Approximate functions for geometric constants of the rectangular cross-section in torsion

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Abstract. Simple approximate functions of constants used for stress and deformation calculations of a straight bar with rectangular cross-section under torsion with sufficient precision were derived. The maximal error of these functions is around 1.4%, but mostly less. Functions are intended for practical or algorithmic use in mentioned calculations, where usually a table of constants needs to be used, which is not very convenient.

Keywords: *rectangular cross-section, torsion, constants, simplified functions.*

1 Introduction

Constants used for calculations of stress – *Eq.1* and deformation – *Eq.2* [1],[2] of rectangular cross-sections under torsion are usually available in the form of a table [1],[2],[3],[4],[5],[6] – *Table 1.*, which is not very practical in view of calculation as an algorithm or by manual calculation.

$$\tau_{max} = \frac{T}{\alpha \cdot b \cdot c^2} \tag{1}$$

Where τ_{max} is maximal shear stress in *MPa*, *T* is torque in *Nm*, *b* is the longer side of the rectangular cross-section in *cm*, *c* is the shorter side of the cross-section in *cm,* and α is the coefficient.

$$\theta = \frac{T \cdot l}{\beta \cdot b \cdot c^3 \cdot G} \tag{2}$$

Where θ is twist angle in *radians*, *T* is torque in *Nm*, *l* is the length of the twisted bar in *m*, *b* is the longer side of the rectangular cross-section in *m*, *c* is the shorter side of the cross-section in *m,* and α is the coefficient.

Table 1. Torsional constants for rectangular cross-section [2]

r	1	1.5	1.75	2	2.5	3	4	6	8	10	∞
α	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333

1.

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β	0.141	0.196	0.214	0.228	0.249	0.263	0.281	0.299	0.307	0.313	0.333

Of course, many modern software can pick a value from a table based on a result. Still, it usually requires advanced functions, and values between the tabular ones must be interpolated, complicating the calculation, and adding an error.

There are several equations for calculations of stiffness constants available as a result of Prandtl's formulation of the Saint-Venant torsion problem [3],[4],[5],[6],[7] which give slightly different results, and what is most important, are always very complicated to input into a software not to mention a handheld calculator, because they often use infinite sums. Therefore, much simpler yet accurate enough equations were derived, which do not require table picking features nor loops needed for the summations.

2 Materials and methods

2.1 Base equations

As basic equations, *Eq.3* and *Eq.4* [3],[6] were used.

$$\beta(r) = \frac{2^8}{\pi^6} \cdot \sum_{k \in N_{odd}} \sum_{l \in N_{odd}} \frac{r^2}{k^2 \cdot l^2 \cdot (k^2 + r^2 \cdot l^2)} \tag{3}$$

$$\alpha(r) = \beta(r) \cdot \frac{\pi^3}{2^5 \cdot r^2} \cdot \left[\sum_{k \in N_{odd}} \sum_{l \in N_{odd}} \frac{(-1)^{\frac{k-1}{2}}}{k \cdot (k^2 + r^2 \cdot l^2)} \right]^{-1} \tag{4}$$

Where r is the ratio between the longer and shorter side of the rectangular cross-section.

These equations were modified for more straightforward computer processing of odd numbers - *Eq.5* and *Eq.6*.

$$\beta(r) = \frac{2^8}{\pi^6} \cdot \sum_{k=1}^{n} \sum_{l=1}^{m} \frac{r^2}{(2k-1)^2 \cdot (2l-1)^2 \cdot ((2k-1)^2 + r^2 \cdot (2l-1)^2)} \tag{5}$$

$$\alpha(r) = \beta(r) \cdot \frac{\pi^3}{2^5 \cdot r^2} \cdot \left[\sum_{k=1}^{n} \sum_{l=1}^{m} \frac{(-1)^{\frac{2k-2}{2}}}{(2k-1) \cdot ((2k-1)^2 + r^2 \cdot (2l-1)^2)} \right]^{-1} \tag{6}$$

2.2 Simple equation derivation

Mathcad 14 software was used for the initial estimation of the functions and graphic outputs. The primary form of equations was created based on the course of *Eq.5* and *Eq.6* and mathematical knowledge.

The initial values of the coefficients were obtained using Python libraries NumPy [9] and SciPy [10]. Then they were manually modified and rounded for more simplification of the final equations.

Finally, statistical analysis was also made using Python libraries – NumPy and Math [11]. Nested loops were used for double sum. Obviously, they were not infinite. Thus $n = m = 1000$ was used for *Eq.5* and *Eq.6*, which was found sufficiently accurate. It means that 1 million loops, thus additions were performed, and the sums for all odd numbers lower than 2000 were used for each ratio value. The step of the ratio *0.1* for values from *1* to *10*, thus *1, 1.1 ... 9.9, 10* was used for statistical analysis.

