# A Theoretical Counterexample of Solutions to Unsaturated Infiltration Equations

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**Abstract.** A common method for obtaining the analytical solution of the unsaturated infiltration Richards equation under the first type of boundary condition is the Laplace transform. By linearizing the differential equation through the Laplace transform, solving it, and then applying the inverse transform, the result can be obtained. In this process, the traditional calculation scheme assumes that the original function converges, which is correct in most engineering cases, but theoretically, counterexamples may exist. This article presents a classic counterexample: when the unsaturated infiltration curve has a power-law form, the original integral diverges, resulting in the non-existence of the Laplace transform and the solution of the equation cannot represent the real infiltration situation. Based on this, this article suggests discussing the convergence of the original function before using the Laplace transform to solve the unsaturated infiltration equation to ensure that the results have mathematical significance. The content discussed in this article can be regarded as an improvement and supplement to the traditional unsaturated infiltration calculation scheme.

**Keywords:** Unsaturated infiltration; Richards equation; Laplace transform; Nonlinear differential equation; Convergence of functions.

## 1 Introduction

How to find the analytical solution of soil water infiltration equation has been a hot research topic(Andre Luis B. C.,2021;Cheng Y. G.,2008,Sharma R. H.,2010;)[1-3] in groundwater dynamics, soil hydrodynamics and unsaturated infiltration theory. Now,there are many existing types of soil water infiltration equations(Feng J. P. ,2016)[4] whose explicit expression are all highly nonlinear,such as Richards equation(Giorgio B.,2020)[5]. And this nonlinearity contains two aspects: One is the nonlinearity of the differential equation itself,the other is the nonlinearity of the main parameters in the equation, (such as unsaturated permeability $K(\theta)$, unsaturated diffusion coefficient $D(\theta)$ , which are all nonlinear functions(Dang F. N.,2022;Ren Z. S.,2021)[6-7] of moisture content, depth $z$, and time $t$) which lead to the exact analytical solution of the infiltration equation not easy to calculate, and in many cases it can not even be calculated.


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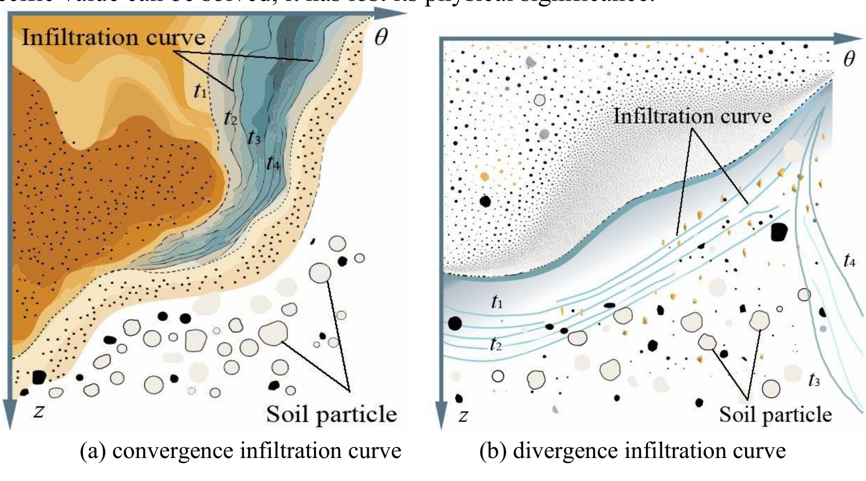
At present, the most common processing means is to use various "transformations" to simplify the infiltration equation. For example, Laplace transformation (Luo G.Y.,2021)[8],Boltzmann transformation(Li J. W., 2021)[9], minimum action principle variational transformation(Zhu Y. L.,2022)[10], etc.In fact these transformations is a linearized program,which in exchange for the acceptable approximate solution and semi-analytical solution(Zhu Y. L.,2022)[11] in the actual engineering by losing a certain accuracy of the equation. However, it is easy to find when reviewing the literature(Aryeni T., Ginting V.,2022;Dolojan N. L. J.,2021;Li L., Ju N.,2020)[12-14].The previous scholars and engineers, may due to the limitation of the academic background, often ignore the preceding mathematical conditions when using various linearization means. For example, in the most widely used linearized Richards equation of Laplace transformation, the premise of the transformation---when $t \to \infty$, the question is not discussed that the divergence speed of the original function does not exceed the exponential function in the historical literature. But all researchers default that the original function meets this condition or directly think that the original function converges, which is true in most engineering cases, but theoretically there may be counterexamples.

