On Rainbow Antimagic Coloring of Graphs

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Abstract. Let G be a simple graph. For a bijective function \( f : V(G) \rightarrow [1, 2, \ldots, |V(G)|] \), the edge weight for any uv \( \in E(G) \) under \( f \) is \( w_f(uv) = f(u) + f(v) \). A path \( P \) in a graph \( G \) is said to be a rainbow path, if for every two edges \( uv, u'v' \in E(P) \), the edges weight satisfies \( w_f(uv) \neq w_f(u'v') \). If for every two vertices \( u \) and \( v \) of \( G \), there exists a rainbow path \( uv \), then \( f \) is called a rainbow antimagic coloring of graph \( G \). The minimum number of colors induced by all edge weight required is called rainbow antimagic connection number, denoted by \( \text{rac}(G) \). In this research, we will find the rainbow antimagic connection number \( \text{rac}(G) \) and new theorems on rainbow antimagic coloring of Double Quadrilateral Windmill Graph \( \text{DQ}_{4(n)} \), Double Quadrilateral Flower Graph \( \text{DFQ}_{4(n-1)} \), Split Graph Of Star \( G_{S_{p³(K_i,n)}} \) and Tunjung Graph \( T_{j,n} \).

Keywords: rainbow antimagic coloring, rainbow antimagic connection number, graph coloring.

INTRODUCTION

Graph theory is one of the mathematical topics that first appeared in 1736. Leonhard Euler a mathematician from Switzerland, was the first to introduce this theory in his writings containing efforts to solve the Konigsberg bridge problem. In this study, the definition of a graph \( G \) is a set pair \( (V(G), E(G)) \), where \( V(G) \) is a finite nonempty set of elements called vertex and \( E(G) \) is a (possibly empty) set of unordered pairs \( (u, v) \) of vertex \( u, v \in V(G) \) are referred to as edges [1] [2]. Then graph theory developed over time, until in 2008 Chartrand introduced a coloring called the rainbow connection. This rainbow connection is defined as coloring the edges of a graph such that no two edges in the path have the same color. The research goal of rainbow coloring is how to find the minimum number of colors called the rainbow connection number \( \text{rc}(G) \) [3] [4] [5] [6]. Recently, the topic of rainbow connection has been combined with antimagic labeling. Antimagic labeling is defined as labeling (giving number) on vertices of graph such that there is no same weight for any two distinct edges. A graph is called an antimagic labeling if there exists a bijective function \( f : E(G) \rightarrow [1, 2, \ldots, |E(G)|] \) such that all its vertex weights are different. The vertex weight of a vertex \( x \) denoted by \( w(x) \) which is the number of labels of each edge associated with \( x \), can be written as \( w(x) = \sum e \in E(x) f(e) \). In this case, \( f \) is called antimagic labeling [7] [8] [9] [10].

Dafik et al. (2019), combined rainbow coloring and antimagic labeling to create a new topic called rainbow antimagic connection number. If for any two vertex \( u \) and \( v \) of a graph \( G \), there exists a rainbow path \( uv \), then the function \( f \) is called rainbow antimagic labeling of the graph \( G \). If for every edge \( uv \) with weight color \( w(uv) \), then it can be called rainbow antimagic coloring. The rainbow antimagic connection number is denoted by \( \text{rac}(G) \), which is to find the minimum number of colors in a graph \( [11] \) [12] [13] [14]. The research in this article will find the rainbow antimagic connection number \( \text{rac}(G) \) on Double Quadrilateral Windmill Graph \( \text{DQ}_{4(n)} \), Double Quadrilateral Flower Graph \( \text{DFQ}_{4(n-1)} \), Split Graph Of Star \( G_{S_{p³(K_i,n)}} \) and Tunjung Graph \( T_{j,n} \). The following are the Definitions, Lemma, and Theorem used in this research.

Definition 1 [3] Let \( c : E(G) \rightarrow \{1, 2, \ldots, k\} \) be an edge \( k \)-coloring of a simple connected graph \( G \) where adjacent edges may have same color. If there is no two edges in a path of \( G \) with same color, then the path is called a rainbow. The edge-colored graph \( G \) is rainbow connected if every two distinct vertices are connected by a rainbow path. The edge \( k \)-coloring in which \( G \) is rainbow-connected is called a rainbow \( k \)-coloring. The minimum number of color \( k \) to make \( G \) rainbow-connected is called the rainbow connection number of \( G \) and denoted by \( \text{rc}(G) \).

Definition 2 [15] Let \( G = (V, E) \) be simple connected graph. The graph \( G \) is called an antimagic if \( G \) has an antimagic labeling. A bijection \( f : E \rightarrow \{1, 2, \ldots, |E|\} \) is called an antimagic labeling if for every \( u \in V(G) \) and weight \( w(u) = \sum e \in E(u) f(e) \), where \( E(u) \) is a set of incident edges to \( u \), there is \( w(u) \neq w(v) \) for two distinct vertices \( u \) and \( v \).
Definition 3 [13] Let $G$ be a simple connected graph. For a bijection $f : V(G) \rightarrow 1, 2, \ldots, |V(G)|$, the associated weight of an edge $uv \in E(G)$ under $f$ is $w_f(uv) \neq w_f(u'v')$. If for every two vertices $u$ and $v$ of $G$, there is a rainbow antimagic if $G$ has a rainbow antimagic labeling.

Lemma 1 [3] Suppose $G$ is a connected graph of size $m$, then $\text{diam}(G) \leq \text{rc}(G) \leq \text{src}(G) \leq m$, where $\text{diam}(G)$ is the diameter of $G$ and $m$ is the number of edges of $G$.

