Check for updates

A Conjecture on the Classification of Symmetric-(m^2+n^2+1, mn+1, λ)-design with mn+1 | m^2+n^2

Delu Tian* and Guiqiu Li
School of Mathematics
Guangdong University of Education
Guangzhou, Guangdong, 510303, P. R. China
tiandelu@gdei.edu.cn

Abstract. Suppose that $v = m^2 + n^2 + 1$, $k = mn + 1$, and $m, n \in N^*$, if $mn + 1$ divides $m^2 + n^2$, then there may be such a family of symmetric (v, k, λ) designs and the minimum value of v is 69.

Keywords: symmetric; block design; flag-transitive

1 Introduction

A balanced incomplete block design, namely a 2- (v, k, λ) design, is an arrangement of v points into b blocks such that each point occurs in r different blocks, each block contains k points and every pair of distinct points occur together in λ blocks. The parameters (v, b, r, k, λ) of a $2\text{-}(v,k,\lambda)$ design satisfy

$$vr = bk \quad (1)$$

$$\lambda(v-1) = r(k-1) \quad (2)$$

$$b \geq v \text{ (Fisher's inequality)} \quad (3)$$

If the block set **B** contains all the k-subset of the point set **P**, then this 2- (v, k, λ) design is called a full design. If $2 < k < v-1$ holds, then we speak of a non-trivial $2\text{-}(v,k,\lambda)$ design, written as simply 2-design [1]. If $b = v$, and thus $k = r$, then the 2-design is called a symmetric design.

In 2013, Tian and Zhou [2] researched flag-transitive point-primitive symmetric (v, k, λ) designs with λ at most 100. In 2015, they [3] completely classified all flag-transitive point-primitive symmetric (v,k,λ) designs with sporadic socle. In [4], Alavi S, Bayat M and Daneshkhah A, classified symmetric designs admitting flag-transitive and point primitive automorphism groups associated to two dimensional projective special groups. In this paper, we study the existence of the symmetric (v, k, λ) designs with $v = m^2 + n^2 + 1$, $k = mn + 1$, and $mn + 1 \mid m^2 + n^2$.

Our main result is the following.

R. Chitrakar et al. (eds.), *Proceedings of the 4th International Conference on New Computational Social Science (ICNCSS 2024)*, Advances in Social Science, Education and Humanities Research 844,
https://doi.org/10.2991/978-2-38476-230-9_14

Conjecture 1. Let $v = m^2 + n^2 + 1$, $k = mn + 1$, and m, $n \in N^*$. If $mn+1$ divides $m^2 + n^2$, then such a family of symmetric (v, k, λ) designs exists and the minimum value of v is 69.

2 Some research on Conjecture 1

We give some research on conjecture 1 in the following three subsections.

2.1 Some Preliminary Results

In this subsection we give some preliminary results which are used throughout the paper.

Lemma 2.1. [5, **PROBLEM 6**] Let m and n be two positive integers such that $mn+1$ divides m^2+n^2, then $\dfrac{m^2+n^2}{mn+1}$ is a perfect square.

Proof. Let $\dfrac{m^2+n^2}{mn+1} = t \in N^*$,

then $m^2 + n^2 = tmn + t$.

Since m and n occur symmetrically we may assume $m \le n$.

We claim that $tm - 1 < n \le tm$.

Firstly, we prove the right-inequality.

Suppose $n > tm$, then $n = tm + c$ with $c \in N^*$.

Thus from $m^2 + n^2 = tmn + t$,

we get $m^2 + (tm + c)^2 = tmn(tm + c) + t$,

so $m^2 + tmc + c^2 = t$.

Such that $t \le tmc < m^2 + tmc + c^2 = t$,

a contradiction.

Thus $n \le tm$.

Secondly, from

$$t = \frac{m^2+n^2}{mn+1} < \frac{m^2+n^2}{mn} = \frac{m}{n} + \frac{n}{m} \le 1 + \frac{n}{m},$$

we get the right-inequality $tm - 1 < n$.