Moisture content $\theta(z,t)$ is a function of depth and time in the unsaturated infiltration process. In general, $\theta$ decreases as $z$ increases and increases when $t$ increases, as shown in Figure 1 (a). But it may exist a special curve in theory that under some extreme conditions, the function formula of the water content curve does not converge or even diverges faster. Mathematically, the value of the binary function $\theta(z,t)$ increases with the increase of $t$ and $z$, as shown in Figure 1 (b); This will lead to the situation that the traditional Laplace transform solution looks reasonable in expression but it is inconsistent with the actual project. At this time, although there is a numerical solution, the premise of Laplace transform no longer exists because of the rapid divergence of the function $\theta(z,t)$, and the simplified linear equation is no longer applicable. Even if a specific value can be solved, it has lost its physical significance.




(a) convergence infiltration curve (b) divergence infiltration curve

**Fig. 1.** Water infiltration curve in different soil texture and soil structure

Based on the above, this paper, for the Laplace linear transformation scheme of non-saturated infiltration Richards equation, puts forward three steps: first fitting the possible moisture content curve form, and then judging the moisture content curve convergence by mathematical means, finally determining whether the simplification transformation can be performed. Improving the theory of unsaturated soil mechanics from the mathematical calculation, clarifying the simplified equation by the mechanism, whether it can represent the most real form of infiltration curve, and give the extreme working conditions that are not suitable for Laplace transformation. Through the counter example to illustrate the significance and necessity of this study in practical engineering.

It should be noted that although in most practical engineering projects, the overall rapid increase of soil moisture content with time and depth is not common, and it is even difficult to provide examples, from a rigorous mathematical perspective, inductive phenomena do not necessarily mean proof. In order for geotechnical engineering to rise from empirical to systematic theoretical science, it is necessary to strive for theoretical completeness, which is also the other significance of this article.

## 2 Laplace transform of Richards infiltration equation

### 2.1 Classical linear transformations and the form of solutions

The infiltration equation for Richards with a constant boundary water content and a constant diffusivity $\overline{D}$ and without considering the gravity term and its boundary conditions are shown in equation (1):

$$\frac{\partial \theta}{\partial t} = \overline{D}\frac{\partial^2 \theta}{\partial z^2} , \quad \begin{cases} \theta = \theta_0, (t > 0, z = 0) \\ \theta = \theta_i, (t > 0, z = \infty) \end{cases} \tag{1}$$

In the equation, $\theta_0$ is the boundary water content of soil mass, $\theta_i$ is the initial water content of soil mass, and *t* is the infiltration time, the Laplace transform with $\theta(z,t)$ as the original function in equation (1) is(Lei Z. D.,1988)[15]:

$$\tilde{\theta}(z,p) = L_t[\theta(z,t)] = \int_0^{\infty} e^{-pt} \theta(z,t)dt \tag{2}$$

The Laplace transform symbol $L_t[\ ]$ above means that the variable *t* is transformed with respect to the function $\theta(z,t)$ , $\tilde{\theta}(z,p)$ is the image function, and *p* is the parameter. Equation (3) is obtained from the Laplace transform property(Hua L. G. I,2009; Qian X. S.,2011)[16-17]:

$$L_t\left[\frac{\partial \theta(z,t)}{\partial t}\right] = p\tilde{\theta}(z,p) - \theta(z,0) \tag{3}$$

The second partial derivative of *z* on both sides of the first equal sign of formula (2) can obtain formula (4):

$$L_t\left[\frac{\partial^2\theta(z,t)}{\partial z^2}\right]=\frac{\partial^2\tilde{\theta}(z,p)}{\partial z^2} \tag{4}$$

Substituting (3) and (4) into (1) and simplifying them respectively, this which can get the Laplace transformed Richards equation. The equation is a second-order linear differential equation with constant coefficients about image function, as shown in (5):

$$\frac{\partial^2\tilde{\theta}(z,p)}{\partial z^2}-\frac{p}{D}\tilde{\theta}(z,p)=-\frac{\theta_i}{D} \tag{5}$$

