Quintic B-spline method for solving Lane-Emden problem

Ulziibayar Vandandoo[1], Zhanlav Tugal[2,3], Mijiddorj Renchin-Ochir[4], Otgondorj Khuder[1]

[1]*School of Applied Sciences, Mongolian University of Science and Technology, Ulaanbaatar, Mongolia*
[2]*Institute of Mathematics and Digital Technology, Mongolian Academy of Sciences, Ulaanbaatar, Mongolia*
[3]*Department of Applied Mathematics, School of engineering and Applied Sciences, National University of Mongolia, Ulaanbaatar, Mongolia*
[4]*Department of Informatics, School of Mathematics and Natural Sciences, Mongolian National University of Education, Ulaanbaatar, Mongolia*
**Correspondence author. Email: otgondorj@gmail.com, ORCID: 0000-0003-1635-7971*

ABSTRACT

This study introduces an efficient and highly accurate quintic B-spline method for solving Lane-Emden type equations which represent many scientific phenomena in astrophysics and mathematical physics. The main advantage of the obtained quintic spline is that it approximates not only a solution of the equation, but also the derivatives of solution with high accuracy. The method is applied on several test problems and the numerical results are compared with some results in the literature. Moreover, the results of numerical experiment show the efficiency of the method computationally.

Keywords: *Quintic B-spline, Multi-point finite-difference scheme, Singular boundary value problem, Solving nonlinear systems.*

1. INTRODUCTION

Several problems in theoretical physics and astrophysics can be modeled by Lane-Emden type equations

$$u'' + \frac{\alpha}{x}u' + f(u) = 0, \ \alpha > 0, \ x > 0, \tag{1}$$

where $f(u)$ is nonlinear function. Equation (1) is singular at $x = 0$ and we consider equation (1) subject to the following boundary conditions

$$u'(0) = 0, \ a_1 u(1) + b_1 u'(1) = c_1, \tag{2}$$

where a_1, b_1 and c_1 are real constants. The singular boundary value problem (1) and (2) describes a variety of phenomena in physics and astrophysics and chemistry such as the problem of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres. The equation (1) also appears in other context, for example, radiative cooling, in the modelling of clusters of galaxies, see [1], [2], [3], [4], [5], [6], [7], [8], [9] and references therein.

In general, equation (1) must be solved numerically. Currently, most techniques in use for equation (1) are based either series solution or perturbation techniques. But in these cases, to increase the accuracy we needed to compute more number of term which is otherwise difficult. Various other non-perturbation approximate analytical methods like Adomian decomposition method (ADM), variational iteration method (VIM), homotopy analysis method (HAM) applied to get the solution of equation (1). For equation (1) involving nonlinear term likes $\exp(u)$ and $\sin(u)$, these methods also have some difficult [4], [5] and are valid for a restricted region.

Note that several authors used finite difference methods (FDM), B-spline, and Hermite spline methods [2], [3], [6], [7], [9] for solving equation (1). When $\alpha = 2$ and $f(u) = u^m$ the equation (1) is called the Lane-Emden equations of the first kind and becomes as

$$u'' + \frac{2}{x}u' + u^m = 0, \ m \geq 0, \ x > 0, \tag{3}$$

that used to model the thermal behaviour of a spherical cloud of gas. The physical interesting range of index m is $0 \leq m \leq 5$. Only for $m = 0, 1$ and 5, the solution of equation (3) can be given in closed form as

$$u = 1 - \frac{x^2}{6}, \ u = \frac{\sin x}{x}, \ u = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}}, \tag{4}$$

respectively. In other cases, there is not any analytic solution. The first zero of $u(x)$ gives the radius of the star, so $u(x)$ must be computed up to this zero. (The formal solution may have additional zeros at larger value of x but $x > R$ is not relevant for stellar models).

