1+z)^{-A}\right] \quad (32)$$

to the above solution, it becomes

$$u(x,t)=\tilde{c}_1(x-t)^s+\tilde{c}_2(x+t)^s,$$

where $\tilde{c}_1=\frac{\Gamma(-s)(c_1+c_2)}{2}$ and $\tilde{c}_2=-\frac{\Gamma(-s)(c_1-c_2)}{2}$.

2) Using the second assertion of Lemma 5.1, we get the following solutions to (11)

$$\varphi(t)=\sum_{k=1}^{n}c_k t^{\alpha-k}E_{\alpha,1+\alpha-k}(t^\alpha).$$

So invariant solutions of (1) with $c(x)=1$ corresponding to U_2 and U_3 are, respectively,

$$u(x,t) = e^x\sum_{k=1}^{n}c_k t^{\alpha-k}E_{\alpha,1+\alpha-k}(t^\alpha), \quad (33)$$

$$u(x,t) = e^{-x}\sum_{k=1}^{n}c_k t^{\alpha-k}E_{\alpha,1+\alpha-k}(t^\alpha). \quad (34)$$

If we set $\alpha=1$ in (33) and (34), we get the following solutions, respectively,

$$u(x,t)=c_1e^{x+t} \text{ and } u(x,t)=c_1e^{-x+t}.$$

When $\alpha=2$ in (33) and (34), by (4.2.2) of [20], we have the following traveling wave solutions, respectively,

$$u(x,t)=\frac{c_1+c_2}{2}e^{x+t}-\frac{c_1-c_2}{2}e^{x-t} \text{ and}$$

$$u(x,t)=\frac{c_1+c_2}{2}e^{-x+t}-\frac{c_1-c_2}{2}e^{-x-t}.$$

3) Using the first assertion of Lemma 5.1, we have the following solutions of (1) with $c(x)=1$ corresponding to U_4,

$$u(x,t)=\varphi(t)=\sum_{k=1}^{n}c_k t^{\alpha-k}. \quad (35)$$

If we take $\alpha=1$ and $\alpha=2$ in (35), then we get the following solutions, respectively,

$$u(x,t)=c_1 \text{ and } u(x,t)=c_1t+c_2.$$

5.3. *Solutions of (1) with* $c(x)=x^2$

Similarly, the invariant solutions of (1) with $c(x)=x^2$ are obtained as follows:

1) Since $D=\frac{1}{4}$ in the equation (13), using Lemma 5.2 we get solutions of (13)

$$\varphi(z)=c_1z^{\frac{\alpha}{2}(1-s)}\Psi\left(-z^{-\frac{\alpha}{2}};-\frac{\alpha}{2},1+\frac{\alpha}{2}(1-s)\right)$$

for $0<\alpha<2$ and

$$\varphi(z)=\sum_{k=1}^{n}c_kz^{\alpha-k}{}_2\Psi_1\left[z^\alpha\left|\begin{matrix}\left(1-\frac{2k}{\alpha}+s,2\right),(1,1)\\(1+\alpha-k,\alpha)\end{matrix}\right.\right]$$

for $\alpha\geq 2$. Then invariant solutions of (1) with $c(x)=x^2$ corresponding to U_6 are

$$u(x,t)=c_1xt^{\frac{\alpha}{2}(1-s)}\Psi\left(-x^{-1}t^{-\frac{\alpha}{2}};-\frac{\alpha}{2},1+\frac{\alpha}{2}(1-s)\right) \quad (36)$$

for $0<\alpha<2$ and

$$u(x,t)=\sum_{k=1}^{n}c_kx^{s+2-\frac{2k}{\alpha}}t^{\alpha-k}\times{}_2\Psi_1\left[x^2t^\alpha\left|\begin{matrix}\left(1-\frac{2k}{\alpha}+s,2\right),(1,1)\\(1+\alpha-k,\alpha)\end{matrix}\right.\right] \quad (37)$$

for $\alpha\geq 2$. When $\alpha=1$, if we take $s=2$ and $s=1$ in (36), we get the following solutions, respectively,

$$u(x,t) = c_1\frac{x}{\sqrt{\pi t}}\exp\left(-\frac{1}{4x^2t}\right) \text{ and}$$

$$u(x,t) = c_1x\left[1-erf\left(\frac{1}{2x\sqrt{t}}\right)\right].$$

By setting $\alpha=2$ in (37), it becomes

$$u(x,t)=c_1x^{s+1}t\,{}_2\Psi_1\left[x^2t^2\left|\begin{matrix}(s,2),(1,1)\\(2,2)\end{matrix}\right.\right]+c_2x^s{}_2\Psi_1\left[x^2t^2\left|\begin{matrix}(s-1,2),(1,1)\\(1,2)\end{matrix}\right.\right].$$

We can rewrite it using (31) and (32) as

$$u(x,t)=\frac{\Gamma(s-1)}{2}(c_1+c_2)x^s(1-xt)^{1-s}-\frac{\Gamma(s-1)}{2}(c_1-c_2)x^s(1+xt)^{1-s}.$$