For the equation (5), there is a well-established mathematical solution to obtain the analytical solution, and the specific solution steps are described in the relevant mathematical textbooks or engineering manuals(Braun M. ,1992) [18], which will not be repeated in this paper. The special solution is:

$$\theta(z,t)=(\theta_0-\theta_i)erfc\left(\frac{z}{2\sqrt{Dt}}\right)+\theta_i \tag{6}$$

In the equation, $erfc(\cdot)$ is the Gaussian error function. The general expression is : $erfc(x)=\frac{2}{\sqrt{\pi}}\int_x^{\infty}e^{-y^2}dy$, where *x* and *y* are variables. Thus we can get the approximate infiltration curve $\theta(z,t)$ of the Richards infiltration equation after Laplace transformation.Since the value of the $erfc(\cdot)$ function does not exceed 1 (Jay L. Devore. ,2011)[19], combined with the definitions of $\theta_0$ and $\theta_i$ and the operational rules, it is easy to obtain the value of equation (6) is constantly less than 1, and the moisture content calculation results converge.

## 2.2 Problems of classical transformation

The above discussion shows that the infiltration equation has an approximate solution in the form of equation (6) after the classical transformation, and it is known that the convergence of the water content calculation can be constrained by the value domain of the Gaussian error function in equation (6). However, in fact, it is unknown in advance whether the infiltration curve $\theta(z,t)$ converges or not in practical engineering. And the essence of the above step is that, within the integral sign at the right end of equation(2), the researcher defaults in advance that $\theta(z,t)$ is continuous and convergent to ensure the subsequent operation. Due to the fixed nature of the solution of second-order linear differential equation with constant coefficients, it is always possible to arrive at a form of the solution like equation (6), which is physically equivalent to using a unique curve form to fit all possible cases of infiltration . But it is obviously not rigorous in mathematics.

Therefore, this paper argues that before Laplace transformation, a priori tests are required to estimate the initial shape of the curve of a certain soil sample in the actual infiltration process and judge its convergence. Only when it is determined that the a priori function $\theta(z,t)$ in equation (2) diverges slower than the exponential function specified by the Laplace transformation condition, the Laplace transformation can be applied to linearize the equation and the results are only physically meaningful when fitted by (6) equation. In the following subsections, this paper will address this issue.

## 3 Several common convergence judgments of infiltration curves

There are many mathematical schemes to compare the divergence degree of different functions. For the actual problem in this study, the water content function value is always greater than zero, so the "comparison limit method" (Tongji University, 2005)[20]can be used to judge : take any two functions $f(t)$ and $g(t)$. When $t \to \infty$, if $\lim f(t)/g(t) \to \infty$, the divergence rate of $f(t)$ is faster than $g(t)$; If $\lim f(t)/g(t) \to 0$, the divergence speed of $f(t)$ is slower than $g(t)$.

### 3.1 Convergence of polynomial morphological infiltration curves

When the $\theta(z,t)$ curve has a polynomial form, take the quadratic polynomial as an example whose standard equation is:

$$\theta(z,t) = a_1 z + a_2 z^2 + a_3 z^3 + b_1 t + b_2 t^2 + b_3 t^3 + c_1 z t^2 + c_2 z^2 t + c_3 z t + d \quad (7)$$

The property of Laplace transformation shows that if (2) is true, it is necessary to have $\theta(z,t) \le He^{\beta t}$, where $H$ and $\beta$ are constants. It is equivalent to proving that the $\theta(z,t)$ divergence rate is slower than $He^{\beta t}$,when $t \to \infty$.Obviously, for polynomial (7), when $t \to \infty$,the above proposition holds, and the Laplace solution satisfies the transformation condition. In this case, it is feasible to apply Laplace transform to linearize the infiltration equation and find the approximate solution of equation (6).

### 3.2 Convergence of exponential form infiltration curves

When $\theta(z,t)$ has an exponential functional form, a common expression of the infiltration equation is equation (8):

$$\theta(z,t) = z^{Nt} \quad (8)$$

In the equation (8), $N$ is a constant. And in the actual infiltration process,if $z$ is constrained by engineering conditions to take a constant, it is easy to know that $zN^t$ and $He^{\beta t}$ have the same order of convergence ,when $t \to \infty$. So, When the infiltration depth is fixed, the exponential form infiltration curve will not disperse faster than $He^{\beta t}$, and this case can also be simplified by Laplace transformation.