U. Vandandoo et al. (eds.), *Proceedings of the International Conference on Applied Sciences and Engineering (ICASE 2023)*
Atlantis Highlights in Engineering 22,
https://doi.org/10.2991/978-94-6463-330-6_7

Some authors were found the zero R for different m. We present it in Table 1. For solution of (3) often used besides of (2) condition

$$u(0) = 1. \tag{5}$$

The condition (2), (5) used as initial conditions for equation (3). In some cases may be useful an approximate formula

$$\begin{aligned} u(x) = 1 &- \frac{x^2}{3!} + \frac{m}{5!}x^4 - \frac{m(8m-5)}{3\cdot 7!}x^6 \\ &+ \frac{m(122m^2 - 183m + 70)}{9\cdot 9!}x^8 \\ &- \frac{a_{10}}{10!}x^8 + \dots \end{aligned} \tag{6}$$

in the small vicinity of singular point $x = 0$. Besides initial value problem, it is often posted boundary values for equation (3) on the interval $[0, L]$. In particular, L may be chosen as R in Table 1, i.e., boundary conditions for equation (3) are

$$u(0) = 1, \ u(L) = 0. \tag{7}$$

When $\alpha = 2$ and $f(u) = e^u$ the equation (1) is called the Lane-Emden equations of the second kind and becomes as

$$u'' + \frac{2}{x}u' + e^u = 0. \tag{8}$$

Equation (8) arises in the study of density distribution in an isothermal gas sphere.

In [12] we proposed the higher accurate multi-point finite-difference scheme for solving two-point boundary value problems and used for numerical solution of the Bratu's problem. It turns out to be also applicable for Lane-Emden type problem (1), (2). The aim of this paper is to apply the proposed method in [12] to problem (1), (2). The paper is composed of four sections. In Sec. 2, we obtain the nonlinear systems of equations for the nonlinear systems of equations for Lane-Emden equation. Furthermore, background on multi-point finite-difference scheme is provided. In Sec. 3, we obtain the quintic spline approximating the solution of Lane-Emden equation. Numerical simulations confirming the result of theoretical are provided in Sec 4. We conclude in Sec 5.

2. THE NONLINEAR SYSTEMS OF EQUATIONS FOR LANE-EMDEN EQUATION

The sixth order multi-point finite-difference scheme [12] for two point BVP

$$u'' = f(x, u), \tag{9}$$

reads as

$$\begin{aligned} &(u_{i-2} + u_{i+2}) + b(u_{i-1} + u_{i+1}) + cu_i \\ &\quad + h^2[d(f(x_{i-2}, u_{i-2}) + f(x_{i+2}, u_{i+2})) \\ &\quad + \theta(f(x_{i-1}, u_{i-1}) + f(x_{i+1}, u_{i+1})) \\ &\quad + \eta f(x_i, u_i)] = 0, \ i = 2(1)N-2, \\ &\quad (x_i = ih, \ i = 0(1)N, \ h = \frac{1}{N}), \end{aligned} \tag{10}$$

where

$$\begin{aligned} b &= 16 + 240d, \ c = -34 - 480d, \\ \theta &= -\frac{8}{3}(1 + 9d), \ \eta = -\frac{44}{3} - 194d. \end{aligned}$$

In order to use the finite difference scheme (10) we need to transform equation (1) to form (9) that does not contain first order derivative. In fact, using the transformation of the dependent variable

$$u = \frac{z}{\sqrt{x^\alpha}}, \tag{11}$$

in (1) we obtain

$$z'' + \frac{\alpha(2-\alpha)}{4x^2}z + \sqrt{x^\alpha} f\left(\frac{z}{\sqrt{x^\alpha}}\right) = 0. \tag{12}$$

The boundary conditions (2) converted to

$$z(0) = 0, \quad \left(a_1 - \frac{\alpha}{2}b_1\right)z(1) + b_1 z'(1) = c_1. \tag{13}$$