2) Using the second assertion of Lemma 5.1, we have the following solutions to (14)

$$\varphi(t)=\sum_{k=1}^{n}c_k t^{\alpha-k}E_{\alpha,1+\alpha-k}(t^\alpha).$$

So invariant solutions of (1) with $c(x)=x^2$ corresponding to U_7 and U_8 are, respectively,

$$u(x,t) = \sum_{k=1}^{n}c_kxe^{-\frac{1}{x}}t^{\alpha-k}E_{\alpha,1+\alpha-k}(t^\alpha), \quad (38)$$

$$u(x,t) = \sum_{k=1}^{n}c_kxe^{\frac{1}{x}}t^{\alpha-k}E_{\alpha,1+\alpha-k}(t^\alpha). \quad (39)$$

By setting $\alpha=1$ in (38) and (39), we get the following solutions, respectively,

$$u(x,t) = c_1xe^{t-\frac{1}{x}} \text{ and}$$

$$u(x,t) = c_1xe^{t+\frac{1}{x}}.$$

When $\alpha=2$ in (38) and (39), by virtue of formula (4.2.2) of [20], we get the following solutions, respectively,

$$u(x,t) = \frac{c_1+c_2}{2}xe^{t-\frac{1}{x}}-\frac{c_1-c_2}{2}xe^{-t-\frac{1}{x}} \text{ and}$$

$$u(x,t) = \frac{c_1+c_2}{2}xe^{t+\frac{1}{x}}-\frac{c_1-c_2}{2}xe^{-t+\frac{1}{x}}.$$

3) Using the first assertion of Lemma 5.1, we have the following invariant solution of (1) with $c(x) = x^2$ corresponding to U_9

$$u(x,t) = x\varphi(t) = \sum_{k=1}^{n} c_k x t^{\alpha-k}. \qquad (40)$$

If we set $\alpha = 1$ and $\alpha = 2$ in (40), we get the following solutions, respectively,

$$u(x,t) = c_1 x \text{ and } u(x,t) = c_1 xt + c_2 x.$$

5.4. Solutions of (1) with $c(x) = x^m$ (here $m \in \mathbb{R}$ and $m \neq 0, 2$)

We give invariant solutions of (1) with $c(x) = x^m$ as follows:

1) If $m = 1$, then the reduced equation given in (17) becomes

$$\frac{d^\alpha \varphi}{dz^\alpha} = s(s-1)\varphi.$$

Using the second assertion of Lemma 5.1, we obtain the following solutions

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k} E_{\alpha,1+\alpha-k}(s(s-1)z^\alpha) \text{ for } \alpha > 0.$$

So, invariant solutions of (1) with $c(x) = x$ corresponding to U_{10} is obtained through (16) as

$$u(x,t) = \sum_{k=1}^{n} c_k x^s t^{\alpha-k} E_{\alpha,1+\alpha-k}(s(s-1)t^\alpha). \quad (41)$$

By setting $\alpha = 1$ in (41), we get

$$u(x,t) = c_1 x^s e^{s(s-1)t}.$$

If $\alpha = 2$ and $s(s-1) > 0$, then (41) becomes

$$u(x,t) = \left(\frac{c_1}{2\sqrt{s(s-1)}} + \frac{c_2}{2}\right) x^s e^{\sqrt{s(s-1)}t} + \left(-\frac{c_1}{2\sqrt{s(s-1)}} + \frac{c_2}{2}\right) x^s e^{-\sqrt{s(s-1)}t}$$

by virtue of (4.2.2) of [20]. This is the exact solution (2.60) of [1].

2) If $m \neq 1$, using the fifth and sixth assertions of Lemma 5.1, we have the following solutions of (17) for $0 < \alpha < 2$

$$\varphi(z) = c_1 H_{1,2}^{2,0}\left[\frac{z^{-\alpha}}{A^2}\middle|\begin{array}{c}(1,\alpha)\\ \left(\frac{s}{A},1\right),\left(\frac{s-1}{A},1\right)\end{array}\right]$$

and for $\alpha \geq 2$

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k} \times {}_3\Psi_1\left[A^2 z^\alpha \middle|\begin{array}{c}\left(B-\frac{k}{\alpha},1\right),\left(C-\frac{k}{\alpha},1\right),(1,1)\\ (1+\alpha-k,\alpha)\end{array}\right],$$

where $A = 2(m-1)$, $B = 1 + \frac{s}{2(m-1)}$ and $C = 1 + \frac{s-1}{2(m-1)}$. Then, by virtue of (16), invariant solutions of (1) with $c(x) = x^m$ corresponding to U_{10} are for $0 < \alpha < 2$

$$u(x,t) = c_1 x^s H_{1,2}^{2,0}\left[\frac{x^{2(1-m)}}{A^2 t^\alpha}\middle|\begin{array}{c}(1,\alpha)\\ \left(\frac{s}{A},1\right),\left(\frac{s-1}{A},1\right)\end{array}\right] \quad (42)$$

and for $\alpha \geq 2$

$$u(x,t) = x^{s+2(m-1)} t^\alpha \sum_{k=1}^{n} c_k \left(\frac{x^{\frac{2(1-m)}{\alpha}}}{t}\right)^k \times {}_3\Psi_1\left[A^2 x^A t^\alpha \middle|\begin{array}{c}\left(B-\frac{k}{\alpha},1\right),\left(C-\frac{k}{\alpha},1\right),(1,1)\\ (1+\alpha-k,\alpha)\end{array}\right]. \quad (43)$$

We include graphical illustrations of the solution (42) with $c_1 = 1$, $s = 2m-1$ and $m = \frac{1}{4}$. From the plots we can see that the solution behaviour changes visibly when α steps over 1.