Lemma 2 [3] If $G$ is a nontrivial connected graph of size $m$, then
(a) $\text{src}(G) = 1$ if and only if $G$ is a complete graph,
(b) $\text{rc}(G) = 2$ if and only if $\text{src}(G) = 2$,
(c) $\text{rc}(G) = m$ if and only if $G$ is a tree graph.

Lemma 3 [11] [12] [13] [14] Suppose $G$ is an arbitrary connected graph. Suppose $\text{rc}(G)$ and $\Delta(G)$ respectively have the rainbow connection number of $G$ and the maximum degree of $G$, such that $\text{rc}(G) \geq \max(\text{rc}(G), \Delta(G))$.

Lemma 4 [3] Suppose $G$ is a connected graph with $d(G) \geq 2$, then $G$ is an interval graph, $\text{diam}(G) \leq \text{rc}(G) \leq \text{diam}(G) + 1$ while if $G$ is an interval graph, $\text{diam}(G) = \text{rc}(G)$.

Theorem 1 [16] Let $G$ be a Tunjung Graph, denoted by $T_{j_n}$. The rainbow connection number of tunjung graph, $\text{rc}(T_{j_n}) = 3$.

THE RESULTS

In this section, we will show the rainbow antimagic coloring of double quadrilateral windmill graph $DQ_{n, n}$. Double Quadrilateral Flower Graph $FDQ_{n, n}$, Split Graph Of Star $Spl(K_{1, n})$, and Tunjung graph $T_{j_n}$.

We will show the cardinality of the Double Quadrilateral Windmill Graph ($DQ_{n, n}$). Suppose $DQ_{n, n}$ is a double quadrilateral windmill graph with $n \geq 2$. The vertex set and edge set of $DQ_{n, n}$ with $n \geq 2$ are $DQ_{n, n} = \{u_i\} \cup \{v_{ij} | 1 \leq i \leq n, j = 1, 2, 3\} \cup \{w_{ij} | 1 \leq i \leq n, j = 1, 2\}$ and $E(DQ_{n, n}) = \{u_0v_{ij} | 1 \leq i \leq n, j = 1, 2, 3\} \cup \{v_{ij}w_{ij} | 1 \leq i \leq n, j = 1, 2\}$ then obtained $|V(DQ_{n, n})| = 5n + 1$ and $|E(DQ_{n, n})| = 7kn$.

Theorem 2 Let $G$ be a double quadrilateral windmill graph, denoted by $DQ_{n, n}$. The rainbow connection number of double quadrilateral windmill graph, $\text{rc}(DQ_{n, n}) = 4$.

Suppose $DQ_{n, n}$ is a double quadrilateral windmill graph with $n \geq 2$. Firstly, will be shown the lower bound of double quadrilateral windmill graph $DQ_{n, n}$. Double quadrilateral windmill graph $DQ_{n, n}$ has $\text{diam}(DQ_{n, n}) = 4$. Based on Lemma 1, it is found $\text{rc}(DQ_{n, n}) \geq \text{diam}(DQ_{n, n})$, then $\text{rc}(DQ_{n, n}) \geq 4$. Second, will be proven upper bound by defining a function $c : E(DQ_{n, n}) \rightarrow 1, 2, \ldots, k$ as follows:

- $f(v_i^1w_i^1) = 1$, for $1 \leq i \leq n, 1 \leq j \leq 2$
- $f(v_i^1w_i^2) = 2$, for $1 \leq i \leq n$
- $f(u_0v_i^1) = 3$, for $j = 1$ and $j = 3, 1 \leq i \leq n$
- $f(u_0v_i^2) = 4$, for $1 \leq i \leq n$

Based on the coloring function above, we get $|f(V)| = 4$ then based on the upper bound and lower bound we get $4 \leq \text{rc}(DQ_{n, n}) \leq 4$ then $\text{rc}(DQ_{n, n}) = 4$. The rainbow path of $u - v$ is found which can be seen in Table 1. The Illustration of rainbow coloring can be seen in Figure 1 which is an example of rainbow coloring in the double quadrilateral windmill graph $DQ_{n, n}$. In Figure 1, there is also a rainbow coloring from the double quadrilateral windmill graph $\text{rc}(DQ_{4, 4}) = 4$. 


TABLE 1. Rainbow connection from u to v in DQ(n)

<table>
<thead>
<tr>
<th>Case</th>
<th>u, v</th>
<th>Rainbow Connection</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>v_2, v_5</td>
<td>u_0, v_1, v_4, v_5</td>
<td>i ≠ k, 1 ≤ i ≤ n, 1 ≤ k ≤ n</td>
</tr>
<tr>
<td>2</td>
<td>w_1, w_3</td>
<td>v_1, u_0, v_2, v_3</td>
<td>1 ≤ i ≤ n</td>
</tr>
<tr>
<td>3</td>
<td>w_2, w_4</td>
<td>v_1, u_0, v_2, v_4</td>
<td>i ≠ k, 1 ≤ i ≤ n, 1 ≤ k ≤ n</td>
</tr>
<tr>
<td>4</td>
<td>w_1, v_3</td>
<td>v_1, u_0, v_2, v_3</td>
<td>i ≠ k, 1 ≤ i ≤ n, 1 ≤ k ≤ n</td>
</tr>
<tr>
<td>5</td>
<td>w_2, v_4</td>
<td>v_1, u_0, v_2, v_4</td>
<td>i ≠ k, 1 ≤ i ≤ n, 1 ≤ k ≤ n</td>
</tr>
</tbody>
</table>

Theorem 3 Let G be a double quadrilateral windmill graph, denoted by DQ(n). The rainbow antimagic connection number of double quadrilateral windmill graph, \(\text{rac}(DQ(n))\) = 3n.