3 Results and discussion

3.1 Final equations

Simple *Eq.7* and *Eq.8* were obtained from the whole process.

$$\alpha_s(r) = \frac{\log(r)}{6.8 \cdot \sqrt[7]{r}} + 0.208 \tag{7}$$

$$\beta_s(r) = \frac{\log(r)}{e \cdot \sqrt[3]{r}} + 0.141 \tag{8}$$

Where r is the ratio between the longer and shorter side of the rectangular cross-section, and α_s and β_s are coefficients obtained by simple equations.

The results of the simplified equation can be seen in *Table 2*. The results of precise equations and simplified equations can be compared in a graphical presentation in *Fig.1* and *Fig.2*. The relative error of simplified calculation can be seen in *Fig.3*.

Table 2. Torsional constants for rectangular cross-section calculated from simple equations

r	1	1.5	1.75	2	2.5	3	4	6	8	10
α	0.208	0.232	0.241	0.248	0.259	0.268	0.281	0.297	0.307	0.314

β	0.141	0.198	0.215	0.229	0.249	0.263	0.281	0.299	0.307	0.312

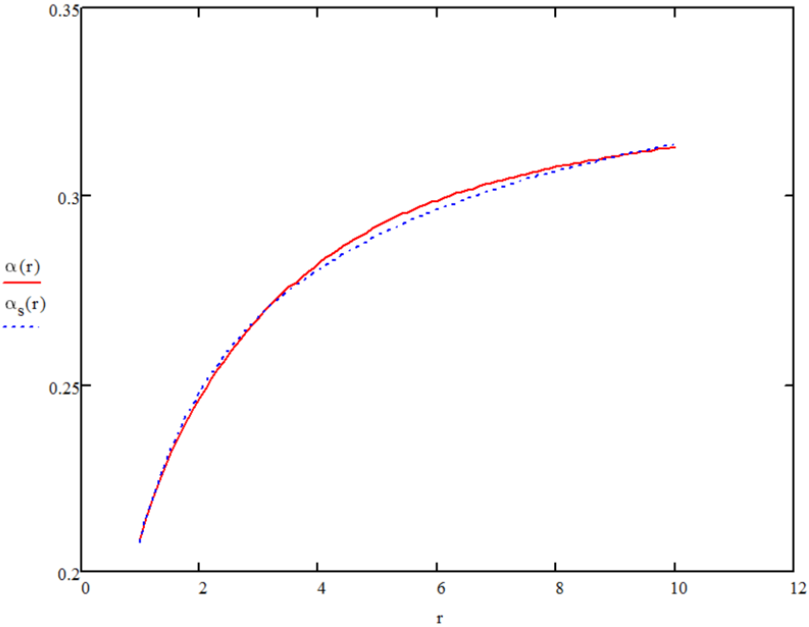

Fig. 1. Comparison between original *(red)* and simplified *(dotted blue)* functions for α.

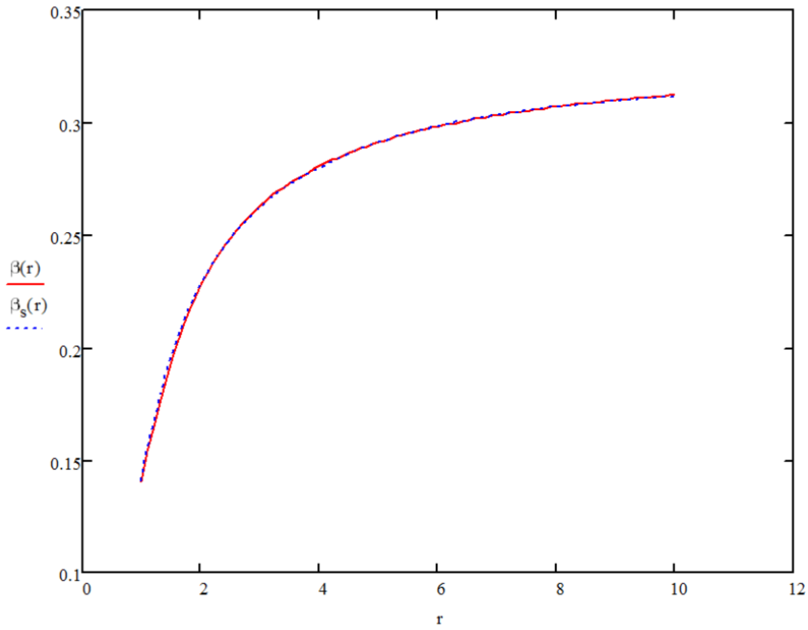

Fig. 2. Comparison between original *(red)* and simplified *(dotted blue)* functions for β.