R. Metzler et.al. [10] studied the anomalous diffusion equation (1) with $c(x) = x^m$ and found solutions for $0 < \alpha < 1$ using the Laplace transformations. The solution (42) coincides with the solutions obtained in [10].

Furthermore, if $\alpha = 1$ then by setting $s = 2m-1$ and $s = 2m-2$ in (42), we obtain through (1.125) of [17] the following solutions, respectively,

$$u(x,t) = (2|m-1|)^{-2-\frac{1}{m-1}} c_1 t^{-1-\frac{1}{2(m-1)}} \times \exp\left(-\frac{x^{2(1-m)}}{4(m-1)^2 t}\right)$$

and

$$u(x,t) = (2|m-1|)^{-2+\frac{1}{m-1}} c_1 x t^{-1+\frac{1}{2(m-1)}} \times \exp\left(-\frac{x^{2(1-m)}}{4(m-1)^2 t}\right).$$

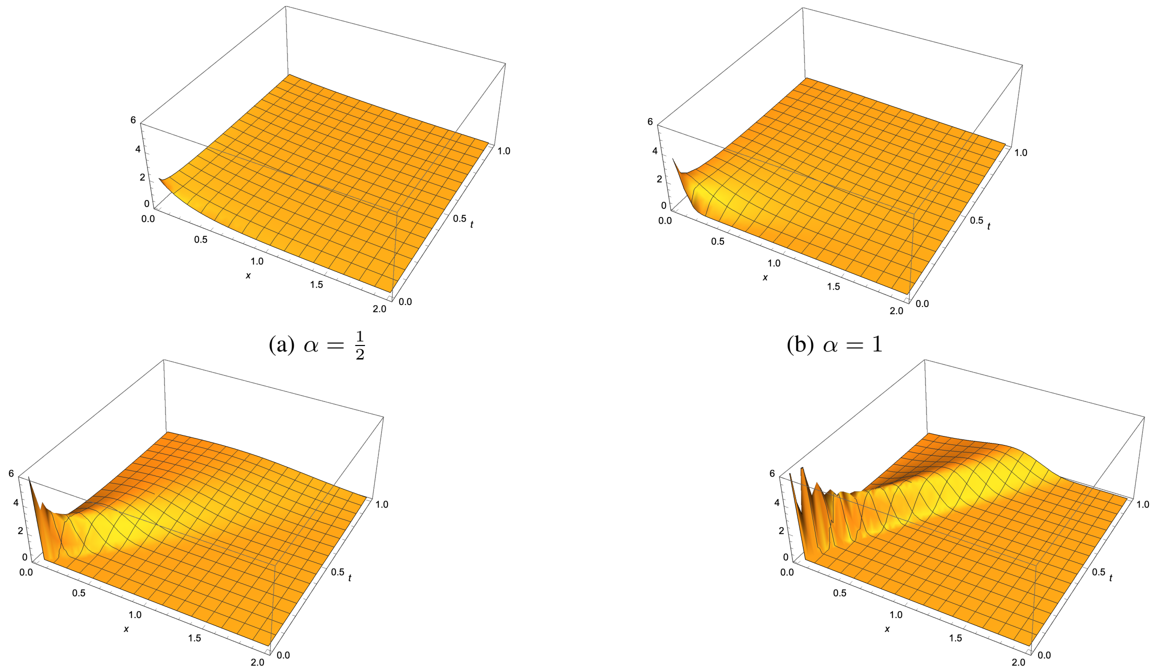
If we set $\alpha = 2$ in (43), then it becomes

$$u(x,t) = c_1 x^s z \times {}_3\Psi_1\left[A^2 z^2 \middle|\begin{array}{c}\left(\frac{1}{2}+\frac{s}{A},1\right),\left(\frac{1}{2}+\frac{s-1}{A},1\right),(1,1)\\ (2,2)\end{array}\right] + c_2 x^s {}_3\Psi_1\left[A^2 z^2 \middle|\begin{array}{c}\left(\frac{s}{A},1\right),\left(\frac{s-1}{A},1\right),(1,1)\\ (1,2)\end{array}\right],$$

where $A = 2(m-1)$ and $z = x^{m-1}t$. By virtue of the following formulas

$${}_3\Psi_1\left[z\middle|\begin{array}{c}(A_1,1),(A_2,1),(1,1)\\ (1,2)\end{array}\right] = \Gamma(A_1)\Gamma(A_2) \times {}_2F_1\left(\begin{array}{c}A_1,A_2\\ \frac{1}{2}\end{array};\frac{z}{4}\right) \text{ for } |z| < 4, \quad (44)$$

$${}_3\Psi_1\left[z\middle|\begin{array}{c}(A_1,1),(A_2,1),(1,1)\\ (2,2)\end{array}\right] = \Gamma(A_1)\Gamma(A_2) \times {}_2F_1\left(\begin{array}{c}A_1,A_2\\ \frac{3}{2}\end{array};\frac{z}{4}\right) \text{ for } |z| < 4, \quad (45)$$