By using L'Hopital's rule in (12) at the singular point $x = 0$, we obtain

$$z'(0) = z''(0) = 0, \ \text{when } \alpha \neq 2, \tag{14}$$

that may be used in sequel. For (12) is applicable the scheme (10) and it has a form

$$\begin{aligned} &\left(1 - \frac{h^2 d\beta}{4x_{i-2}^2}\right)z_{i-2} + \left(1 - \frac{h^2 d\beta}{4x_{i+2}^2}\right)z_{i+2} \\ &\quad + \left(b - \frac{\theta h^2\beta}{4x_{i-1}^2}\right)z_{i-1} + \left(b - \frac{\theta h^2\beta}{4x_{i+1}^2}\right)z_{i+1} \\ &\quad + \left(c - \frac{\eta h^2\beta}{4x_i^2}\right)z_i - h^2 d\Bigg(\sqrt{x_{i-2}^\alpha} f\Big(\frac{z_{i-2}}{\sqrt{x_{i-2}^\alpha}}\Big) \\ &\quad + \sqrt{x_{i+2}^\alpha} f\Big(\frac{z_{i+2}}{\sqrt{x_{i+2}^\alpha}}\Big)\Bigg) \\ &\quad - h^2\theta\Bigg(\sqrt{x_{i-1}^\alpha} f\Big(\frac{z_{i-1}}{\sqrt{x_{i-1}^\alpha}}\Big) \\ &\quad + \sqrt{x_{i+1}^\alpha} f\Big(\frac{z_{i+1}}{\sqrt{x_{i+1}^\alpha}}\Big)\Bigg) \\ &\quad - h^2\eta\sqrt{x_i^\alpha} f\Big(\frac{z_i}{\sqrt{x_i^\alpha}}\Big) = 0, \\ &\quad i = 2(1)N-2, \ \beta = \alpha(2-\alpha). \end{aligned} \tag{15}$$

When $\alpha = 2$ the equation (15) has a simple form

$$\begin{aligned} &z_{i-2} + z_{i+2} + b(z_{i-1} + z_{i+1}) + cz_i \\ &\quad - h^2\Bigg(d\Big(x_{i-2} f\Big(\frac{z_{i-2}}{x_{i-2}}\Big) + x_{i+2} f\Big(\frac{z_{i+2}}{x_{i+2}}\Big)\Big) \\ &\quad + \theta\Big(x_{i-1} f\Big(\frac{z_{i-1}}{x_{i-1}}\Big) + x_{i+1} f\Big(\frac{z_{i+1}}{x_{i+1}}\Big)\Big) \\ &\quad + \eta x_i f\Big(\frac{z_i}{x_i}\Big)\Bigg) = 0, \ i = 2(1)N-2. \end{aligned} \tag{16}$$

Table 1.

m	0	1	2	3	4	5
R	$\sqrt{6}$	π	4.35287460	6.89684862	14.9715463	∞

Besides of (13) additional boundary equations [12] look like:

$$\sum_{k=0}^{3} a_k z_k - h^2 \sum_{k=0}^{5} b_k \Big(\frac{\beta}{4x_k^2} z_k + \sqrt{x_k^\alpha} f\Big(\frac{z_k}{\sqrt{x_k^\alpha}}\Big)\Big) = 0, \quad (17a)$$

$$\sum_{k=0}^{3} \tilde{a}_k z_{N-k} - h^2 \sum_{k=0}^{5} \tilde{b}_k \Big(\frac{\beta}{4x_{N-k}^2} z_k + \sqrt{x_{N-k}^\alpha} f\Big(\frac{z_{N-k}}{\sqrt{x_{N-k}^\alpha}}\Big)\Big) = 0, \quad (17b)$$

where $a_0 = -4$, $a_1 = 7$, $a_2 = -2$, $a_3 = -3$, $b_0 = \frac{71}{240}$, $b_1 = \frac{43}{12}$, $b_2 = \frac{7}{8}$, $b_3 = \frac{1}{3}$, $b_4 = -\frac{5}{48}$, $b_5 = \frac{1}{60}$.