From $tm - 1 < n \le tm$, we get $n = tm - r_1$ with $0 \le r_1 < m$ and $r_1 \in N$.

From $m^2 + n^2 = tmn + t$ and $n = tm - r_1$,

we get $\dfrac{m^2 + r_1^2}{mr_1 + 1} = t$.

Thus if one pair (m,n) satisfies $\frac{m^2+m^2}{mn+1}=t$, then also the pair (r_1,m) satisfies $\frac{r_1^2+m^2}{r_1m+1}=t$.

But we have $0\le r_1<m\le n$,so that each component of (r_1,m) is smaller than each component of (m,n). Since there is no infinite strictly monotonic decreasing sequence of positive integers, by applying twiddle and divide algorithm, we will get smallest pair (r_N,r_{N-1}) until $r_N=0$. From $\frac{r_N^{\ 2}+r_{N-1}^{\ \ 2}}{r_{N-1}r_N+1}=t$ and $r_N=0$, we get $t=r_{N-1}^{\ \ 2}$, thus t is a perfect square. □

NOTE: (1) The problem was submitted to the IMO in 1988 by the West Germany. In [6], Arthur Engel wrote the following note about its difficulty:

Nobody of the six members of the Australian problem committee could solve it. Two of the members were Georges Szekeres and his wife, both famous problem solvers and problem posers. Because it was a number theory problem, it was sent to the four most renowned Australian number theorists. They were asked to work on it for six hours. None of them could solve it in this time. The problem committee submitted it to the jury of the 29th IMO marked with a double asterisk, which meant a superhard problem, possibly too hard to pose. After a long discussion, the jury finally had the courage to choose it as the last problem of the competition.

Eleven students gave perfect solutions.

(2) We can find a large number of examples that satisfy the conditions of Lemma 2.1. For example,

let $n=m^3\in N^*$,

then $\frac{m^2+n^2}{mn+1}=\frac{m^2+(m^3)^2}{m\cdot m^3+1}=m^2$ is a perfect square.

Give another general example, let *m*=240, *n*=2133, then $\frac{m^2+n^2}{mn+1}=9$ is a perfect square. In fact, by applying twiddle and divide algorithm, $2133=9\times240-27$, $240=9\times27-3$, $27=9\times3-0$. Then the perfect square 9 is $r_2^{\ 2}$.

Lemma 2.2. [1, **Bruck-Ryser-Chowla Theorem**] Let v,k and λ be positive integers. If a 2-(v,k,λ) symmetric design exists and v is even, then $k-\lambda$ must be a square.

Lemma 2.3. [3, **Lemma 2.3**] Let **D** = (**P**, **B**) be a nontrivial 2-(v,k,λ) design and G be an automorphism group of **D**. For any point $x\in$ **P** and block $B\in$ **B**, the following three statements are equivalent:

(i) G acts flag-transitively on **D**;

(ii) G acts point-transitively on **D** and G_x acts transitively on B(x), where B(x) denotes the set of all blocks that are incident with a point *x*;

(iii) G acts block-transitively on **D** and G_B acts transitively on the points of *B*.

Lemma 2.4. Let *m* and n be two positive integers such that $mn+1$ divides m^2+n^2. If $m=n$, then $m=n=1$.

Proof. From Lemma 2.1, we get $\frac{m^2+n^2}{mn+1}$ is a perfect square, and assume that $\frac{m^2+n^2}{mn+1}=t^2$.

If $m=n$, then $\frac{2m^2}{m^2+1}=t^2$, and so $(2-t^2)m^2=t^2$.

Since t is a positive integer, we get $2-t^2>0$, then $t=1$, $m=n=1$.

□

2.2 Feasible Parameters of 2-designs

In this subsection we describe briefly our approach to search for feasible parameters of symmetric (v,k,λ) designs.