(a) $\alpha = \frac{1}{2}$ (b) $\alpha = 1$

(c) $\alpha = \frac{3}{2}$ (d) $\alpha = \frac{7}{4}$

Figure 1: Solution plots of (1) with $c(x) = x^{\frac{1}{4}}$

the above solution equals to

$$u(x,t) = \Gamma\left(\frac{1}{2}+\omega_1\right)\Gamma\left(\frac{1}{2}+\omega_2\right)c_1 x^s z \times {}_2F_1\left(\begin{matrix}\frac{1}{2}+\omega_1, \frac{1}{2}+\omega_2\\ \frac{3}{2}\end{matrix}; (m-1)^2z^2\right) + \Gamma(\omega_1)\Gamma(\omega_2)c_2 x^s {}_2F_1\left(\begin{matrix}\omega_1,\omega_2\\ \frac{1}{2}\end{matrix}; (m-1)^2z^2\right), \tag{46}$$

where $\omega_1 = \frac{s}{2(m-1)}$, $\omega_2 = \frac{s-1}{2(m-1)}$. We recall the formulae (37) and (38) in [2]

$$\frac{2\Gamma\left(\frac{1}{2}\right)\Gamma\left(a+b+\frac{1}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)}{}_2F_1\left(\begin{matrix}a,b\\ \frac{1}{2}\end{matrix}; z^2\right) = \left[1+\frac{\Gamma\left(a+b+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-a-b\right)}{\Gamma\left(a-b+\frac{1}{2}\right)\Gamma\left(b-a+\frac{1}{2}\right)}\right] \times {}_2F_1\left(\begin{matrix}2a, 2b\\ a+b+\frac{1}{2}\end{matrix}; \frac{1+z}{2}\right) + \frac{\Gamma\left(a+b-\frac{1}{2}\right)\Gamma\left(a+b+\frac{1}{2}\right)}{\Gamma(2a)\Gamma(2b)}\left(\frac{1+z}{2}\right)^{\frac{1}{2}-a-b} \times {}_2F_1\left(\begin{matrix}a-b+\frac{1}{2}, b-a+\frac{1}{2}\\ \frac{3}{2}-a-b\end{matrix}; \frac{1+z}{2}\right) \tag{47}$$

and

$$\frac{2\Gamma\left(-\frac{1}{2}\right)\Gamma\left(a+b-\frac{1}{2}\right)}{\Gamma\left(a-\frac{1}{2}\right)\Gamma\left(b-\frac{1}{2}\right)} z\, {}_2F_1\left(\begin{matrix}a,b\\ \frac{3}{2}\end{matrix}; z^2\right) = \left[\frac{\Gamma\left(a+b-\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-a-b\right)}{\Gamma\left(a-b+\frac{1}{2}\right)\Gamma\left(b-a+\frac{1}{2}\right)}-1\right] \times {}_2F_1\left(\begin{matrix}2a-1, 2b-1\\ a+b-\frac{1}{2}\end{matrix}; \frac{1+z}{2}\right) + \frac{\Gamma\left(a+b-\frac{3}{2}\right)\Gamma\left(a+b-\frac{1}{2}\right)}{\Gamma(2a-1)\Gamma(2b-1)}\left(\frac{1+z}{2}\right)^{\frac{3}{2}-a-b} \times {}_2F_1\left(\begin{matrix}a-b+\frac{1}{2}, b-a+\frac{1}{2}\\ \frac{5}{2}-a-b\end{matrix}; \frac{1+z}{2}\right). \tag{48}$$

Using (47) and (48), we can rewrite the solution (46) as

$$u(x,t) = \tilde{c}_1 x^s F_1(x^{m-1}t) + \tilde{c}_2 x^s F_2(x^{m-1}t),$$

where

$$\tilde{c}_1 = 4^{-(\omega_1+\omega_2)}\sqrt{\pi}\Gamma(2\omega_1)\Gamma(2\omega_2)\left(\frac{c_1+2c_2}{\Gamma(\omega_1+\omega_2+\frac{1}{2})} - \frac{\Gamma(\frac{1}{2}-\omega_1-\omega_2)(c_1-2c_2)}{\Gamma(\omega_1-\omega_2+\frac{1}{2})\Gamma(\omega_2-\omega_1+\frac{1}{2})}\right),$$

$$\tilde{c}_2 = -4^{-(\omega_1+\omega_2)}\sqrt{\pi}\Gamma\left(\omega_1+\omega_2-\frac{1}{2}\right)(c_1-2c_2),$$

$$F_1(z) = {}_2F_1\left(\begin{matrix}2\omega_1, 2\omega_2\\ \omega_1+\omega_2+\frac{1}{2}\end{matrix}; \frac{1+(m-1)z}{2}\right),$$

$$F_2(z) = \left(\frac{1+(m-1)z}{2}\right)^{\frac{1}{2}-\omega_1-\omega_2} \times {}_2F_1\left(\begin{matrix}\omega_1-\omega_2+\frac{1}{2}, \omega_2-\omega_1+\frac{1}{2}\\ \frac{3}{2}-\omega_1-\omega_2\end{matrix}; \frac{1+(m-1)z}{2}\right),$$

$$\omega_1 = \frac{s}{2(m-1)}, \qquad \omega_2 = \frac{s-1}{2(m-1)}.$$

This is the exact solution (2.47) of [1].