Analogously, the coefficients $\tilde{a}_k$, $\tilde{b}_k$ are determined in the same way as (17a). The equations (15), (13) and (17a) , (17b) provide a nonlinear system $(N+1)$ equations with $(N+1)$ unknowns $z_0, z_1, \dots, z_N$. This system can be solved for the unknown vector $z = (z_0, z_1, \dots, z_N)^T$ by using any non-linear solver, like Newton's method.

It should be noted that when $b_1 \neq 0$, $z'(1)$ in (13) can be approximated by linear combination of z_i, $i = N, N-1, \dots$, with accuracy $O(h^6)$. If $b_1 \neq 0$ and $\left(a_1 - \frac{\alpha}{2} b_1\right) \neq 0$ in (13), we need two additional equations. Using the Taylor expansion in the neighborhood of $x = x_1$, we get the following equation

$$z_1 - 2z_2 + z_3 + h^2\Big(-\frac{1}{15} z_1'' - \frac{131}{144} z_2'' + \frac{1}{18} z_3'' - \frac{1}{8} z_4'' + \frac{1}{18} z_5'' - \frac{7}{720} z_6''\Big) = 0.$$

Likewise, for $x = x_{N-1}$ one more can be provided in the same fashion.

3. APPROXIMATION OF SOLUTION OF LANE-EMDEN EQUATIONS BY QUINTIC SPLINE

The consistency relation for $M_i = S''(x_i)$ of quintic spline interpolating smooth function $z(x)$ is [7]

$$M_{i-2} + 26M_{i-1} + 66M_i + 26M_{i+1} + M_{i+2} = \frac{20}{h^2}\left[z_{i-2} + 2z_{i-1} - 6z_i + 2z_{i+1} + z_{i+2}\right], \quad i = 2(1)N-2,\ z_i = z(x_i). \quad (18)$$

For M_i, we have

$$M_i = z_i'' + \frac{h^4}{720} z_i^{(6)} + O(h^6),\ i = 0(1)N. \quad (19)$$

For smooth function $z(x)$ hold the following relations

$$z_{i-2}'' + 26z_{i-1}'' + 66z_i'' + 26z_{i+1}'' + z_{i+2}'' = 120z_i'' + 30h^2 z_i^{(4)} + \frac{7}{2} h^4 z_i^{(6)} + O(h^6), \quad (20)$$

$$z_{i-2}^{(6)} + 26z_{i-1}^{(6)} + 66z_i^{(6)} + 26z_{i+1}^{(6)} + z_{i+2}^{(6)} = 120z_i^{(6)} + O(h^2), \quad (21)$$

$$z_i^{(4)} = \frac{1}{12h^2}(-z_{i-2}'' + 16z_{i-1}'' - 30z_i'' + 16z_{i+1}'' - z_{i+2}'') + O(h^4), \quad (22)$$

$$z_i^{(6)} = \frac{1}{h^4}(z_{i-2}'' - 4z_{i-1}'' + 6z_i'' - 4z_{i+1}'' + z_{i+2}'') + O(h^2). \quad (23)$$

Substituting (19) into (18) and using the relations (20), (21), (22) and (23) we get

$$z_{i-2} + 2z_i - 6z_{i-1} + 2z_{i+1} + z_{i+2} = \frac{h^2}{120}\Big(7z_{i-2}'' + 152z_{i-1}'' + 402z_i'' + 152z_{i+1}'' + 7z_{i+2}''\Big) + O(h^8). \quad (24)$$