Suppose \(DQ(n)\) is a double quadrilateral windmill graph with \(n \geq 2\). Firstly, will be shown the lower bound of double quadrilateral windmill graph \(DQ(n)\). Double quadrilateral windmill graph \(DQ(n)\) has \(\Delta(DQ(n)) = 3n\) and has rainbow connection number \(\text{rc}(DQ(n)) = 4\). Based on Lemma 3, it is found \(\text{rac}(DQ(n)) \geq \max(\text{rc}(DQ(n)), \Delta(DQ(n)))\), then \(\text{rac}(DQ(n)) \geq \max(4, 3n)\). Second, there will be shown the upper bound of Double quadrilateral windmill graph \(\text{rac}(DQ(n)) \geq 3n\). Next we will prove that \(\text{rac} \leq 3n\) by defining the vertex label function \(f(V(G)) \rightarrow 1, 2, 3, \ldots |V(G)|\) as follows:

\[
\begin{align*}
    f(v_0) &= 2n \\
    f(w_1) &= 2n + i \\
    f(w_2) &= 2n - i + 1, \text{for } 2 \leq i \leq n \\
    f(v_1') &= 2n + 3i + 1, \text{for } 1 \leq i \leq n \\
\end{align*}
\]

Based on a vertex label function, the edge weight of double quadrilateral windmill graph is as follows:

\[
\begin{align*}
    w(u_0v_0^2) &= 4n + 3i - 1, \text{for } 1 \leq i \leq n \\
    w(u_0v_0^3) &= 4n + 3i + 1, \text{for } 1 \leq i \leq n \\
    w(v_1'w_1^2) &= 4n + 3i - j + 1, \text{for } 2 \leq i \leq n \\
    w(v_1'w_1^3) &= 4n + i + 1, \text{for } 2 \leq i \leq n \\
    w(v_1'w_1^4) &= 4n + i + 2, \text{for } 2 \leq i \leq n \\
\end{align*}
\]

The next step is to see that the edge weights induce rainbow antimagic coloring on the double quadrilateral windmill graph \(DQ(n)\). Let \(W(DQ_n)\) be the set of the edge weights of graph. To determine the upper bound, we should find the cardinality of the set \(W(DQ_n)\). In this step, we use an arithmetic sequence to know the cardinality of the set. To show the number of edge weights in the double quadrilateral windmill graph \(DQ(n)\) we can classify the edges into two types namely edges connected to vertex \(u_0\) and edges not connected to vertex \(u_0\) which we denote by \(W_2 = W_3, W_4, W_6, W_8, W_9\). First we will calculate the sum of the weights of \(W_1\), the weights of \(W_1\) can be described through the following arithmetic sequence \(\{4n + 2, 4n + 4, 4n + 4, \ldots, 7n + 1\}\):

\[
\begin{align*}
    a + (n - 1)b &= U_n \\
    4n + 2 + (W_1 - 1)1 &= 7n + 1 \\
    W_1 &= 3n
\end{align*}
\]

From the first step we get the number of \(|W_1|\) is 3n. Second, we will show that \(W_2 \subset W_1\). Based on the edge weight function we will show that \(W_3 \subset W_1, W_4 \subset W_1, W_5 \subset W_1, W_6 \subset W_1, W_7 \subset W_1, W_8 \subset W_1\), and \(W_9 \subset W_1\). \(W_3 = W(v_1'w_1^3) = 4n + 3i + 1\), then we get the edge weight \(W_1 = \{4n + 5\}\). \(W_4 = W(v_1'w_1^2) = 4n + 3i - j + 1\), for \(2 \leq i \leq n\), then we get the set of edge weight \(W_4 = \{4n + 3, 4n + 3, 4n + 3, \ldots, 5n + 1\}\). \(W_5 = W(v_1'w_1^1) = 4n + 4i - 1\), then we get the edge weight \(W_5 = \{4n - 3\}\). From the second step we get that \(W_4 \subset W_3, W_5 \subset W_3, W_6 \subset W_3, W_7 \subset W_3\), and \(W_8 \subset W_3\). Based on the set of edge weights, the number of edge weights is 3n, so \(\text{rac}(DQ(n)) \leq 3n\). We know from the upper bound and lower bound that \(3n \leq \text{rac}(DQ(n)) \leq 3n\) we get that the rainbow antimagic connection number of double quadrilateral windmill graph is \(\text{rac}(DQ(n)) = 3n\).

Based on the edge weight above, the rainbow path of \(u-v\) is found which can be seen in table 2, and illustration of rainbow antimagic coloring of graph \(DQ(n)\) is in Figure 2, with \(\text{rac}(DQ_4) = 3n = 3 \times 4 = 12\).
TABLE 2. Rainbow connection from \( u \) to \( v \) in \( DQ(n) \)

<table>
<thead>
<tr>
<th>Case</th>
<th>( u )</th>
<th>( v )</th>
<th>Rainbow Connection Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( v_i )</td>
<td>( v_j )</td>
<td>( i \neq k, 1 \leq i \leq n, 1 \leq k \leq n )</td>
</tr>
<tr>
<td>2</td>
<td>( w_i )</td>
<td>( w_j )</td>
<td>( i \leq n, 1 \leq i \leq n, i = k )</td>
</tr>
<tr>
<td>3</td>
<td>( w_i )</td>
<td>( w_j )</td>
<td>( i \leq n, 1 \leq i \leq n, i \neq k )</td>
</tr>
<tr>
<td>4</td>
<td>( w_i )</td>
<td>( w_j )</td>
<td>( i \leq n, 1 \leq i \leq n, i \neq k )</td>
</tr>
<tr>
<td>5</td>
<td>( w_i )</td>
<td>( w_j )</td>
<td>( i \leq n, 1 \leq i \leq n, i \neq k )</td>
</tr>
</tbody>
</table>