5.5. Solutions of (1) with $c(x) = e^{-\frac{x}{2}}$

Using the fifth and sixth assertions of Lemma 5.1, we get the following solutions of (18)

$$\varphi(z) = c_1 H_{1,2}^{2,0}\left[z^{-\alpha}\middle|\begin{array}{c}(1,\alpha)\\(-s,1),(-s,1)\end{array}\right]$$

for $0 < \alpha < 2$ and

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k}$$
$$\times {}_3\Psi_1\left[z^{\alpha}\middle|\begin{array}{c}\left(1-\frac{k}{\alpha}-s,1\right),\left(1-\frac{k}{\alpha}-s,1\right),(1,1)\\(1+\alpha-k,\alpha)\end{array}\right]$$

for $\alpha \geq 2$. Then, invariant solutions of (1) with $c(x) = e^{-\frac{x}{2}}$ corresponding to U_{11} are for $0 < \alpha < 2$

$$u(x,t) = c_1 e^{sx} H_{1,2}^{2,0}\left[\frac{e^x}{t^{\alpha}}\middle|\begin{array}{c}(1,\alpha)\\(-s,1),(-s,1)\end{array}\right] \quad (49)$$

and for $\alpha \geq 2$

$$u(x,t) = e^{(s-1)x} t^{\alpha} \sum_{k=1}^{n} c_k \left(\frac{e^{\frac{x}{\alpha}}}{t}\right)^k$$
$$\times {}_3\Psi_1\left[\frac{t^{\alpha}}{e^x}\middle|\begin{array}{c}\left(1-\frac{k}{\alpha}-s,1\right),\left(1-\frac{k}{\alpha}-s,1\right),(1,1)\\(1+\alpha-k,\alpha)\end{array}\right]. \quad (50)$$

By setting $\alpha = 1$, $s = -1$ in (49), we get the following solution through (1.125) of [17]

$$u(x,t) = c_1 \frac{1}{t}\exp\left(-\frac{e^x}{t}\right).$$

By setting $\alpha = 2$ in (50), we obtain

$$u(x,t) = c_1 e^{sx} z$$
$$\times {}_3\Psi_1\left[z^2\middle|\begin{array}{c}\left(\frac{1}{2}-s,1\right),\left(\frac{1}{2}-s,1\right),(1,1)\\(2,2)\end{array}\right]$$
$$+ c_2 e^{sx} {}_3\Psi_1\left[z^2\middle|\begin{array}{c}(-s,1),(-s,1),(1,1)\\(1,2)\end{array}\right],$$

where $z = e^{-\frac{x}{2}}t$. Applying (44) and (45) into the above solution, it becomes

$$u(x,t) = \Gamma\left(\frac{1}{2}-s\right)^2 c_1 e^{sx} z\, {}_2F_1\left(\begin{array}{c}\frac{1}{2}-s,\frac{1}{2}-s\\\frac{3}{2}\end{array};\frac{z^2}{4}\right)$$
$$+\Gamma(-s)^2 c_2 e^{sx} {}_2F_1\left(\begin{array}{c}-s,-s\\\frac{1}{2}\end{array};\frac{z^2}{4}\right). \quad (51)$$

Substituting (47) and (48) into (51), we derive

$$u(x,t) = \tilde{c}_1 e^{sx} F_1\left(e^{-\frac{x}{2}}t\right) + \tilde{c}_2 e^{sx} F_2\left(e^{-\frac{x}{2}}t\right),$$

where

$$\begin{aligned}\tilde{c}_1 &= 2^{4s+1}\sqrt{\pi}\Gamma(-2s)^2 \\ &\quad\times\left(\frac{c_1+c_2}{\Gamma\left(\frac{1}{2}-2s\right)} - \frac{\Gamma\left(\frac{1}{2}+2s\right)(c_1-c_2)}{\pi}\right),\\ \tilde{c}_2 &= -2^{4s+1}\sqrt{\pi}\Gamma\left(-\frac{1}{2}-2s\right)(c_1-c_2),\\ F_1(z) &= {}_2F_1\left(\begin{array}{c}-2s,-2s\\\frac{1}{2}-2s\end{array};\frac{2+z}{4}\right),\\ F_2(z) &= \left(\frac{2+z}{4}\right)^{\frac{1}{2}+2s} {}_2F_1\left(\begin{array}{c}\frac{1}{2},\frac{1}{2}\\\frac{3}{2}+2s\end{array};\frac{2+z}{4}\right).\end{aligned}$$

This is the exact solution (2.71) of [1].