Substituting (12) into (24) and ignoring $O(h^8)$ term we obtain

$$\begin{aligned}
&\Big(1 + \frac{7h^2\beta}{120 \cdot 4x_{i-2}^2}\Big) z_{i-2} + \Big(1 + \frac{7h^2\beta}{120 \cdot 4x_{i+2}^2}\Big) z_{i+2} + \Big(2 + \frac{152h^2\beta}{120 \cdot 4x_{i-1}^2}\Big) z_{i-1} \\
&+ \Big(2 + \frac{152h^2\beta}{120 \cdot 4x_{i+1}^2}\Big) z_{i+1} + \Big(-6 + \frac{402h^2\beta}{120 \cdot 4x_i^2}\Big) z_i \\
&+ \frac{7h^2}{120}\Big(\sqrt{x_{i-2}^\alpha} f\big(\frac{z_{i-2}}{\sqrt{x_{i-2}^\alpha}}\big) + \sqrt{x_{i+2}^\alpha} f\big(\frac{z_{i+2}}{\sqrt{x_{i+2}^\alpha}}\big)\Big) \\
&+ \frac{152h^2}{120}\Big(\sqrt{x_{i-1}^\alpha} f\big(\frac{z_{i-1}}{\sqrt{x_{i-1}^\alpha}}\big) + \sqrt{x_{i+1}^\alpha} f\big(\frac{z_{i+1}}{\sqrt{x_{i+1}^\alpha}}\big)\Big) \\
&+ \frac{402h^2}{120} \sqrt{x_i^\alpha} f\big(\frac{z_i}{\sqrt{x_i^\alpha}}\big) = 0, \\
&i = 2(1)N-2.
\end{aligned} \quad (25)$$

When $d = -\dfrac{7}{120}$, $b = 2$, $c = -6$, $\theta = -\dfrac{152}{120}$, $\eta = -\dfrac{402}{120}$ the equation (15) coincides with (25). This means that the quintic spline interpolation satisfies the nonlinear system (15) i.e., in order to find quintic spline interpolating the solution of equation (12) it suffices to solve nonlinear systems of equations (15) subject to appropriate boundary equations.

Assume that the solution of system (13), (15) is found by any of iterative method i.e., z_i for $i = 0, 1, \dots, N$ are known. Then it is easy to find quintic spline interpolation $S(x)$

$$S(x) = \sum_{j=-2}^{N+2} C_j B_j(x), \quad (26)$$

such that

$$S_i = z_i, \ i = 0, 1, \dots, N \quad (27)$$

$$S_i'' = A_i, \ i = 1, 2 \dots, N-2, N-1, \quad (28)$$

where

$$A_i = -\frac{\alpha(2-\alpha)}{4x_i^2} z_i - \sqrt{x_i^{\alpha}} f\Big(\frac{z_i}{\sqrt{x_i^{\alpha}}}\Big), \quad i = 1, 2 \dots, N. \quad (29)$$

Moreover, it is possible to find quintic spline approximation to z without solving the linear system (27) and (28). Indeed, using the following formula [12]

$$C_j = z_j + \frac{h^2}{30}\left(z_{j-1}'' - \frac{19}{2} z_j'' + z_{j+1}''\right) + O(h^6), \quad (30)$$

we obtain

$$C_j = z_j + \frac{h^2}{30}\left(A_{j-1} - \frac{19}{2} A_j + A_{j+1}\right), \quad j = 1, 2 \dots, N-1. \quad (31)$$

According to (14) and (29) we have $A_0 = 0$. The remainder coefficients of (26) are determined from (27) and (28). Since z_i are found by system (13) and (15) with accuracy $O(h^6)$ then for (26) hold the following relations [12], [13]

$$z_i^{(r)} - S_i^{(r)} = O(h^{6-r}), \ i = 0, 1 \dots, N, \ r = 0, 1, 2. \quad (32)$$

