The didactic transposition of probabilistic metric space concept

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Abstract. A space is a set which in defined a structure. Many typologies of spaces are provided in Mathematics. Before studying a lesson, teachers may know the structure of a lesson and their knowledge scholar. Two related lessons, probability and statistics, are everywhere jointly thought, however they are completely separated in their knowledge origin. In this paper, we will present the metric spaces related to the statistics and we will introduce the didactic transposition of the probabilistic metric spaces which are a generalization of usual metric spaces, the idea is to replace the distance between two points by a distribution function, it means that we do not know exactly the distance between two points but we know the probabilities of possible values of the distance. We offer an approach by the didactic transposition for how to use this type of spaces in the scholarly knowledge. Through this paper, we give an epistemological analysis of this new concept with a thematization of knowledge about the concept of probabilistic metric spaces.

Keywords: Probabilistic metric spaces, didactic transposition, thematisation.

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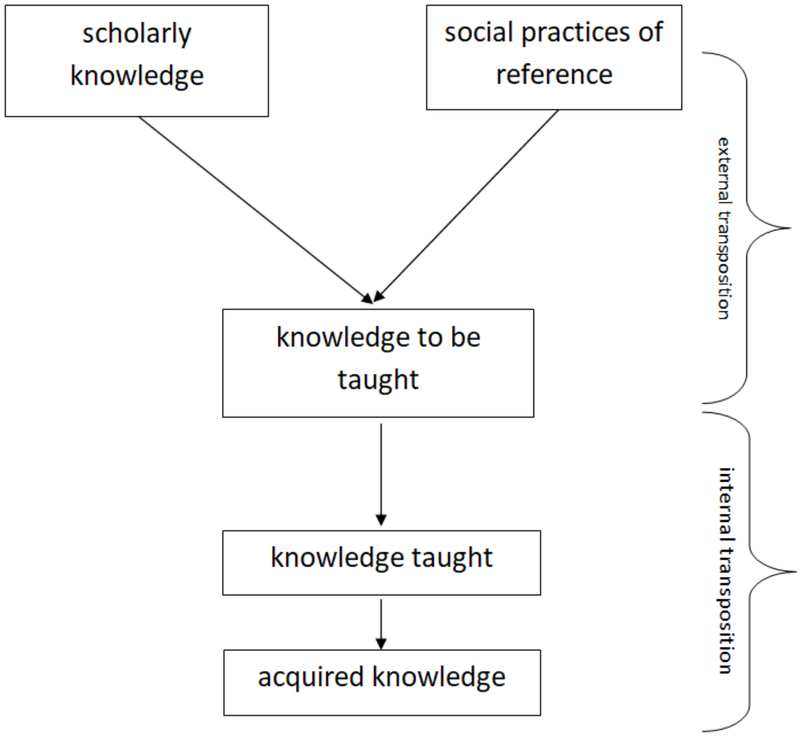
1 Introduction

The theory of didactic transposition was approached by Yves Chevallard [3] at the beginning of the years 1980, which is defined as a process who allows to transform and to pass from learned knowledge to taught knowledge, or the definition of Chevallard [4] is as the following “mathematics of the mathematicians to the taught mathematics”.

And among the great problems posed at the time of didactic transpsition, it is the distance or the gap between learned knowledge and taught knowledge, and the transposition consists in reducing this gap as well as possible, for that there exists two reasons, first is “because no taught knowledge would know how to authorize itself”, says Chevallard, it means that the existence of learned knowledge provokes the existence of school knowledge. It is this knowledge that epistemologically [1] guarantees school knowledge in the eyes of society. Secondly, if the mathematics taught is not obviously identical to that of mathematicians, it remains that the mathematics taught must allow those to whom it is taught to later find that of mathematicians.

We can divide the process of transposition in two stages, External transposition and internal transposition, the internal transposition means the transformation of scholarly knowledge into knowledge to be taught without forgetting the social practices of reference, and the internal transposition is the continuous process of the external transposition and which means the transformation of knowledge to be taught into taught knowledge and then into acquired knowledge, For the external transposition the problem is how a knowledge built outside the school (scholarly knowledge) combining with social practices will be transformed into knowledge to be taught which implies the problem of adaptation of the scholarly knowledge. Therefore, it is necessary to respect some rules so that the external transposition is effective, and the role of internal transposition consists in transforming the knowledge to be taught into knowledge taught and then into knowledge acquired by the learner, the process of transposition is seen as in the following picture.



From the diagram of didactic transposition, we notice that the probabilistic metric space is the learned knowledge concerned in this work, which is introduced by K. Menger [7] in 1942. And the role of the designer is the external transposition which is to transform and reorganize this learned knowledge in knowledge to be taught (manuals or programs), in this frame, it is going to reorganize the concept of the probabilistic metric spaces to the context of the school teachings. The taught knowledge is the concept of the probabilistic metric spaces taught by the teacher to the learners according to a set of conditions (prerequisites, difficulties, means,…). Finally the acquired knowledge is the concept of the probabilistic metric spaces learned and conceived by the students, and to interpret the school mathematical knowledge it is necessary to take into account the phenomena related to the reconstruction of the school mathematical knowledge [2] from the learned mathematical knowledge whose external transposition plays an essential role in mathematics teaching, thus, the internal transposition is a very important process to make the transition of programs and manuals of mathematics teaching to the appropriate sequences of learning. For this reason, this research is made to elaborate and analyze the internal and external transitional form the concept of the probabilistic metric spaces from the learned knowledge to the taught knowledge.