We will show the cardinality of the Double Quadrilateral Flower Graph \( (FDQ_n) \). Suppose \( FDQ_n \) is a double quadrilateral flower graph with \( n \geq 2 \). The vertex set and edge set of \( FDQ_n \) have \( FDQ_n \)

have \( V(FDQ_n) = \{u_i|1 \leq i \leq 2n \} \cup \{v_i|1 \leq i \leq n, j = 1, 2\} \cup \{u_j|1 \leq j \leq 2n, i = 1, 2\} \cup \{v_j|1 \leq j \leq n, i = 1\} \)

then obtained \( |V(FDQ_n)| = 4n + 2 \) and \( |E(FDQ_n)| = 6n + 1 \) \[18].

**Theorem 4** Let \( G \) be a Double Quadrilateral Flower Graph, denoted by \( FDQ_n \). The rainbow connection number of Double Quadrilateral Flower Graph, \( rc(FDQ_n) \) = 4.

Suppose \( FDQ_n \) is a double quadrilateral flower graph with \( n \geq 2 \). Firstly, will be shown the lower bound of double quadrilateral flower graph \( FDQ_n \). Double quadrilateral flower graph \( FDQ_n \) has \( diam(FDQ_n) = 4 \). Based on Lemma 1, it is found \( rc(FDQ_n) \geq diam(FDQ_n) \), then \( rc(FDQ_n) \geq 4 \). Second, will be proven upper bound by defining a function \( c : E(FDQ_n) \rightarrow 1, 2, \ldots, k \) as follows:

\[
\begin{align*}
  f(u_1 v_1) &= 1, \text{ for } 1 \leq i \leq n, j = 1, 2 \\
  f(u_1 v_1) &= 2, \text{ for } 1 \leq i \leq n, j = 2 \\
  f(u_0 u_i) &= 3, \text{ for } i = \text{odd} \\
  f(u_0 v_i) &= 4, \text{ for } i = \text{even}
\end{align*}
\]

Based on the rainbow connection function above, we get \(|f(V)| = 4\) then based on the upper bound and lower bound we get \( 4 \leq rc(FDQ_n) \leq 4 \) then \( rc(FDQ_n) = 4 \). The rainbow path of \( u - v \) is found which can be seen in table 3.

The illustration of rainbow coloring can be seen in figure 3 which is an example of rainbow coloring in the double quadrilateral flower graph \( FDQ_n \). In figure 3, there is also a rainbow coloring from the double quadrilateral flower graph \( rc(FDQ_n) = 4 \).
<table>
<thead>
<tr>
<th>Case</th>
<th>u vs. v</th>
<th>Rainbow Connection</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$u_{2i}$, $v_{2k}$, $u_{2(k+1)}$, $v_{2k}$</td>
<td>$i \neq k$, $1 \leq i \leq n$, $1 \leq k \leq n$, $j = 1$, $2$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$v_{2i}$, $v_{2k}$</td>
<td>$i \neq k$, $1 \leq i \leq n$, $1 \leq k \leq n$, $j = 1$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$v_{2i}$, $u_{2k}$, $v_{2(k+1)}$, $v_{2k}$</td>
<td>$i \neq k$, $1 \leq i \leq n$, $1 \leq k \leq n$, $j = 2$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$v_{2i}$, $u_{2k}$, $u_{2(k+1)}$, $v_{2k}$</td>
<td>$i \neq k$, $1 \leq i \leq n$, $1 \leq k \leq n$, $j = 1$, $2$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$u_{2i}$, $v_{2k}$, $u_{2(k+1)}$, $v_{2k}$</td>
<td>$i \neq k$, $1 \leq i \leq n$, $1 \leq k \leq n$, $j = 1$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$u_{2i}$, $v_{2k}$, $u_{2(k+1)}$, $v_{2k}$</td>
<td>$i \neq k$, $1 \leq i \leq n$, $1 \leq k \leq n$, $j = 2$</td>
<td></td>
</tr>
</tbody>
</table>

**Theorem 5** Let $G$ be a Double Quadrilateral Flower Graph, denoted by $FDQ_{(n)}$. The rainbow antimagic connection number of Double Quadrilateral Flower Graph, $rac(FDQ_{(n)}) = 2n + 1$.

Suppose $FDQ_{(n)}$ is a double quadrilateral flower graph with $n \geq 2$. Firstly, will be shown the lower bound of double quadrilateral flower graph $FDQ_{(n)}$. Double quadrilateral flower graph $FDQ_{(n)}$ has $\Delta(FDQ_{(n)}) = 2n + 1$ and has rainbow connection number $rc(FDQ_{(n)}) = 4$. Based on Lemma 3, it is found $rac(FDQ_{(n)}) \geq \max(rc(FDQ_{(n)}), \Delta(FDQ_{(n)}))$, then $rac(FDQ_{(n)}) \geq \max(4, 2n + 1)$. Second, there will be shown the upper bound of Double quadrilateral flower graph $FDQ_{(n)} \geq 2n + 1$. Next we will prove that $rac \leq 2n + 1$ by defining the vertex label function $f(V(G)) \rightarrow 1, 2, 3, ..., |V(G)|$ as follows:

\[
\begin{align*}
    f(u) &= 2n \\
    f(v) &= 2i, \text{ for } 1 \leq i \leq n - 1 \\
    f(u) &= 4n + 3 - i, \text{ for } 1 \leq i \leq 2n - 2 \\
    f(u) &= 4n + 2 - i, \text{ for } 2n - 1 \leq i \leq 2n + 1
\end{align*}
\]