According to (11) approximate solution of the original Lane-Emden (1) is defined by

$$\tilde{u} = \frac{S}{\sqrt{x^{\alpha}}}. \quad (33)$$

From (11) and (33) we obtain

$$\begin{aligned} u - \tilde{u} &= \frac{u - S}{\sqrt{x^{\alpha}}} : \\ u' - \tilde{u}' &= x^{\frac{\alpha}{2}-1}(z' - S' - \frac{\alpha}{2x}(z - S)); \\ u'' - \tilde{u}'' &= (\frac{\alpha}{2} - 1)x^{\frac{\alpha}{2}-2}(z' - S' - \frac{\alpha}{2x}(z - S)) \\ &+ x^{\frac{\alpha}{2}-1}\big(z'' - S'' - \frac{\alpha}{2}\frac{(z' - S')x - (z - S)}{x^2}\big) \\ &= x^{\frac{\alpha}{2}-1}(z'' - S'') + \big((\frac{\alpha}{2} - 1)x^{\frac{\alpha}{2}-2} \\ &- \frac{\alpha}{2}x^{\frac{\alpha}{2}-2}\big)(z' - S') + x^{\frac{\alpha}{2}-3}(z - S). \end{aligned}$$

From these relations, by virtue of (32) we have

$$\begin{aligned} u - \tilde{u} = O(h^6), \ u' - \tilde{u}' = O(h^5), \\ u'' - \tilde{u}'' = O(h^4), \ i = 0, 1 \dots, N. \end{aligned} \quad (34)$$

It means that the function $\tilde{u}$ given by (33) actually is quintic spline approximation for the solution to (1).

4. NUMERICAL RESULTS

To demonstrate the effectiveness of the presented algorithms (15), (17) and to test its convergence computationally, several Lane-Emden type equations are considered. The maximum absolute errors and the computational order of convergence obtained using our methods are tabulated in Tables 2, 3. Additionally, the absolute errors of the approximate the derivative of solution are listed in Table 4. The maximum absolute error E_n is computed by

$$E_n = \max_{0 \le i \le n} |u(x_i) - u_i|,$$

where $u(x_i)$ and u_i denote the exact solution and approximate solution at the $i-$th grid point, respectively. The computational order of convergence is checked applying the following formula [3]

$$\hat{r} = \left|\frac{\ln(E_n) - \ln(E_{2n})}{\ln(2)}\right|.$$

All calculations have been performed using MATLAB. A test examples we consider the following real life problems [1], [2], [3], [7], [8].

Example 1: As a first example, we consider Lane-Emden type problem describing the equilibrium of the isothermal gas sphere [7], [8], [2]:

$$u'' + \frac{2}{x}u' + u^5 = 0, \quad (35)$$

subject to the following BC:

$$u'(0) = 0, \quad u(1) = \sqrt{\frac{3}{4}}.$$

The exact solution is given by $u = \sqrt{\dfrac{3}{3 + x^2}}$.

This problem has been solved using the method (25) with different values of N and the results of the experiment are tabulated in Table 2. An addition, for this problem our results are compared with results in [2], [3], [15] which shown the superiorly of presented technique.

Example 2: Next, we consider the following BVP

$$u'' + \frac{\alpha}{x}u' + \lambda e^u = 0, \quad 0 < x < 1 \quad (36)$$

with boundary conditions

$$u'(0) = 0, \quad u(1) = 0.$$

Equation (36) with $\alpha = 1$ and $\lambda = -1$ arises in the study of thermal explosions. The exact solution is given by $u(x) = 2\ln\left(\frac{d+1}{dx^2+1}\right)$, where $d = -5 + 2\sqrt{6}$. We calculate an approximate solution of this problem for $\alpha = 1, \lambda = -1$ and different values of N. The results of the experiment are also displayed in Table 2.

Example 3: The following two point BVP.

$$\begin{aligned} &u'' + \frac{2}{x}u'(x) - e^{-u} = 0, \\ &u'(0) = 0, \ 2u(1) + u'(1) = 0 \end{aligned}$$

which describes the thermal distribution profile in the human head [1].