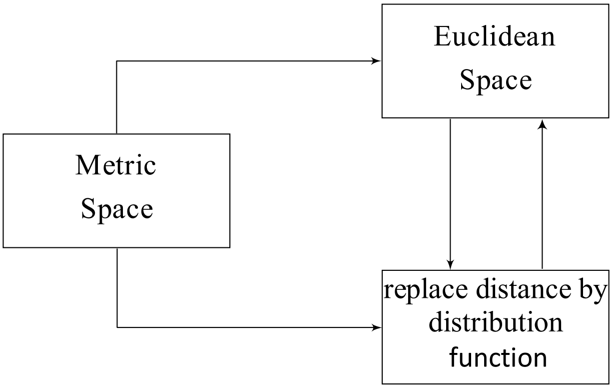
2 Problem of the notion of distance

Let E be a set, we can define the distance on E in the actual mathematical sense, is an application of E into E, verifying some axioms, this notion is presented to the students as a notion of the topology of the real line, or the mathematicians have forged it. At the beginning of the century, in order to extend the notion to the functional spaces that are generally of infinite dimension, the notion of topology acquired for a long time for the spaces of finite dimension. One of the most guiding ideas for the elaboration of this notion is to measure the resemblance or certain resemblance between two objects and which is totally absent from the current apprehension of the ordinary geometrical space, the resemblance between two points does not always mean that they are closer to each other, what causes a big gap between the distance of mathematical knowledge and the distance of school knowledge.

3 Method and Results

Our work consists in making an external transposition of the probabilistic metric spaces in school knowledge to be taught, and to transpose this concept, we thought of using Euclidean space with the Euclidean metric (Euclidean distance) as an example of metric space already transposed in school knowledge, by replacing this distance with a function of distribution distance in order to approach the notion of probabilistic metric space to the learners, during the choice and the elaboration of this function, we tried to adapt this transposition to the cognitive styles of the learners [9] which they can manage their learning, and so that it is well conceived by the learners. For the internal transposition, we will give an example of a didactic sequence that can be used in textbooks and a summary

that shows the process of transposition of this concept.
For the external transposition of the concept of probabilistic metric space, we have used a Euclidean space replacing the Euclidean metric with a well-selected function of distance distribution function as it is shown in the following plan.



And for internal transposition, we have proposed activities of construction of this concept, which are simple to acquire by the learners.

3.1 External transposition

3.1.1 Scholarly knowledge

We introduce here the notion of a probabilistic semi-metric space and before stating this, let's recall first the concept of a distance distribution function.

Definition 1. [5] Let Δ^+ be the space of all distance distribution functions

$f:[0,+\infty] \to [0,1]$

such that:

1 f is left continuous on $[0,+\infty]$,

2 f is non-decreasing.

3 $f(0)=0$ and $f(+\infty)=1$.
The subset $D^+ \subset \Delta^+$ is the set $D^+ = \{f \in \Delta^+ : \lim_{x\to\infty} f(x) = 1\}$.

As a special element of D^+ is the function ε_0 defined by

$$\varepsilon_0(x) = \begin{cases} 0 & if \quad x \leq 0, \\ 1 & if \quad x > 0. \end{cases}$$

Definition 2. [5] A Probabilistic semi-metric space is a couple (X, F) where X is nonempty set, F is a function from $X \times X$ into Δ^+, and the following conditions are satisfied for all $p, q \in X$:

1 $F_{p,p} = \varepsilon_0$,

2 $F_{p,q} \neq \varepsilon_0$ if $p \neq q$,

3 $F_{p,q} = F_{q,p}$.

3.1.2 knowledge to be taught

Formal Curriculum (Objectives): General guidelines for teaching this concept

It is necessary to teach the notion of the probabilistic metric space, based on Euclidean space by replacing the Euclidean metric by a function of distribution distance chosen and which must be simple for the learners.

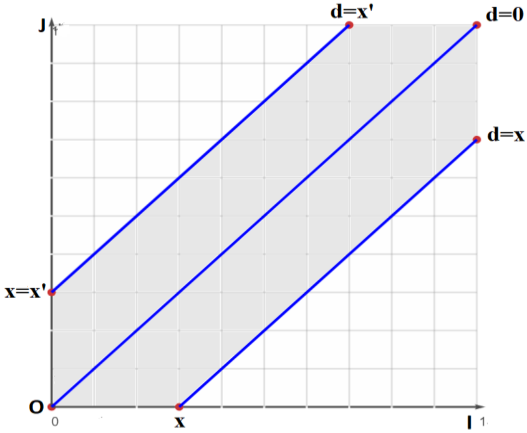
The learner must know that we can calculate the distance between two random points by using the distance distribution function.

