Analytical solutions different analytical solutions in the nonlinear vibration of bridge panels

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Abstract. To study the nonlinear vibration response of a bridge panel, a mechanical model based on Von Karman theory was established. This model led to the formulation of the vibration equation for a large deformation plate, resulting in the Duffing equation through Galerkin integration. Subsequently, a multi-scale analytical solution was employed to analyze the system's dynamic response and investigate the impact of material nonlinearity and external excitation parameters on the system response. Finally, the accuracy of the analytical solution was confirmed using the incremental harmonic balance method and the Runge-Kutta method.

Keywords: nonlinear vibration, harmonic balance method, multiple scales, Duffing equation.

1 Introduction

1.1 Nonlinear vibration phenomena

For a Duffing system, in addition to the main resonance, secondary resonances occur when the excitation frequency is far away from the natural frequency of the derived system, including 3 superharmonic resonances and 1/3 subharmonic resonances[1]. When there are different modes, the two modes will interact to form an internal resonance, which will affect the complexity of the system[2].

1.2 Numerical solution of nonlinear vibration

At present, there are many kinds of nonlinear vibration solving algorithms at home and abroad. For example, the generalized average method adopted by Seth R. Sanders and J. Mark Noworolski[3] can find applications in simulation and design, and may be applicable to a wider range of circuits and systems. Wang and Zu[4] used the harmonic balance method to study the large amplitude vibration of material thin plate and obtained the nonlinear vibration response of the structure. Ji and Wu[5] used the multi-scale method to demonstrate the influence of nonlinearity on the free vibration and

G. Zhao et al. (eds.), *Proceedings of the 2023 5th International Conference on Civil Architecture and Urban Engineering (ICCAUE 2023)*, Atlantis Highlights in Engineering 25,
https://doi.org/10.2991/978-94-6463-372-6_25

forced vibration of a nonlinear rotor-bearing system by numerical simulation. H Ambriz-Perez et al. [7] discussed the Newton-Raphson method and the Newtonian optimal power flow SVC model. Liu et al. [8] proposed a new semi-analytic method, namely the time-domain minimum residual method. Different from the existing approximate analytic methods, this method does not rely on small parameters and can quickly converge to an exact analytic solution. JHE Cartwright and O Piro[9] used Runge-Kutta method to apply dynamical system theory techniques to numerical analysis. AR Narayanan[6] used the incremental harmonic balance method to study the periodic motion of a nonlinear gear-bearing system.

1.3 The research idea of this paper

This paper establishes the vibration equation of plates with large deformation based on Von Karman theory and derives the Duffing equation through Galerkin integration. It analyzes the dynamic response using a multi-scale analytical solution and assesses the impact of nonlinear and external excitation parameters of various materials on the system's response. The accuracy of the analytical solution is verified using the harmonic balance method and the Runge-Kutta method.

2 Nonlinear solution example

2.1 Method of multiple scales

Set the external load $P = F\cos(\Omega\tau)$ as simple harmonic excitation and adopt dimensionless:

$$\tau = \omega_0 t, \omega_0 = \sqrt{\Lambda_2}, \bar{\Omega} = \frac{\Omega}{\omega_0}, \mu = \frac{\Lambda_1}{\sqrt{\Lambda_2}}, u = \frac{q}{h}, \ddot{u} = \frac{d^2 u}{d\tau^2}, \dot{u} = \frac{du}{d\tau} \tag{1}$$

Then the dimensionless equation of the system is:

$$\ddot{u}(\tau) + \delta_1 \dot{u}(\tau) + u(\tau) + \delta_3 u^3(\tau) = f\cos\bar{\Omega}\tau \tag{2}$$

Where:

$$\delta_1 = \frac{\Lambda_1}{\omega_0}, \delta_3 = \frac{\Lambda_3 h^2}{\omega_0^2}, f = \frac{F}{\omega_0^2 h} \tag{3}$$

When the external excitation frequency is close to the natural frequency, the resonance phenomenon is the main resonance, and the dimensionless small parameters ε are introduced, we get:

$$\ddot{u}(\tau) + u(\tau) = \varepsilon f\cos\bar{\Omega}\tau - \varepsilon\delta_1 \dot{u}(\tau) - \varepsilon\delta_3 u^3(\tau) \tag{4}$$