Example 4: The next SBVP occurs when studying radial stress on a rotationally symmetric shallow membrane cap [1]

$$u'' + \frac{3}{x}u' = \frac{1}{8u^2} - \frac{1}{2}, \quad 0 < x < 1,$$

with boundary conditions:

$$u'(0) = 0, \quad u(1) = 1.$$

Tables 2, 3 show that the rate of convergence obtained by our method is compatible with the theoretical results. Furthermore, the numerical results in the tables demonstrate that the convergence order of the method (15) is six.

From Table 4, we also show that our method (15) approximates not only the solution of Lane-Emden type equations but also approximate the $u_j^{(r)}$ derivative of solution with accuracy $O(h^{6-r})$ for $r = 1, 2, 3, 4$.

Table 5 shows the comparison of the absolute error obtained by the quintic spline method (15) and the methods in [2], [3], [15]. In particular, it gives similar results to the method in [3] and better results than the methods given [2] and [15]. Finally, numerical experiments have shown that the introduced method is applicable and approximate the solution very well for Lane-Emden problem.

Table 2. Maximum absolute errors of method (15)

	Example 1		*Example 2*	
N	**Error**	**Order**	**Error**	**Order**
10	8.08e-09	-	2.61e-09	-
20	9.02e-10	3.16	3.27e-11	6.31
40	5.42e-11	4.05	3.42e-13	6.57
80	9.22e-13	5.87	4.62e-15	6.20

Table 3. Maximum absolute errors of method (15)

	Example 3		*Example 4*	
N	**Error**	**Order**	**Error**	**Order**
10	3.13e-07	-	4.21e-09	-
20	3.36e-09	6.54	2.98e-11	7.14
40	4.52e-11	6.21	3.37e-13	6.46
80	8.98e-13	5.65	6.98e-15	6.00

5. CONCLUSIONS

We propose new nonlinear multi-point finite-difference scheme for Lane-Emden type equations. Solving the obtained nonlinear system, we actually construct the quintic B-spline approximation for the solution of Lane-Emden equation. As a consequence, our method approximates not only the solution of the equation but also approximate the derivatives of the solution with higher accuracy. To illustrate the high efficiency and accuracy of proposed algorithm, the numerical experiments are carried out on some real-life problems. A comparison of the proposed method with other methods was made. Numerical outcomes show that our algorithm is very promising and can be applied accurately and efficiently to solve a similar class of two-point BVPs.

Table 4. Error for derivative of solution

	N	**Error**	$\hat{r}$	**Error**	$\hat{r}$
		Example 1		***Example 2***	
	10	3.21e-06	-	2.21e-0.4	-
$\|S_j' - u_j'\|$	20	1.32e-07	4.6	9.86e-06	4.4
	40	3.70e-08	5.1	2.92e-07	5.0
	10	3.21e-05	-	9.11e-04	-
$\|S_j'' - u_j''\|$	20	2.86e-06	3.4	8.16e-05	3.4
	40	2.22e-07	3.7	5.20e-06	3.9
	10	3.21e-03	-	7.21e-02	-
$\|S_j''' - u_j'''\|$	20	7.16e-04	2.1	1.96e-02	2.0
	40	8.72e-05	3.3	2.35e-03	2.9
	10	4.33e-02	-	1.21e-01	-
$\|S_j^{(4)} - u_j^{(4)}\|$	20	6.96e-03	2.6	1.11e-01	1.0
	40	1.95e-03	1.8	4.95e-02	1.6

Table 5. Maximum absolute errors of *Example 1*

	Method (15)	**Method [3]**	**Method [2]**	**Method [15]**
N	**Error**	**Error**	**Error**	**Error**
8	3.22e-08	5.33e-08	$\cdots$	$\cdots$
16	2.57e-09	1.59e-09	7.34e-08	2.43e-07
32	3.53e-11	2.67e-11	4.54e-09	2.59e-08
64	5.48e-13	4.08e-13	2.83e-10	1.80e-09

ACKNOWLEDGMENT

The authors would like to sincerely thank the reviewers and the editor for their comments and valuable suggestions, which greatly improved the readability of the article. The work was supported partially by the Foundation of Science and Technology of Mongolia University Science and Technology research project MFUND-03/2022.