Based on a vertex label function, the edge weight of Double quadrilateral flower graph is as follows:

\[
\begin{align*}
    w(u_1) &= 4n + 2i - 2, \text{ for } 1 \leq i \leq 2n - 3, \text{ and } 1 \leq k \leq n - 1 \\
    w(u_2) &= 4n + 2k - i + 3, \text{ for } i = \text{ even and } 1 \leq k \leq n - 1 \\
    w(u_3) &= 4n + 2k - i + 3, \text{ for } 1 \leq i \leq 2n - 3 \text{ and } 1 \leq k \leq n - 2 \\
    w(u_4) &= 4n + 5, \\
    w(u_5) &= 4n + 6, \\
    w(u_6) &= 4n + 1
\end{align*}
\]

The next step is to see that the edge weights induce rainbow antimagic coloring on the double quadrilateral flower graph ($FDQ_{(n)}$). Let $W(FDQ_{(n)})$ be the set of the edge weights of graph. To determine the upper bound, we should find the cardinality of the set $W(FDQ_{(n)})$. In this step, we use an arithmetic sequence to know the cardinality of the set. To show the number of edge weights in the double quadrilateral flower graph ($FDQ_{(n)}$) we can classify the edges into two types namely edges connected to vertex $u_0$ which we denoted by $W_1 = \{W_2, W_3\}$ and edges not connected to vertex $u_0$ which we denoted by $W_2 = \{W_4, W_5, W_6, W_7, W_8, W_9\}$. First, we will calculate the sum of the weights of $|W_1|$. The weights of the $W_1 = \{W_2, W_3\}$ can be described through the following arithmetic sequences $W_2 = \{6n + 2, 6n + 1, 6n, 6n - 1, ..., 4n + 5\}$ for $1 \leq i \leq 2n - 2$ and $W_3 = \{4n + 3, 4n + 2, 4n + 1\}$ for $2n - 1 \leq i \leq 2n + 1$.

\[
\begin{align*}
    U_n &= a + (n - 1)b \\
    4n + 5 &= 6n + 2 + (|W_2| - 1)(-1) \\
    |W_2| &= 2n - 2 \\
    4n + 1 &= 4n + 3 + (|W_3| - 1)(-1) \\
    |W_3| &= 3
\end{align*}
\]

From the first step we get the number of edge weights $|W_1| = |W_2| + |W_3| = 2n + 1$. Second, we will show that $W_4 \subseteq W_4$. Based on the edge weight function we will show that $W_5 \subseteq W_4$, $W_6 \subseteq W_4$, $W_7 \subseteq W_4$, $W_8 \subseteq W_4$, $W_9 \subseteq W_4$, and $W_{11} \subseteq W_4$, $W_5 = W\{u_1v_1\} = 4n + 2k - i + 2$, for $1 \leq i \leq 2n - 3$, and $1 \leq k \leq n - 1$, then we get the edge weight $W_5 = \{4n + 3\}$. $W_6 = W\{u_1v_2\} = 4n + 2k - i + 3$, for $1 \leq i \leq 2n - 3$ and $1 \leq k \leq n - 2$, then we get the edge weight $W_6 = \{4n + 2\}$. $W_7 = W\{u_1v_3\} = 4n + 5$, then we get the edge weight $W_7 = \{4n + 5\}$. $W_8 = W\{u_1v_4\} = 4n + 2k - i + 2$, for $i = \text{ even}$ and $1 \leq k \leq n - 1$ then we get the edge weight $W_8 = \{4n + 3\}$. $W_9 = W\{u_1v_5\} = 4n + 2k - i + 3$,
for $i = \text{even}$ and $1 \leq k \leq n - 1$ then we get the edge weight $W_9 = \{4n + 2\}$, $W_{10} = W(u_{2n}v_n) = 4n + 1$, then we get the edge weight $W_{10} = \{4n + 1\}$. $W_{11} = W(u_{2n}v_0) = 4n + 6$, then we get the edge weight $W_{11} = \{4n + 6\}$. From the second step we get that $W_9 \subset W_5 \subset W_7$, $W_5 \subset W_4 \subset W_7$, $W_6 \subset W_4 \subset W_7$, and $W_{10} \subset W_4$. Based on the set of edge weights, the number of edge weights is $2n + 1$, so $\text{rac}(FDQ_n) \leq 2n + 1$. We know from the upper bound and lower bound that $2n + 1 \leq \text{rac}(FDQ_n) \leq 2n + 1$ we get that the rainbow antimagic connection number of double quadrilateral flower graph is $\text{rac}(FDQ_n) = 2n + 1$.

Based on the edge weight above, the rainbow path of $u,v$ is found which can be seen in Table 4 and for illustration of rainbow antimagic coloring of graph $FDQ_n$ is in Figure 4, with $\text{rac}(FDQ_n) = 2n + 1 = 2.5 + 1 = 11$.