Let $m=n=1$, $v=m^2+n^2+1$, $k=mn+1$, we can get a 2-(3, 2, 1) design. Since this design's block set contains all the *k*-subset of point set, then this design is a full design. In fact, by $k=2$, it is also a trivial 2-(v,k,λ) design. Thus, by Lemma 2.4, we only need to consider the case that $m \neq n$. As *m* and *n* have equal status, it can be assumed that $m<n$.

We compute possible parameters (m,n) satisfying the following conditions:

(i) $1 \le m < n \le 10000$;

(ii) $mn+1 \mid m^2+n^2$.

By using the computer algebra system GAP [7], we get 30 two-tuples of parameters (m,n) listed in TABLE I.

TABLE I. FEASIBLE PARAMETERS (m,n)

m	2	3	4	5	6
n	8	27	64	125	216

m	7	8	8	9	10
n	343	30	512	729	1000

m	11	12	13	14	15
n	1331	1728	2197	2744	3375

m	16	17	18	19	20
n	4096	4913	5832	6859	8000

m	21	27	30	64	112
n	9261	240	112	1020	418

m	125	216	240	418	1560
n	3120	7770	2133	1560	5822

Now, compute possible parameters (v,k,λ) satisfying the following conditions:

(i) $v=m^2+n^2+1,\ k=mn+1$;

(ii) $\lambda=\dfrac{k(k-1)}{v-1}=\dfrac{mn(mn+1)}{m^2+n^2}$;

(iii) if v is even, then $k-\lambda$ is a square(see **Lemma 2.2**).

By using the computer algebra system GAP, we get 27 five-tuples of parameters (v,k,λ,m,n) listed in Table 2. In the following, "*No. i*" denotes the *i*-th "Feasible parameters" of TABLE II.

TABLE II. Feasible Parameters (v,k,λ,m,n)

No.	(v,k,λ,m,n)
1	(69,17,4,2,8)
2	(739,82,9,3,27)
3	(4113,257,16,4,64)
4	(15651,626,25,5,125)
5	(46693,1297,36,6,216)
6	(117699,2402,49,7,343)
7	(965,241,60,8,30)
8	(262209,4097,64,8,512)
9	(531523,6562,81,9,729)
10	(1000101,10001,100,10,1000)
11	(1771683,14642,121,11,1331)
12	(2986129,20737,144,12,1728)
13	(4826979,28562,169,13,2197)
14	(7529733,38417,196,14,2744)
15	(11390851,50626,225,15,3375)
16	(16777473,65537,256,16,4096)

17	(24137859,83522,289,17,4913)
18	(34012549,104977,324,18,5832)
19	(47046243,130322,361,19,6859)
20	(64000401,160001,400,20,8000)
21	(85766563,194482,441,21,9261)
22	(13445,3361,840,30,112)
23	(1044497,65281,4080,64,1020)
24	(187269,46817,11704,112,418)
25	(60419557,1678321,46620,216,7770)
26	(2608325,652081,163020,418,1560)
27	(36329285,9082321,2270580,1560,5822)

2.3 The Existence of Designs

Now, we study the existence of designs on conjecture 1.

Lemma 2.5. [8, **Construction** 3.18]&[9] Let $\lambda-1$ and $\lambda^2-\lambda+1$ be prime powers. Then, there exists a family of symmetric block designs with parameters $(v,k,\lambda)=(\lambda^3+\lambda+1,\lambda^2+1,\lambda)$.

Lemma 2.6. [10, **Corollary 5.4**] **Let** q and $r=(q^d-1)/(q-1)$ be prime powers. Then, for any positive integer a, there exists a symmetric design with parameters

$$(v,k,\lambda)=\left(1+\frac{qr(r^a-1)}{r-1},r^a,\frac{r^{a-1}(r-1)}{q}\right).$$

Corollary 2.7. There exist symmetric (69, 17, 4) design and symmetric (739,82,9) design.