5.6. Solutions of (1) with $c(x) = (x^2+1)e^{2m\arctan x}$ (here $m \in \mathbb{R}$)

In the special case of $m = 0$, the reduced equation given in (19) becomes

$$\frac{d^{\alpha}\varphi}{dz^{\alpha}} = (4s^2+1)\varphi.$$

Solutions of which are given by the second assertion of Lemma 5.1

$$\varphi(z) = \sum_{k=1}^{n} c_k z^{\alpha-k} E_{\alpha,1+\alpha-k}\left((4s^2+1)z^{\alpha}\right).$$

Thus, invariant solutions of (1) with $c(x) = x^2+1$ corresponding to U_{12} are

$$u(x,t) = \sqrt{x^2+1}e^{2s\arctan x}$$
$$\times\sum_{k=1}^{n} c_k t^{\alpha-k} E_{\alpha,1+\alpha-k}\left((4s^2+1)t^{\alpha}\right). \quad (52)$$

By taking $\alpha = 1$ in (52), we get

$$u(x,t) = c_1\sqrt{x^2+1}e^{2s\arctan x+(4s^2+1)t}.$$

If we set $\alpha = 2$ in (52), then we get the following solution by virtue of (4.2.2) of [20]

$$u(x,t) = \sqrt{(x^2+1)}e^{2s\arctan x}$$
$$\times\left(\tilde{c}_1 e^{\sqrt{4s^2+1}t} + \tilde{c}_2 e^{-\sqrt{4s^2+1}t}\right),$$

where $\tilde{c}_1 = \frac{c_1}{2\sqrt{4s^2+1}} + \frac{c_2}{2}$, $\tilde{c}_2 = -\frac{c_1}{2\sqrt{4s^2+1}} + \frac{c_2}{2}$. This is the exact solution (2.89) of [1].

If $m \neq 0$, then the discriminant of (19) is less than zero and we cannot apply the lemmas to give a solution.

5.7. Solutions of (1) with $c(x) = \frac{(1-x)^{m+1}}{(1+x)^{m-1}}$ (here $m \in \mathbb{R}$ and $m \neq \pm 1$)

1. If $m \neq 0$, we have the following solutions of (20)

$$\varphi(z) = c_1 H_{1,2}^{2,0}\left[\frac{z^{-\alpha}}{16m^2}\middle|\begin{array}{c}(1,\alpha)\\\left(\frac{2s+1}{4m},1\right),\left(\frac{2s-1}{4m},1\right)\end{array}\right]$$

for $0<\alpha<2$ and

$$\varphi(z)=\sum_{k=1}^{n} c_k z^{\alpha-k} \times {}_3\Psi_1\left[16m^2 z^{\alpha}\left|\begin{array}{c}\left(B-\frac{k}{\alpha},1\right),\left(C-\frac{k}{\alpha},1\right),(1,1)\\(1+\alpha-k,\alpha)\end{array}\right.\right]$$

for $\alpha\geq 2$, where $B=1+\frac{2s+1}{4m}$, $C=1+\frac{2s-1}{4m}$. Then, when $0<x<1$, the invariant solutions of (1) with $c(x)=\frac{(1-x)^{m+1}}{(1+x)^{m-1}}$, $m\neq 0$, corresponding to U_{13} are for $0<\alpha<2$

$$u(x,t)=c_1\sqrt{1-x^2}g(x)^s \times H^{2,0}_{1,2}\left[\frac{1}{16m^2 g(x)^{2m}t^{\alpha}}\left|\begin{array}{c}(1,\alpha)\\\left(\frac{2s+1}{4m},1\right),\left(\frac{2s-1}{4m},1\right)\end{array}\right.\right] \tag{53}$$

and for $\alpha\geq 2$

$$u(x,t)=\sqrt{1-x^2}g(x)^{s+2m}t^{\alpha}\sum_{k=1}^{n}c_k\left(\frac{1}{g(x)^{\frac{2m}{\alpha}}t}\right)^k \times {}_3\Psi_1\left[\frac{16m^2t^{\alpha}}{g(x)^{-2m}}\left|\begin{array}{c}\left(B-\frac{k}{\alpha},1\right),\left(C-\frac{k}{\alpha},1\right),(1,1)\\(1+\alpha-k,\alpha)\end{array}\right.\right]. \tag{54}$$

where $g(x)=\frac{1-x}{1+x}$. By setting $\alpha=1$ in (53), we get

$$u(x,t)=c_1\sqrt{1-x^2}g(x)^s \times H^{2,0}_{1,2}\left[\frac{1}{16m^2 g(x)^{2m}t}\left|\begin{array}{c}(1,1)\\\left(\frac{2s+1}{4m},1\right),\left(\frac{2s-1}{4m},1\right)\end{array}\right.\right]. \tag{55}$$

If we take $s=2m-\frac{1}{2}$ and $s=2m+\frac{1}{2}$ in (55), by (1.125) of [17], it becomes, respectively,

$$u(x,t)=c_1(4m)^{\frac{1}{m}-2}(1-x)t^{\frac{1}{2m}-1} \times\exp\left(-\frac{1}{16m^2}\left(\frac{1+x}{1-x}\right)^{2m}\frac{1}{t}\right)$$

and

$$u(x,t)=c_1(4m)^{-\frac{1}{m}-2}(1+x)t^{-\frac{1}{2m}-1} \times\exp\left(-\frac{1}{16m^2}\left(\frac{1+x}{1-x}\right)^{2m}\frac{1}{t}\right).$$