### 3.3 Convergence and divergence of power exponential form infiltration curves

If under certain conditions, the depth $z$ varies continuously, at which point equation (8) is transformed from an exponential function to a power exponential function(In this case, $z$ and $t$ are independent variables) , when $z \to \infty$ , $t \to \infty$ ,We can get the equation (9):

$$\lim_{\substack{z\to\infty \\ t\to\infty}} \frac{z^{Nt}}{He^{\beta t}} = \infty \tag{9}$$

This shows that the infiltration curve $\theta(z,t)$ dissipates faster than $He^{\beta t}$ .At this point, the integral of the rightmost equal sign of equation (2) which is applied the Laplace transform does not converge.So the transform cannot hold, and in this case, the Laplace transform cannot be used to linearize the Richards equation, because its solution has lost its physical meaning.

Although this section only proposes a method to determine the convergence of the original function, it is evident that this method is not unique mathematically. The method of judgment varies depending on the type of function. This article only adopts a commonly used discrimination method, with the aim of introducing counterexamples. The simplicity of the method does not affect the main idea and conclusion of this article.

## 4 Counterexamples and comparisons

The soil in a certain study area has a power exponential infiltration curve form: $\theta(z,t) = z^{0.8t}$ , without considering its convergence, the Laplace transform is applied to simplify the infiltration equation for the following working conditions: $\theta_i = 0.037$ , $\theta_0 = 0.64$ , $\overline{D} = 0.42$ , and the calculated water content values at different depths $z$ for times $t$ of 20 min, 25 min, 30 min, 35 min, and 40 min, respectively.And calculating water content values at different depths $z$ for times $t$ of 20 min, 25 min, 30 min, 35 min and 40 min, respectively. The calculation results based on the linearized equation of equation (6) are shown in Table 1, which only lists the calculation results at each depth $z$ at $t$=35min for verification.The rest of the data are similar and are not listed because of the limited space.

**Table 1.** Linear Laplace Transform Solution

| $\theta_i$ | $\theta_0$ | $\overline{D}$ | *z*/cm | *t*/min | *q* | *erfc*(*q*) | Calculated value of moisture content |
|---|---|---|---|---|---|---|---|
| 0.037 | 0.64 | 0.42 | 5 | 35 | 0.65 | 0.3579 | 0.253 |
| 0.037 | 0.64 | 0.42 | 7 | 35 | 0.91 | 0.1981 | 0.156 |
| 0.037 | 0.64 | 0.42 | 9 | 35 | 1.17 | 0.0979 | 0.096 |
| 0.037 | 0.64 | 0.42 | 11 | 35 | 1.43 | 0.0431 | 0.063 |
| 0.037 | 0.64 | 0.42 | 13 | 35 | 1.70 | 0.0168 | 0.047 |
| 0.037 | 0.64 | 0.42 | 15 | 35 | 1.96 | 0.0058 | 0.041 |
| 0.037 | 0.64 | 0.42 | 17 | 35 | 2.22 | 0.0017 | 0.038 |
| 0.037 | 0.64 | 0.42 | 19 | 35 | 2.48 | 0.0004 | 0.037 |
| 0.037 | 0.64 | 0.42 | 21 | 35 | 2.74 | 0.0001 | 0.037 |

$$q = z / 2\sqrt{\overline{D}t}$$

The results of Table 1 calculations were plotted as a function curve at $\theta - z$, as shown in Figure 2. The curve is a cluster of concave function graphs, indicating that the water content decreases with increasing depth and increases with increasing infiltration time at the same depth. With the blue vertical line as a separation in the figure, in the calculation time, the linearized equation calculation results show that the infiltration depth is about 15 cm, and the water content is 0.037 of the initial water content in the $z$ >15 cm region, which indicates that the infiltration has not yet reached that region. In many engineering cases, the infiltration curve of this form is often found(Shi Z. N.,2022; Zhang C.,2020; Zhu Y. L.,2023) [21-23], and this calculation results "seems" not to be a problem.