Proof. Let $\lambda=4$, then both $\lambda-1=3$ and $\lambda^2-\lambda+1=13$ are prime powers. By **Lemma 2.5**, there exists a symmetric (69, 17, 4) design.

Let $\lambda=9$, then both $\lambda-1=8$ and $\lambda^2-\lambda+1=73$ are prime powers. By **Lemma 2.5**, there exists a symmetric (739,82,9) design. □

NOTE: (1) Let $m=2,n=8$, then $v=m^2+n^2+1=69$, $k=mn+1=17$, and $mn+1$ divides m^2+n^2. This is the "No. 1" in TABLE II. The same to "No. 2" in TABLE II.

(2) Now, some information about symmetric (69, 17, 4) design is given from reference [1]. Let $A,B,C,D,1_i,2_i,3_i,4_i,5_i,i\equiv 0,1,\cdots,12(\mathrm{mod}13)$, then the automorphisms are

$\rho=(A)(B)(C)(D)(I_0,I_1,\cdots,I_{12}),I\equiv 1,2,\cdots,5,$

$\tau = (ABCD)(1_i 2_i 3_i 4_i)(5_i), i \equiv 0,1,\cdots,12(\mathrm{mod}\,13)$.

The base blocks are:

$ABCD1_0 1_1 1_2 1_3 1_4 1_5 1_6 1_7 1_8 1_9 1_{10} 1_{11} 1_{12}$,

$A1_1 1_3 1_9 2_2 2_6 2_5 3_4 3_{12} 3_{10} 4_8 4_{11} 4_7 5_0 5_4 5_{12} 5_{10}$,

$1_0 1_1 1_3 1_9 2_0 2_1 2_3 2_9 3_0 3_1 3_3 3_9 4_0 4_1 4_3 4_9 5_0$.

(3) In the future, **Lemma 2.3** and **Lemma 2.6** can be used to study the classification problem of this family of symmetric (v, k, λ) designs.

Acknowledgement. This work was supported in part by NSFC (Grant Nos. 12071092,11801092), and the Fund of Guangzhou Science and Technology (No.201804010088).

The authors would like to thank the referees for their valuable suggestions and comments which helped to improve this paper.

References

1. Colbourn C and Dinitz J, 2007, *The CRC Handbook of Combinatorial Designs* (CRC press, Boca Raton, FL).
2. Tian D., Zhou S., 2013, Flag-transitive point-primitive symmetric (v, k, λ) designs with λ at most 100, *J. Combin. Designs*, **21(4** , 127-141.
3. Tian D and Zhou S, 2015, Flag-transitive 2-(v,k,λ) symmetric designs with sporadic socle. *J. Combin. Designs,* **23(4)**, 140-150.
4. Alavi S, Bayat M and Daneshkhah A, 2016, Symmetric designs admitting flag-transitive and point primitive automorphism groups associated to two dimensional projective special groups, Designs*, Codes and Cryptography*, **79**,337-351.
5. Alfaro L, 2011, Another perspective on a famous problem, IMO 1988: The equation $\frac{x^2+y^2}{xy+1} = n^2$. *Boletín de la Asociación Matemática Venezolana,* **XVIII(2)** , 143-151.
6. Engel A, 1999, *Problem-Solving Strategies*, Springer.
7. The GAP Group, *GAP-Groups, Algorithms, and Programming*, Version 4.4, 2005, (http://www.gap-system.org).
8. Rajkundlia D. P., 1983, Some techniques for constructing infinite families of BIBD's, *The Electronic Journal of Combinatorics Discrete Math.*, **44**, 61-96.
9. Shrikhande S. S., Singhi N. M., 1975, Construction of geometroids, *Utilitas Math.,* **8**, 187-192.
10. Ionin J., 2001, Applying balanced generalized weighing matrices to construct block designs, *The Electronic Journal of Combinatorics*, R**12(8)**, 1-15.s