<table>
<thead>
<tr>
<th>Case</th>
<th>$u$</th>
<th>$v$</th>
<th>Rainbow Connection</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$u_l$</td>
<td>$u_k$</td>
<td>$u_l, u_k$</td>
<td>$i = \text{even}, k = \text{even} \ 1 \leq i \leq n, 1 \leq k \leq n,$</td>
</tr>
<tr>
<td>2</td>
<td>$u_l$</td>
<td>$v_j$</td>
<td>$u_l, u_{j-1}, v_j$</td>
<td>$i = \text{even}, 1 \leq i \leq n, 1 \leq k \leq n,$</td>
</tr>
<tr>
<td>3</td>
<td>$u_l$</td>
<td>$v_j$</td>
<td>$u_l, u_{j+1}, v_j$</td>
<td>$i = \text{even}, 1 \leq i \leq n, 1 \leq k \leq n,$</td>
</tr>
<tr>
<td>4</td>
<td>$v_j$</td>
<td>$v_k$</td>
<td>$u_{j+k}$</td>
<td>$1 \leq i \leq n, j = 1, k = 2,$</td>
</tr>
<tr>
<td>5</td>
<td>$v_j$</td>
<td>$v_k$</td>
<td>$u_{j+k}, u_{j-1}, v_j$</td>
<td>$1 \leq i \leq n, j = 1, 1 \leq l \leq n,$</td>
</tr>
<tr>
<td>6</td>
<td>$v_j$</td>
<td>$v_k$</td>
<td>$u_{j+k}, u_0, u_{j-1}, v_j$</td>
<td>$1 \leq i \leq n, j = 1, 1 \leq l \leq n,$</td>
</tr>
<tr>
<td>7</td>
<td>$v_j$</td>
<td>$v_k$</td>
<td>$u_{j+k}, u_0, u_{j-1}, v_j$</td>
<td>$1 \leq i \leq n, j = 1, 1 \leq l \leq n,$</td>
</tr>
<tr>
<td>8</td>
<td>$v_j$</td>
<td>$v_k$</td>
<td>$u_{j+k}, u_0, u_{j-1}, v_j$</td>
<td>$1 \leq i \leq n, j = 1, 1 \leq l \leq n,$</td>
</tr>
</tbody>
</table>

**FIGURE 3.** Rainbow coloring $FDQ_5$

**FIGURE 4.** Rainbow antimagic coloring $FDQ_5$

We will show the cardinality of the split graph of star ($\text{Spl}(K_{1,n})$). Suppose ($\text{Spl}(K_{1,n})$) is a split graph of star with $n \geq 3$. The vertex set and edge set of $\text{Spl}(K_{1,n})$ with $n \geq 3$ are $\text{Spl}(K_{1,n})$ have $V(\text{Spl}(K_{1,n})) = \{v\} \cup \{v'\} \cup \{v''\}$ for $1 \leq i \leq n$ and $E(\text{Spl}(K_{1,n})) = \{v_i\} \cup \{v_i'\} | 1 \leq i \leq n \} \cup \{v_i''\} | 1 \leq i \leq n \} \cup \{v_i'\} | 1 \leq i \leq n \}$ then obtained $|V(\text{Spl}(K_{1,n}))| = 2n + 2$ and $|E(\text{Spl}(K_{1,n}))| = 3n$.[19]

Theorem 6 Let $G$ be a Split Graph Of Star, denoted by $\text{Spl}(K_{1,n})$. The rainbow connection number of Split Graph Of Star, $\text{rc}(\text{Spl}(K_{1,n})) = n$. For every integer $n \geq 3$.

Suppose $\text{Spl}(K_{1,n})$ is a split graph of star with $n \geq 3$. Firstly, will be shown the lower bound of split graph of star $\text{Spl}(K_{1,n})$. Split graph of star $\text{Spl}(K_{1,n})$ has $\text{diam}(\text{Spl}(K_{1,n})) = 3$. Based on Lemma 1, it is found $\text{diam}(\text{Spl}(K_{1,n})) \geq \text{rc}(\text{Spl}(K_{1,n}))$, then $\text{rc}(\text{Spl}(K_{1,n})) \geq 3$. Secondly, this graph will not be able to satisfy the upper bound in Lemma 1 which is $\text{Spl}(K_{1,n}) \geq 3$, because the split star graph has $n$ vertices of degree one, then like the tree graph in Lemma 2, all vertices of degree one will have different colors. So a split graph of star will have $n$ different colors. Will be
proven upper bound by defining a function $c : E(Spl(K_{1,n})) \to 1, 2, \ldots, k$ as follows:

\[
\begin{align*}
  f(v'v_i) &= 1 & \text{for } 1 \leq i \leq n & f(vv_i) &= 2 & \text{for } 1 \leq i \leq n \\
  f(v'_i) &= 3 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor & f(vv'_i) &= 4 & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor
\end{align*}
\]

From the above function we get a rainbow connection number of $n$ because the set of edges $f(v'v_i) = 1$, $f(vv_i) = 2$, $f(v'_i) = 3$, $f(vv'_i) = 4$ will always equal the set of edges $f(v'v'_i) = i$ when $i = 1, 2, 3, 4$. Based on the rainbow connection function above, we get $|f(V)| = n$ then based on the upper bound and lower bound we get $n \leq rc(Spl(K_{1,n})) \leq n$ then $rc(Spl(K_{1,n})) = n$. The rainbow path of $u-v$ is found which can be seen in Table 5. The illustration of rainbow coloring can be seen in Figure 5 which is an example of rainbow coloring in the split graph of star $spl(K_{1,n})$. In Figure 5, there is also a rainbow coloring from the split graph of star $rc(spl(K_{1,n})) = n$.