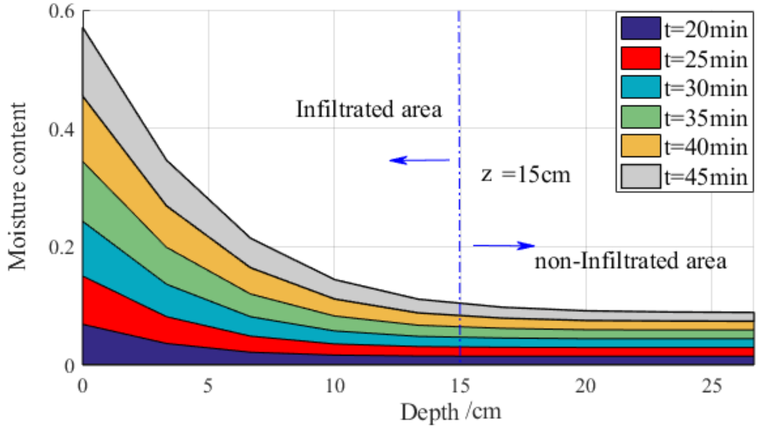



**Fig. 2.** Infiltration curve of linearized equation solution

Using the same data, the actual infiltration curve in the calculation example $\theta(z,t) = z^{0.8t}$ is substituted into the calculation, the results are shown in Figure 3.

Since the power exponential function grows too fast, Figure 3 uses a single logarithmic coordinate system in order to examine. But this does not affect the increase or decrease of the function; it can be seen that in the calculation time, The function $\theta(z,t)$ value increases with the increase of $z$ and $t$, All depths are permeable zones, and the result also shows that this extreme situation objectively exists mathematically.

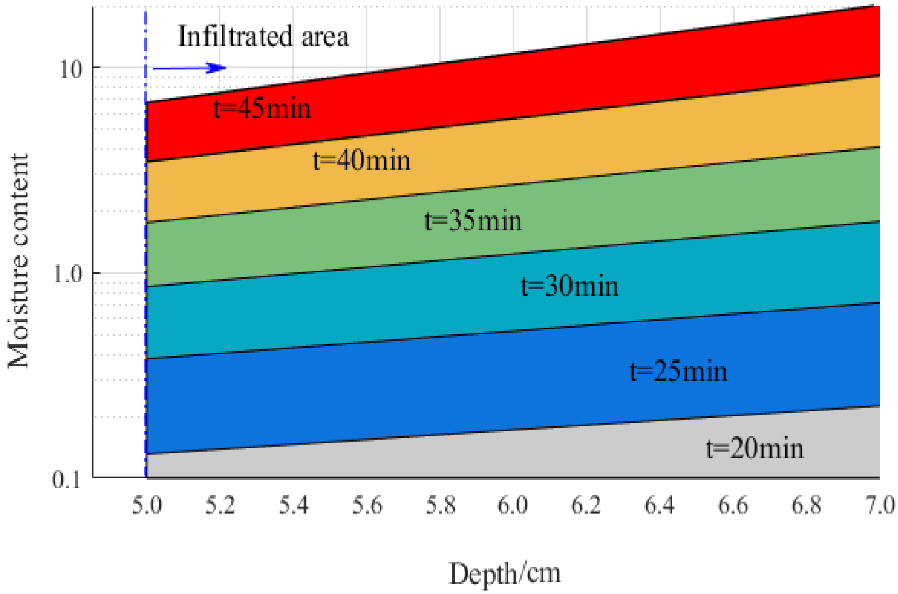



**Fig. 3.** Actual infiltration curve

In order to more intuitively demonstrate the above results, the linearized equation calculation results and the original equation calculation results are plotted in the three-dimensional coordinate system $\theta - z - t$ , as shown in figures 4 and 5. At this time, it can be clearly seen that the water content infiltration surface in both cases varies with *z and t*, showing the opposite concavity,which indicates that, at this time, no matter what method is used to improve the calculation accuracy of the linearized equation (6) equation itself, it is no longer effective to simulate the water content variation, and other methods need to be used. The mathematical essence of this phenomenon is that the traditional calculation scheme reduces the unsaturated infiltration problem to a generalized equation, and then converts this kind of equation into a definite solution problem only related to the initial conditions and boundary conditions. However, this differential equation that does not contain the information of the seepage field itself has theoretical flaws in some applications.