If we take $\alpha=2$ in (54), by (44), (45), (47) and (48), it becomes

$$u(x,t)=\tilde{c}_1\sqrt{1-x^2}\left(\frac{1-x}{1+x}\right)^s F_1\left(\left(\frac{1-x}{1+x}\right)^m t\right) + \tilde{c}_2\sqrt{1-x^2}\left(\frac{1-x}{1+x}\right)^s F_2\left(\left(\frac{1-x}{1+x}\right)^m t\right),$$

where

$$\tilde{c}_1=\frac{\sqrt{\pi}\Gamma(2\omega_1)\Gamma(2\omega_2)}{2^{2(\omega_1+\omega_2)}}\left[\frac{c_1+2c_2}{\Gamma\left(\omega_1+\omega_2+\frac{1}{2}\right)} -\frac{\Gamma\left(\frac{1}{2}-\omega_1-\omega_2\right)(c_1-2c_2)}{\Gamma\left(\omega_1-\omega_2+\frac{1}{2}\right)\Gamma\left(\omega_2-\omega_1+\frac{1}{2}\right)}\right],$$

$$\tilde{c}_2=-2^{-2(\omega_1+\omega_2)}\sqrt{\pi}\Gamma\left(\omega_1+\omega_2-\frac{1}{2}\right)(c_1-2c_2),$$

$$F_1(z) = {}_2F_1\left(\begin{array}{c}2\omega_1,2\omega_2\\\omega_1+\omega_2+\frac{1}{2}\end{array};\frac{1+2mz}{2}\right),$$

$$F_2(z)=\left(\frac{1+2mz}{2}\right)^{\frac{1}{2}-\omega_1-\omega_2} \times {}_2F_1\left(\begin{array}{c}\omega_1-\omega_2+\frac{1}{2},\omega_2-\omega_1+\frac{1}{2}\\\frac{3}{2}-\omega_1-\omega_2\end{array};\frac{1+2mz}{2}\right),$$

$$\omega_1=\frac{2s+1}{4m}\text{ and }\omega_2=\frac{2s-1}{4m}.$$

This is the exact solution (2.104) of [1].

2. If $m=0$, then the reduced equation (20) becomes

$$\frac{d^{\alpha}\varphi}{dz^{\alpha}}=(4s^2-1)\varphi.$$

Solutions of which are given by the second assertion of Lemma 5.1

$$\varphi(z)=\sum_{k=1}^{n}c_k z^{\alpha-k}E_{\alpha,1+\alpha-k}\left((4s^2-1)z^{\alpha}\right).$$

Thus, for $0<x<1$, invariant solutions of (1) with $c(x)=1-x^2$ corresponding to U_{13} are

$$u(x,t)=\sqrt{1-x^2}\left(\frac{1-x}{1+x}\right)^s t^{\alpha} \times\sum_{k=1}^{n}c_k\frac{1}{t^k}E_{\alpha,1+\alpha-k}\left((4s^2-1)t^{\alpha}\right). \tag{56}$$

If we take $\alpha=1$ in (56), we get

$$u(x,t)=c_1\sqrt{1-x^2}\left(\frac{1-x}{1+x}\right)^s e^{(4s^2-1)t}.$$

If $\alpha=2$ and $4s^2-1>0$, then (56) becomes

$$u(x,t)=\sqrt{1-x^2}\left(\frac{1-x}{1+x}\right)^s\left[\left(\frac{c_1}{2\sqrt{4s^2-1}}+\frac{c_2}{2}\right) \times e^{\sqrt{4s^2-1}t}-\left(\frac{c_1}{2\sqrt{4s^2-1}}-\frac{c_2}{2}\right)e^{-\sqrt{4s^2-1}t}\right].$$

by virtue of (4.2.2) of Ref. [20]. This is the exact solution (2.107) of [1].

5.8. Solutions of (1) with $c(x)=x^2e^{\frac{1}{x}}$

We can give solutions to (18) by fifth and sixth assertions of Lemma 5.1

$$\varphi(z)=c_1H^{2,0}_{1,2}\left[\frac{z^{-\alpha}}{4}\left|\begin{array}{c}(1,\alpha)\\\left(\frac{s}{2},1\right),\left(\frac{s}{2},1\right)\end{array}\right.\right]$$

for $0<\alpha<2$,

$$\varphi(z)=\sum_{k=1}^{n}c_k z^{\alpha-k} \times {}_3\Psi_1\left[4z^{\alpha}\left|\begin{array}{c}\left(1-\frac{k}{\alpha}+\frac{s}{2},1\right),\left(1-\frac{k}{\alpha}+\frac{s}{2},1\right),(1,1)\\(1+\alpha-k,\alpha)\end{array}\right.\right]$$

for $\alpha \geq 2$. Then invariant solutions of (1) with $c(x) = x^2 e^{\frac{1}{x}}$ corresponding to U_{14} are for $0 < \alpha < 2$

$$u(x,t) = c_1 x e^{\frac{s}{x}} H^{2,0}_{1,2}\left[\frac{1}{4}e^{-\frac{2}{x}}t^{-\alpha}\middle| \begin{matrix} (1,\alpha) \\ \left(\frac{s}{2},1\right),\left(\frac{s}{2},1\right)\end{matrix}\right] \tag{57}$$