\begin{table}[h]
\centering
\caption{Rainbow connection from $u$ to $v$ in $spl(K_{1,n})$}
\begin{tabular}{ccc}
\hline
Case & $u$ & $v$ & Rainbow Connection Condition \\
\hline
1 & $v$ & $v'_i$ & $v_i,v'_i,v'_i$ & $1 \leq i \leq n, i \neq 1,2,3,4$ \\
2 & $v$ & $v'_i$ & $v'_i,v'_i,v'_i,v'_i$ & $1 \leq i \leq n, i = 1, 2$ \\
3 & $v$ & $v'_i$ & $v_i,v'_i,v'_i,v'_i$ & $1 \leq i \leq n, i = 3, 4$ \\
4 & $v'_i$ & $v'_i$ & $v'_i,v'_i,v'_i,v'_i$ & $i \neq k, 1 \leq i \leq n, 1 \leq k \leq n$ \\
5 & $v_i,v_k$ & $v'_i,v'_i,v'_i,v'_i,v'_i,v'_i,v'_i$ & $i \neq k, 1 \leq i \leq n, 1 \leq k \leq n$ \\
\hline
\end{tabular}
\end{table}

**Theorem 7** Let $G$ be a Split Graph Of Star, denoted by $Spl(K_{1,n})$. The rainbow antimagic connection number of Split Graph Of Star, $rac(Spl(K_{1,n})) = 2n$. For every integer $n \geq 3$.

Suppose $Spl(K_{1,n})$ is a split graph of star with $n \geq 3$. Firstly, will be shown the lower bound of split graph of star $Spl(K_{1,n})$. Split graph of star $Spl(K_{1,n})$ has $\Delta(Spl(K_{1,n})) = 2n$ and has rainbow connection number $rc(Spl(K_{1,n})) = n$. Based on Lemma 3, it is found $rac(Spl(K_{1,n})) \geq \max(rc(Spl(K_{1,n})), \Delta(Spl(K_{1,n})))$, then $rac(Spl(K_{1,n})) \geq \max(n, 2n)$. Second, there will be shown the upper bound of split graph of star $Spl(K_{1,n}) \geq 2n$. Next we will prove that $rac(spl(K_{1,n})) \leq 2n$ by defining the vertex label function $f(V(G)) \to 1, 2, 3, \ldots, |V(G)|$ as follows:

\[
\begin{align*}
  f(v) &= 2 \\
  f(v_i) &= 2i + 1 & \text{for } 1 \leq i \leq n \\
  f(v'_i) &= 1 \\
  f(v'_i) &= 2i + 2 & \text{for } 1 \leq i \leq n
\end{align*}
\]

Based on a vertex label function, the edge weight of split graph of star is as follows:

\[
\begin{align*}
  w(vv_i) &= 2i + 3 & \text{for } 1 \leq i \leq n \\
  w(v'_i) &= 2i + 3 & \text{for } 1 \leq i \leq n \\
  w(v'v'_i) &= 2i + 2 & \text{for } 1 \leq i \leq n
\end{align*}
\]

The next step is to see that the edge weights induce rainbow antimagic coloring on the split graph of star $Spl(K_{1,n})$. Let $W(Spl(K_{1,n}))$ be the set of the edge weights of graph. To determine the upper bound, we should find the cardinality of the set $W(Spl(K_{1,n}))$. Based on the edge weight function of the split graph of star, we get that $W(vv_i)$ and $W(v'_i)$ have the same weight as $n$. While for the weight function $W(v'v'_i)$ has a different edge weight with another $n$. So the number of edge weights on the split graph of star $|W(Spl(K_{1,n}))| = 2n$. Based on the set of edge weights, the number of edge weights is $2n$, so $rac(Spl(K_{1,n})) \leq 2n$. We know from the upper bound and lower bound that $2n \leq rac(Spl(K_{1,n})) \leq 2n$ we get that the rainbow antimagic connection number of split graph of star is $rac(Spl(K_{1,n})) = 2n$.

Based on the edge weight above, the rainbow path of $u-v$ is found which can be seen in Table 6 and for illustration of rainbow antimagic coloring of graph $Spl(K_{1,n})$ is in Figure 4, with $rac(Spl(K_{1,n})) = 2n = 14$.

\begin{table}[h]
\centering
\caption{Rainbow connection from $u$ to $v$ in $spl(K_{1,n})$}
\begin{tabular}{ccc}
\hline
Case & $u$ & $v$ & Rainbow Connection Condition \\
\hline
1 & $v$ & $v'_i$ & $v_i,v'_i,v'_i$ & $1 \leq i \leq n$ \\
2 & $v'_i$ & $v'_i$ & $v'_i,v'_i$ & $i \neq k, 1 \leq i \leq n, 1 \leq k \leq n$ \\
\hline
\end{tabular}
\end{table}
We will show the cardinality of the Tunjung Graph $T_{jn}$. Suppose $(T_{jn})$ is a tunjung graph with $n \geq 3$. The vertex set and edge set of $(T_{jn})$ with $n \geq 3$ are $(T_{jn})$ have $V(T_{jn}) = \{c\} \cup \{v_1|1 \leq i \leq n\} \cup \{v_2|1 \leq i \leq n\}$ and $E(T_{jn}) = \{c_i|1 \leq i \leq n\} \cup \{v_1v_2|1 \leq i \leq n\} \cup \{v_1v_2|1 \leq i \leq n\} \cup \{v_1x_i|1 \leq i \leq n-1\} \cup \{v_2x_i|1 \leq i \leq n-1\} \cup \{a_{n-1}\}$ then obtained $|V(T_{jn})| = 3n + 1$ and $|E(T_{jn})| = 7n$ [20].

**Theorem 8** Let $G$ be a Tunjung Graph, denoted by $T_{jn}$. The rainbow antimagic connection number of Tunjung Graph, $rac(T_{jn}) = 3n$.