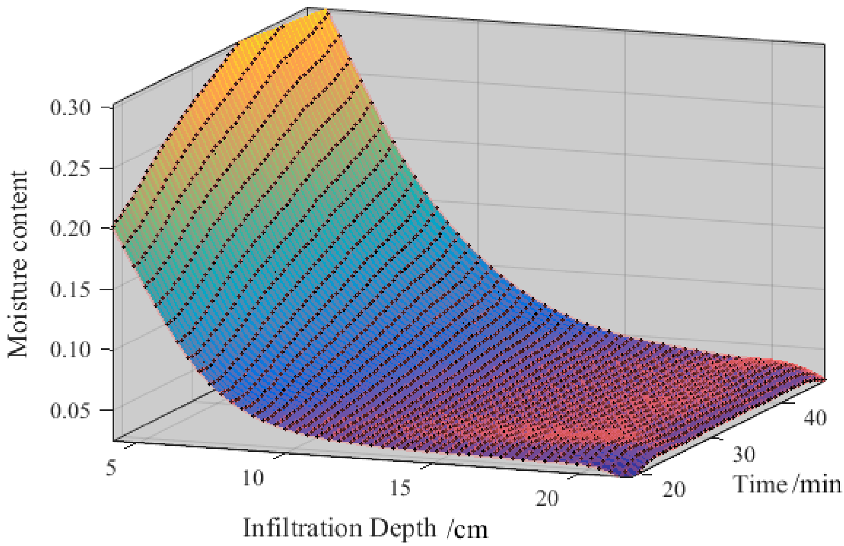



**Fig. 4.** Infiltration curves of linearized equation solutions in 3-dimensional coordinates

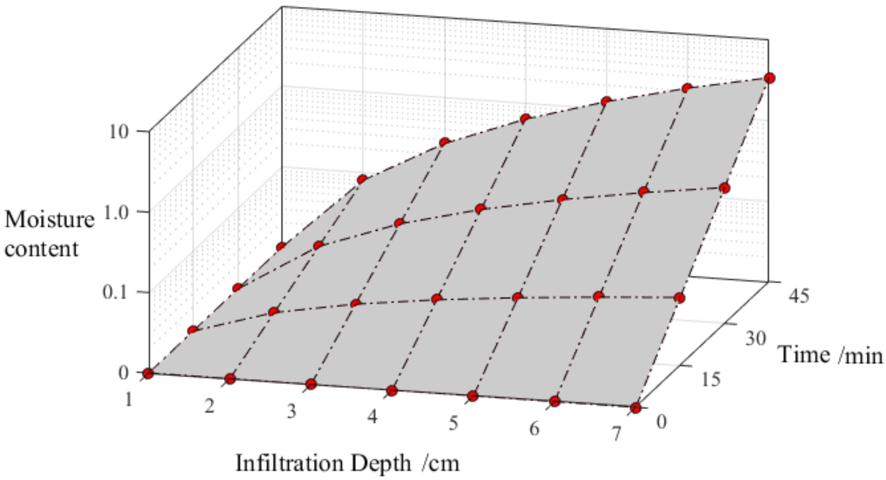



**Fig. 5.** Actual infiltration curve in 3-dimensional coordinates

The above counterexample proves that the Laplace transform simplifying Richards equation is not applicable to all cases. In fact, in actual engineering, due to the different soil properties, pore structure and engineering conditions, the true unsaturated infiltration curve shape varies greatly.And the corresponding infiltration curve equation can mathematically satisfy the strict convergence condition is also a minority. Therefore, the convergence judgment of the original function proposed in this paper is necessary both mathematically and physically before the simplification transformation.

# 5 Conclusions

In this study, the convergence of the original function during the Laplace transformation of the unsaturated infiltration Richards equation is discussed and the following conclusions are drawn.

(1) Laplace transform existence theorem still applies in unsaturated soil mechanics. But in the process of transformation, the scheme that directly default convergence of the original function or its divergence rate is less than $He^{\beta t}$ .This is not mathematically rigorous.

(2) In practical engineering, the infiltration curve $\theta(z,t)$ has polynomial form, exponential form, and the original function divergence speed are less than $He^{\beta t}$ divergence speed, so the linearized equation can be used to solve the infiltration curve after simplification. If the infiltration curve has a power exponential form, the original function divergence faster than $He^{\beta t}$ , then the Laplace transform integral divergence, the transform does not hold, so other methods need to be used to solve.

(3) The Laplace linearized solution scheme does not hold for any a priori test in which the divergence rate of the moisture content curve $\theta(z,t)$ is greater than that of $He^{\beta t}$ . The counterexample in this paper effectively verifies this conclusion.

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