Let the first order approximate solution of (4) be:

$$u(\tau,\varepsilon)=u_0(\Gamma_0,\Gamma_1)+\varepsilon u_1(\Gamma_0,\Gamma_1) \tag{5}$$

Where, $\Gamma_n=\varepsilon^n\tau$ is the time variable of different scales:

$$\frac{d}{d\tau}=\frac{\partial}{\partial\Gamma_0}\frac{d\Gamma_0}{d\tau}+\frac{\partial}{\partial\Gamma_1}\frac{d\Gamma_1}{d\tau}+\frac{\partial}{\partial\Gamma_2}\frac{d\Gamma_2}{d\tau}+\ldots \tag{6}$$

The coefficients of the same ε on both sides of the equation are equal, we get:

$$\begin{cases} D_0^2u_0+u_0=0 \\ D_0^2u_1+u_1=-2D_0D_1u_0-\delta_1D_0u_0-\delta_3u_0^3+f\cos(\Gamma_0+\sigma\Gamma_1) \end{cases} \tag{7}$$

Substituting into the two equations and combining with Euler's formula, we get:

$$D_0^2u_1+u_1=-2i\dot{A}e^{\bar{i}\Gamma_0}-i\delta_1Ae^{\bar{i}\Gamma_0}-3\delta_3A^2\bar{A}e^{\bar{i}\Gamma_0}-A^3\bar{A}e^{\bar{i}3\Gamma_0}+\frac{f}{2}e^{\bar{i}\sigma\Gamma_0}e^{\bar{i}\Gamma_0}+cc \tag{8}$$

Let the duration term to equal nothing, then we have:

$$-2\bar{i}\dot{A}-\bar{i}\,\delta_1A-3\delta_3A^2\bar{A}+\frac{f}{2}e^{\bar{i}\sigma\Gamma_1}=0 \tag{9}$$

Substituting $A=\frac{1}{2}\alpha e^{\bar{i}\beta}$ in yields:

$$-2\bar{i}\left(\frac{1}{2}\dot{\alpha}+\frac{1}{2}\alpha\bar{i}\dot{\beta}\right)-\frac{1}{2}\delta_1\bar{i}\,\alpha-3\delta_3\frac{1}{8}\alpha^3+\frac{f}{2}e^{\bar{i}\sigma\Gamma_1-\bar{i}\beta}=0 \tag{10}$$

Separate the real and imaginary parts and combine Euler's formula, $e^{\bar{i}\sigma\Gamma_1-\bar{i}\beta}=\cos(\sigma\Gamma_1-\beta)+\bar{i}\sin(\sigma\Gamma_1-\beta)$, $\varphi=\sigma\Gamma_1-\beta,\dot{\varphi}=\sigma-\dot{\beta}$, to have:

$$\begin{cases} \alpha\dot{\varphi}=\alpha\sigma-\frac{3}{8}\delta_3\alpha^3+\frac{f}{2}\cos\varphi \\ \dot{\alpha}=-\frac{1}{2}\delta_1\alpha+\frac{f}{2}\sin\varphi \end{cases} \tag{11}$$

For the steady-state response, $\dot{\alpha}=0,\alpha\dot{\varphi}=0$, the sum of squares of the two equations of φ, is eliminated, and the amplitude-frequency response equation of the main resonance is obtained:

$$\alpha^2\left[\frac{\delta_1^2}{4}+\left(\sigma-\frac{3\alpha^2}{8}\delta_3\right)^2\right]=\frac{f^2}{4} \tag{12}$$

Phase-frequency response equation:

$$\varphi=\arctan\frac{\frac{\delta_1}{2}}{\frac{3\alpha^2}{8}\delta_3-\sigma}=\arctan\frac{\frac{\delta_1}{2}}{\frac{3\alpha^2}{8}\delta_3-\Omega+1} \tag{13}$$