and for $\alpha \geq 2$

$$u(x,t) = x e^{\frac{2+s}{x}} t^{\alpha} \sum_{k=1}^{n} c_k \left(e^{\frac{2}{\alpha x}} t\right)^{-k} \times {}_3\Psi_1\left[4e^{\frac{2}{x}}t^{\alpha}\middle| \begin{matrix} \left(B-\frac{k}{\alpha},1\right),\left(B-\frac{k}{\alpha},1\right),(1,1) \\ (1+\alpha-k,\alpha)\end{matrix}\right], \tag{58}$$

where $B = 1 + \frac{s}{2}$. If we take $\alpha = 1$, $s = 2$ in (57), by (1.125) of Ref. [17], we get

$$u(x,t) = \frac{c_1}{4} x t^{-1} \exp\left(-\frac{1}{4}e^{-\frac{2}{x}}t^{-1}\right).$$

When $\alpha = 2$ in (58), by (44), (45), (47) and (48), the solution (58) becomes

$$u(x,t) = \tilde{c}_1 x e^{\frac{s}{x}} F_1\left(e^{\frac{1}{x}}t\right) + \tilde{c}_2 x e^{\frac{s}{x}} F_2\left(e^{\frac{1}{x}}t\right),$$

where

$$\tilde{c}_1 = \frac{\sqrt{\pi}\Gamma(s)^2}{2^{2s-1}}\left[\frac{c_1+c_2}{\Gamma\left(\frac{1}{2}+s\right)} - \frac{\Gamma\left(\frac{1}{2}-s\right)(c_1-c_2)}{\pi}\right],$$

$$\tilde{c}_2 = -2^{1-2s}\sqrt{\pi}\Gamma\left(-\frac{1}{2}+s\right)(c_1-c_2),$$

$$F_1(z) = {}_2F_1\left(\begin{matrix} s,s \\ \frac{1}{2}+s\end{matrix};\frac{1+z}{2}\right),$$

$$F_2(z) = \left(\frac{z+1}{2}\right)^{\frac{1}{2}-s} {}_2F_1\left(\begin{matrix} \frac{1}{2},\frac{1}{2} \\ \frac{3}{2}-s\end{matrix};\frac{1+z}{2}\right).$$

This is the exact solution (2.116) of [1].

Remark. From the invariant solutions that we obtained, we can see the following equivalent relationships:

1) If we introduce a new unknown function $v = \frac{1}{x}u$ and a new independent variable $y = \frac{1}{x}$, then $u(x,t)$ solves $\frac{\partial^\alpha}{\partial t^\alpha}u = x^{2m}u_{xx}$ if and only if $v(y,t)$ solves $\frac{\partial^\alpha}{\partial t^\alpha}v = y^{2(2-m)}v_{yy}$. In other words, the equation (1) with $c(x) = x^m$ is equivalent to the equation (1) with $c(x) = x^{2-m}$.
2) By introducing $v = \frac{1}{1+x}u$ and $y = 2^{\frac{1}{m}}\frac{1-x}{1+x}$, we can see that $u(x,t)$ solves $\frac{\partial^\alpha}{\partial t^\alpha}u = \frac{(1-x)^{2(m+1)}}{(1+x)^{2(m-1)}}u_{xx}$ if and only if $v(y,t)$ solves $\frac{\partial^\alpha}{\partial t^\alpha}v = y^{2(m+1)}v_{yy}$.
3) Again by introducing $v = \frac{1}{x}v$ and $y = -\frac{2}{x} - \ln 4$, $u(x,t)$ solves $\frac{\partial^\alpha}{\partial t^\alpha}u = x^4 e^{\frac{2}{x}}u_{xx}$ if and only if $v(y,t)$ solves $\frac{\partial^\alpha}{\partial t^\alpha}v = e^{-y}v_{yy}$.

6. CONCLUSION

We have studied a class of linear diffusion-wave equations with variable coefficients via Lie symmetry analysis. The group invariant classifications of the equations under study have been systemically done and exact invariant solutions that correspond to each symmetry in the optimal systems of infinitesimal symmetries have been derived. We obtain the invariant solutions explicitly in means of the special functions. The invariant solutions can be considered as generalizations of the well-known solutions of corresponding diffusion and wave equations in means of order of time differentiation. We have also done analysis revealing some equivalent relationships between the solutions obtained. Interested readers may continue doing analysis on the given solutions, such as plotting graphs of the solutions.

In the Lie symmetry analysis, the fractional differential operator under consideration is global in means of the order of the derivative. So, the Lie transformations of fractional differential equations were obtained fewer than the corresponding transformations, which were obtained in G. Bluman, S. Kumei [1]. As a result, the reduced equations were fewer than those of wave equations. On the other hand, when it comes to give solutions to reduced equations, the fractional differential equations are more complicated than the corresponding wave equations. For example, the reduced equations of wave equations have solutions expressed in terms of hypergeometric functions. In our case, we give solutions of reduced equations in terms of generalized Wright functions and Fox H-functions, which can be considered as generalizations of hypergeometric functions.

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