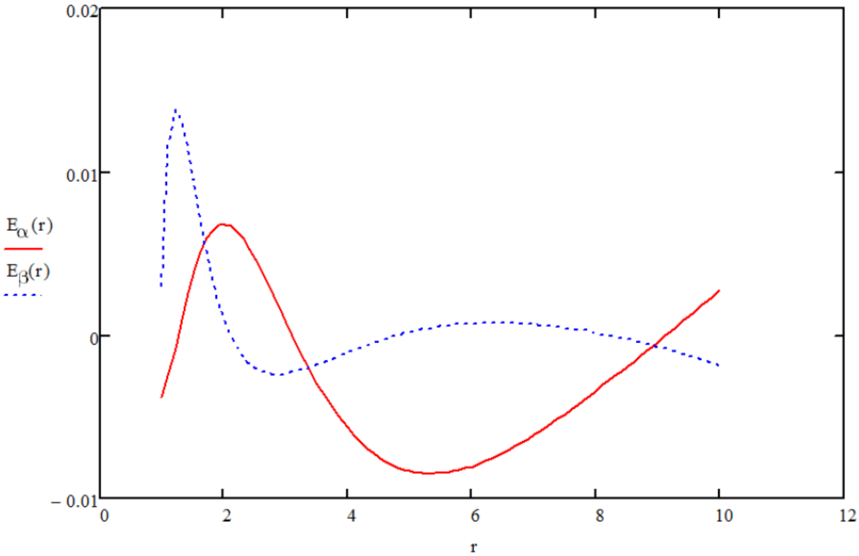

Fig. 3. Relative error for α *(red)* and β *(dotted blue)* simplified functions.

3.2 Discussion

Obviously, *Eq.7* and *Eq.8* are much simpler and more suitable for practical use than the infinite sums shown in *Eq.4* and *Eq.3* [3],[6]. Also, other solutions, which even use hyperbolic functions in infinite sums [4],[5] do not represent handy options.

Before statistical analysis, all values were rounded to 3 decimal places as it is usual precision practically used – *Tab.2*. Maximal relative error for the coefficient α is *-0.85%* at $r = 5.333$ and maximal error for the coefficient β is *1.4%* at $r = 1.2$ – *Fig.3*. Statistical analysis overview can be seen in *Tab.3*.

Table 3. Statistical analysis overview

	MSE	MAD	RMSE	MAPE	R^2	χ^2	EF
α	0.001325	0.000868	0.036765	0.471408	0.997199	0.000003	0.999991
β	0.000363	-0.000077	0.019043	0.157602	0.999686	0	1

Single factor ANOVA test with level of confidence 95% for α and α_s gives results F-value = 0.0486 < F-crit = 3.8936 and P-value = 0.8258 > 0.05 which confirms the hypothesis that the curves are significantly similar. For β and β_s the results are F-value = 0.0002 < F-crit = 3.8936 and P-value = 0.9894 > 0.05 which also confirms significant similarity of simple formula for the second coefficient.

There is an even more straightforward equation already known for α - *Eq.9* [1],[2],[8], but it is much less accurate, with the maximum error of *4.1%* around $r = 2$.

$$\alpha(r) = \frac{1}{3 + \frac{1.8}{r}} \tag{9}$$

Derived equations can also be used for values of ratio $r > 10$ with approximately linearly increasing error when at ratio $r = 20$, the maximal relative error for α is *2.8%* and for β the error is *-1.7%*. For $r = 50$, the maximal error for α is *6.3%,* and for β the error is *-5.6%*.

For values of ratio greater than 10, it is also appropriate to use linear interpolation between values $r = 10$, $\alpha = 0.314$, $\beta = 0.312$ and $r = 200$, $\alpha = 1/3$, $\beta = 1/3$. Around value *200*, the asymptotes become almost horizontal. However, thin rectangular bars with ratios significantly greater than *10* are not usually used for torsion in praxis.

4 Conclusion

Simple formulae for calculating coefficients used for strength and deformation calculation of straight bars with rectangular cross-sections were derived. Despite its simplicity, their accuracy in the side ratio ranges from *1* to *10* is very high, with a maximal relative error peak of *1.4%*, but mostly much less. These equations are suitable for manual calculations and computer calculations where automatic readout from a table or loops for nested sums are not available or too impractical. Also, when the table of these values is not available or particular value of the ratio is not in the table, these formulae give more accurate results than linear interpolation.

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