Multiply both sides of ε^2 , then the original system parameters can be expressed as:

$$\alpha^2\varepsilon^2\left[\frac{\delta_1^2\varepsilon^2}{4}+\left(\Omega-1-\frac{3\alpha^2\varepsilon^2}{8}\delta_3\right)^2\right]=\frac{\varepsilon^2 f^2}{4} \tag{14}$$

In the amplitude-frequency response diagram obtained using the multi-scale method, damping and external excitation amplitude impact the peak of the response curve such as Figure 1 and Figure 3, and nonlinear stiffness affects its shape such as Figure 2. When $\sigma \le 0$, different excitation amplitudes f correspond to unique response amplitudes; When $\sigma > 0$, the value of f is small, the response amplitude appears multiple solutions and jumps, and with the increase off, the response amplitude returns to the single solution, seeing Figure 4.

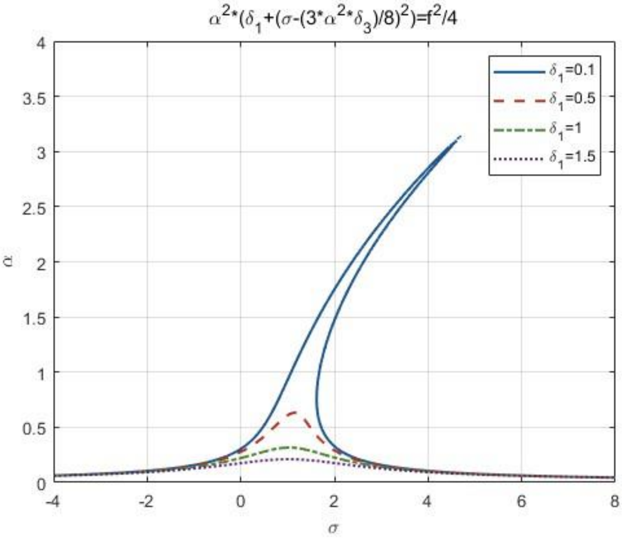

Fig. 1. The amplitude-frequency response curve of the steady-state main resonance with damping variation.

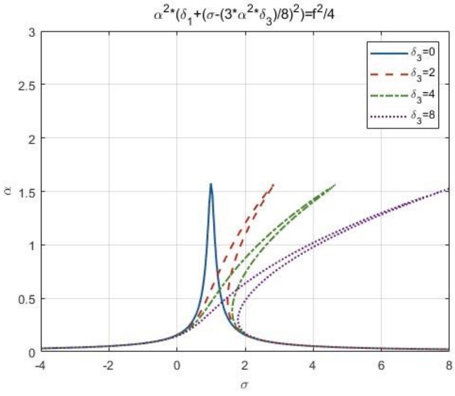

Fig. 2. Amplitude-frequency response curve of steady-state main resonance with nonlinear stiffness change.

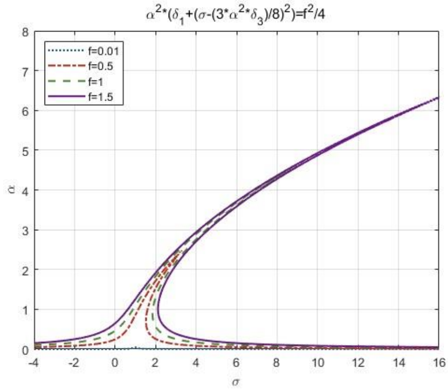

Fig. 3. Amplitude-frequency response curve of steady-state main resonance with external excitation amplitude change.

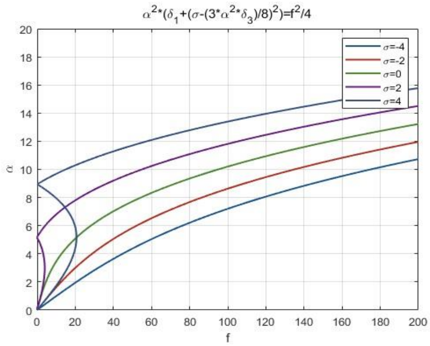

Fig. 4. Relation between excitation amplitude and steady-state response amplitude.