REFERENCES

[1] M. Singh, Amit K. Verma, An effective computational technique for a class of Lane-Emden equations, J. Math. Chem. 54 (2016), pp. 231-251. https://doi.org/10.1007/s10910-015-0557-8

[2] P. Roul, K. Thula, A fourth-order B-spline collocation method and its error analysis for Bratu-type and Lane–Emden problems, Int. J. Comput. Math. 96 (2019), pp. 85-104. https://doi.org/10.1080/00207160.2017.1417592

[3] P. Roul, K. Thula, V.M.K. Prasad Goura, An optimal sixth-order quartic B-spline collocation method for solving Bratu-type and Lane-Emden–type problems, Math. Methods Appl. Sci. 42 (2019), pp. 2613-2630. https://doi.org/10.1002/mma.5537

[4] N. Das, R. Singh, A.M. Wazwaz, J. Kumar, An algorithm based on the variational iteration technique for the Bratu-type and the Lane–Emden problems, J. Math. Chem. 54 (2016), pp. 527–551. https://doi.org/10.1007/s10910-015-0575-6

[5] A. Wazwaz, R. Rach, Comparison of the Adomian decomposition method and the variational iteration method for solving the Lane-Emden equations of the first and second kinds, Kybernetes, 40: 9-10 (2011), pp. 1305–1318. https://doi.org/10.1108/03684921111169404

[6] J. Goh, A. A. Majid, A. Izani, M. Ismail, A quartic B-spline for second-order singular boundary value problems, Comput. Math. Appl. 64 (2012), pp. 115–120. https://doi.org/10.1016/j.camwa.2012.01.022

[7] R. Mohammadzadeh, M. Lakestani, M. Dehghan, Collocation method for the numerical solutions of Lane-Emden type equations using cubic Hermite spline functions, Math. Meth. Appl. Sci, 37 (2014), pp. 1303-1317. https://doi.org/10.1002/mma.2890

[8] M. Lakestani, M. Dehghan, Four techniques based on the B-spline expansion and the collocation approach for the numerical solution of the Lane–Emden equation, Math. Meth. Appl. Sci, 36 (2013), pp. 2243-2253. https://doi.org/10.1002/mma.2755

[9] J. Rashidinia , Z. Mahmoodi, M. Ghasemi, Parametric spline method for a class of singular two-point boundary value problems, Appl. Math. Comput. 188 (2007), pp. 58-63, . https://doi.org/10.1016/j.amc.2006.09.084

[10] G. P. Horedt, Polytropes: Applications in Astrophysics and Related Fields, Kluwer Academic Publishers, Dordrecht, 2004.

[11] Om. P. Singh, R.K. Pandey, V.K. Singh, An analytic algorithm of Lane–Emden type equations arising in astrophysics using modified Homotopy analysis method, Comp. Phys. Commun. 180 (2009), pp. 1116-1124. https://doi.org/10.1016/j.cpc.2009.01.012

[12] T. Zhanlav, R. Mijiddorj, Kh. Otgondorj, Quintic B-spline method for solving Bratu's problem, Submitted.

[13] T. Zhanlav, R. Mijiddorj, H. Behforooz, Construction of local integro quintic splines, Communications in Numerical Analysis, 2 (2016), pp. 167-179. https://doi.org/10.5899/2016/cna-00267

[14] F. G. Lang, X. P. Xu, Quintic B-spline collocation method for second order mixed boundary value problem, Computer Physics Communications, 183 (2012), pp.913-921, . https://doi.org/10.1016/j.cpc.2011.12.017

[15] S.A. Khuri, A. Sayfy, Numerical solutions for the nonlinear Emden–Fowler type equations by a fourth-order adaptive method. Int. J. Comput. Methods 11-1 (2014), 1350052. https://doi.org/10.1142/S0219876213500527