3.2 Internal transposition

3.2.1 Actual Curriculum (Content): Knowledge Taught

Example of a didactic sequence

Let $(O; \vec{\imath}; \vec{\jmath})$ be an orthonormal frame of reference. We want to calculate the distance between two points M(x;0) and M'(0,x') chosen randomly on the two segments [OI] and [OJ], and let us consider d the random variable measuring the distance between x and x', we have colored the area that corresponds to d < x where x and x' are real numbers of [0;1]. We should find the distribution function and apply it. To achieve that we give the following activity.



1) Determine the area of the part colored in gray as a function of x.

2) Express this area in terms of the integral

3) We consider the expression of this area is the expression of the function g ,

Using the function g show that the probability of getting $0.11 < d < 0.15$ (magnitude 0.04) is greater than the probability of getting $0.65 < d < 0.79$ (magnitude 0.14).

4) Consider the function f such that f(0)=0 and for x greater than 0

$f(x) = e^{g(x)} + 1 = e^{-x^2+2x} + 1$

a) Calculate the limit of f in $+\infty$

b) Calculate the image of 0.12 and 0.70 by the function f and compare the two values

Solution elements

The learner should be able to determine the distribution function of the d distribution using the areas, and determine the probability between two randomly selected points on two axes, and should be able to use the distance distribution function as an implicit tool to solve certain exercises and problems.

1 $A(x) = 1 - (1 - x)^2 = -x^2 + 2x$

2 $\int_0^x h(t)dt = g(x) = A(x)$

3 $0,078 > 0,069$

4 a) $\lim_{+\infty} f(x) = 1$

5 b) $g\,(0.12) = e^{-(0.12)^2+0.12}$

4 Discussion

The evolution of learned mathematics influences school mathematics, therefore it requires us to think about adapting this knowledge which requires modifying the structure, the sequence, the form, and the context of knowledge.

For the transposition realized in the previous part, we have realized a transposition at the level of form, structure and context, also we have chosen an example of a very familiar to the learners. Next, we are going to try to cover all the chain of process, therefore, we will divide this discussion according to the stages of transposition as the following.

4.1 External transposition

Didacticians confirm that external didactic transposition is a "schooling", or a "didactization" of scholarly knowledge. This process consists in a decontextualization followed by a recontextualization of the knowledge to be taught. The scholarly knowledge is then removed from its initial epistemological context to be positioned in a pedagogical context.

The transposed scholarly knowledge which is a concept validated and labelled by the scientific community around 1942. And among the transformations to be made to scholarly knowledge, Paun [8] cites the simplification, the introduction of certain terminological equivalents, and the introduction of figurative aspects.

4.1.1 Simplification

This is a simplified representation of the scientific reference model. The simplification must be carried out while respecting both the basic scientific meaning and the conceptual identity of the scientific referent.

For that, we tried to introduce this concept with a simplification that does not deform this notion as well as possible, and this simplification appears by choosing an example of simple and important metric space which is Euclidean space.

4.1.2 Terminological transposition

the terminological transposition is designed to determine the equivalent terms to render the content accessible to learning. This operation must be carried out with no semantic change of scientific concepts.

4.1.3 Figurative aspects

The presentation of figurative aspects in the didactic texts allows the learners to imagine the abstract concepts of the scientific texts.

The textual and geometrical figures used in the example help learners to better conceive this concept.

4.2 Internal transposition

The internal word means that this transposition occurs inside the teacher-student relationship, it composes the objectification of the differences in the relationship between them and the formal curriculum. This relationship is personalized, ideologized, axiologized, and sociologized. We can consider the internal didactic transposition as a process of specification and new curricular meaning.

In order to understand the transformations concerning the internal transposition, we must look at the framework of the analyses concerning the didactic contract that describes the action and the reciprocal relation - in the cognitive and socio-affective plan - between the teacher and his pupil.

The result of these transformations (which form the substance of the internal didactic transposition) is materialized by two types of curriculum:

a) The real curriculum or "the taught knowledge": represents in this case, the new notion that is a function of distance distribution that is introduced by the teacher and constructed by the student using a simple function and a geometric and functional framework.

b) The realized curriculum or, according to Chevallard's expression, "the learned and retained knowledge", which represents a personalized curriculum, expressing the particular relationship of the student to the taught concept, the objective is to unbalance the cognitive and sociocultural conceptions of the students on the notion of distance and to approach another more general notion (function distribution distance).

5 Conclusion

Every day, mathematical knowledge evolves and develops, of which there is appearance of new objects and tools which are going to play an essential role in the field of sciences and engineering, what requires to crush the gap between this knowledge and school knowledge.

In the example introduced in this work, we are limited to the semi-metric spaces, in our next work, we are going to give other examples of transposition of the probabilistic metric spaces by approaching the triangular norm, and by improving certain aspects concerning the construction, acquisition, and sharing.

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