2.2 Strong nonlinear vibration (harmonic balance method)

The Duffin equation with the same parameters as the multi-scale method is:

$$\ddot{u}(t)+\delta_1\dot{u}(t)+u(t)+\delta_3u^3(t)=f\cos\omega t \tag{15}$$

When the external excitation frequency is close to the natural frequency, the resonance phenomenon is the main resonance. The dimensionless small parameter is introduced. The equation (16) can be rewritten as the corresponding value of the harmonic balance method formula:

$$\ddot{u}(t)+\omega_0^2u(t)=\varepsilon\left[f\cos\omega t-\delta_1\dot{u}(t)-\delta_3u^3(t)\right] \tag{16}$$

Where $\omega_0=1$

Let the first order approximate solution of (25) be:

$$u(t)=A_0\cos\omega t+B_0\sin\omega t \tag{17}$$

Substituting formula (26) into (25) gives:

$$\begin{aligned}&\cos\omega t\left(f\varepsilon-B_0\varepsilon\omega\delta_1-\frac{3}{4}A_0^3\varepsilon\delta_3-\frac{3}{4}A_0B_0^2\varepsilon\delta_3\right)+\cos3\omega t\left(-\frac{1}{4}A_0^3\varepsilon\delta_3+\frac{3}{4}A_0B_0^2\varepsilon\delta_3\right)\\&+\sin\omega t\left(A_0\varepsilon\omega\delta_1-\frac{3}{4}B_0^3\varepsilon\delta_3-\frac{3}{4}A_0^2B_0\varepsilon\delta_3\right)+\sin3\omega t\left(\frac{1}{4}B_0^3\varepsilon\delta_3-\frac{3}{4}A_0^2B_0\varepsilon\delta_3\right)=0\end{aligned} \tag{18}$$

Harmonic balance method obtains the same harmonic coefficient, ignores the third harmonic term, makes the coefficient of the first harmonic term 0, and only considers the main resonance case, is $\omega=\omega_0+\varepsilon\sigma$

$$\begin{cases}f\varepsilon-B_0\varepsilon\omega\delta_1-\frac{3}{4}A_0^3\varepsilon\delta_3-\frac{3}{4}A_0B_0^2\varepsilon\delta_3=0\\A_0\varepsilon\omega\delta_1-\frac{3}{4}B_0^3\varepsilon\delta_3-\frac{3}{4}A_0^2B_0\varepsilon\delta_3=0\end{cases} \tag{19}$$

Let $\omega=\omega_0+\varepsilon\sigma$, $A_0=a\cos\phi, B_0=-a\sin\phi$ and ignore ε^2 to get:

$$\begin{cases} a\cos\phi\sigma + \dfrac{a\sin\phi\delta_1}{2} - \dfrac{3}{8\omega_0}\delta_3 a^2\cos^3\phi - \dfrac{3}{8\omega_0}\delta_3 a^2\cos\phi\sin^2\phi = -\dfrac{f}{2\omega_0} \\ a\sin\phi\sigma + \dfrac{a\cos\phi\delta_1}{2} - \dfrac{3}{8\omega_0}\delta_3 a^2\sin\phi\cos^2\phi - \dfrac{3}{8\omega_0}\delta_3 a^2\sin^3\phi = 0 \end{cases} \tag{20}$$

Add the squares of the two formulas at the same time and eliminate ϕ:

$$\alpha^2\left[\frac{\delta_1^2}{4} + \left(\sigma - \frac{3\alpha^2}{8\omega_0}\delta_3\right)^2\right] = \frac{f^2}{4\omega_0} \tag{21}$$

3 Conclusions

In this work, the vibration equation of plates with large deformation based on Von Karman theory and derives the Duffing equation through Galerkin integration. It analyzes the dynamic response using a multi-scale analytical solution and assesses the impact of nonlinear and external excitation parameters of various materials on the system's response. Through the harmonic balance method and the Runge-Kutta method analysis, several major remarks can be summarized as follows:

- It is the same as the principal common amplitude frequency response equation of the multi-scale method, which verifies the accuracy of the harmonic balance method.
- In the amplitude-frequency response diagram obtained using the multi-scale method, damping and external excitation amplitude impact the peak of the response curve, and nonlinear stiffness affects its shape.

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