Suppose $T_{jn}$ is a tunjung graph with $n \geq 4$. Firstly, will be shown the lower bound of tunjung graph $T_{jn}$. Tunjung graph $T_{jn}$ has $\Delta(T_{jn}) = 3n$ and based theorem 1 has rainbow connection number $rc(T_{jn}) = 3$. Based on Lemma 3, it is found $rac(T_{jn}) \geq max(rc(T_{jn}), \Delta(T_{jn}))$, then $rac(T_{jn}) \geq max(3, 3n)$. Second, there will be shown the upper bound of tunjung graph $T_{jn} \geq 3n$. Next we will prove that $rac \leq 3n$ by defining the vertex label function $f(V(G)) \rightarrow 1, 2, 3, ..., |V(G)|$ as follows:

$$f(c) = 3, f(x_i) = \begin{cases} \frac{3i+7}{6n-3i+10}, & i = \text{odd} \\ \frac{i}{6n-3i+12}, & i = \text{even} \end{cases} \quad f(y_i) = \begin{cases} \frac{3n-3i+1}{2}, & i = \text{odd}, i \neq n \\ \frac{3n+2}{2}, & i = \text{even}, i \neq n \end{cases} \quad f(z_i) = \begin{cases} \frac{3n+9}{2}, & i = \text{odd}, i \neq n \\ \frac{6n-3i+6}{2}, & i = \text{even}, i \neq n \\ 3n+1, & i = n \end{cases}$$

Based on a vertex label function above, then it will be shown the edge weight of tunjung graph is as follows:

$$W(c_x) = \begin{cases} \frac{3i+7}{6n-3i+10}, & i = \text{odd} \\ \frac{i}{6n-3i+12}, & i = \text{even} \end{cases} \quad W(c_y) = \begin{cases} \frac{3n-3i+5}{2}, & i = \text{odd}, i \neq n \\ \frac{3n+8}{2}, & i = \text{even}, i \neq n \end{cases} \quad W(c_z) = \begin{cases} \frac{3n+15}{2}, & i = \text{odd}, i \neq n \\ \frac{3n+12}{2}, & i = \text{even}, i \neq n \end{cases} \quad W(x_{y_i}) = \begin{cases} \frac{3n+3}{2}, & i = \text{odd}, i \neq n \\ \frac{3n+6}{2}, & i = \text{even}, i \neq n \end{cases}$$

The next step is to see that the edge weights induce rainbow antimagic coloring on the tunjung graph $(T_{jn})$. Let $W(T_{jn})$ be the set of the edge weights of graph. To determine the upper bound, we should find the car-
dinality of the set $W(T_jn)$. In this step, we use an arithmetic sequence to know the cardinality of the set. To show the number of edge weights in the tunjung graph $(T_jn)$ we can classify the edges into two types namely edges connected to vertex $c$ which we denoted by $W_1$ and edges not connected to vertex $c$ which we denoted by $W_2 = \{W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{12}, W_{13}, W_{14}, W_{15}, W_{16}\}$. First we will calculate the sum of the weights of the $|W_1|$. Based on the cardinality of the edge set, it is known that all vertices in the tunjung graph are adjacent to vertex $c$ and $f(c) = 3$ with labeling using a bijective function then the $|W_1|$ will always start from $4$ to $3n+4$ and it is not possible to have an edge weight of $6$ for every edge connected to vertex $c$. From this analysis, the number of $|W_1|$ can be found using the following arithmetic sequence $W_1 = \{4, 5, 6, 7, \ldots, 3n, 3n+1, 3n+2, 3n+3, 3n+4\}$.

$$U_n = a + (n-1)b$$

$$3n + 4 = 4 + (|W_1| - 1)1$$

$$3n + 1 = |W_1|$$

From the first step we get the number of $|W_1|$ is $3n+1$, but the result obtained in the first step must be minus $1$ because the edge weight $6$ is not possible. So the sum of $|W_1|$ is $3n$. Second, we will show that $W_2 \subset W_1$. $W_5 = W(x_iy_i) = 3n$ for $i = odd$ and $i \neq n$ then we get the edge weight $W_5 = 3n + 3$ for $i$ = even and $i \neq n$ then we get the edge weight $W_4 = 3n$. $W_5 = W(x_iy_i) = 3n + 3$ for $n = odd$ then we get the edge weight $W_3 = \frac{3n + 3}{2}$, $W_6 = W(x_iy_i) = \frac{3n + 6}{2}$ for $n = even$ then we get the edge weight $W_6 = \frac{3n + 6}{2}$. $W_7 = W(x_iy_i) = 3n + 4$ for $i \neq n$ then we get the edge weight $W_7 = \frac{3n + 4}{2}$. $W_8 = W(x_iy_i) = 3n + 2$ for $i = n$ then we get the edge weight $W_9 = \frac{3n + 2}{2}$. $W_9 = W(x_iy_i) = 3n + 1$ for $i$ = odd and $i \neq n$ then we get the edge weight $W_10 = \frac{3n + 1}{2}$, $W_11 = W(x_iy_i) = 3n + 1$ for $n = odd$ then we get the edge weight $W_11 = \frac{3n + 5}{2}$, $W_12 = W(x_iy_i) = \frac{3n + 8}{2}$ for $n = even$ then we get the edge weight $W_12 = \frac{3n + 8}{2}$, $W_13 = W(x_iy_i) = 3n + 2$ for $i = odd$ and $i \neq n$ then we get the edge weight $W_13 = \frac{3n + 2}{2}$, $W_14 = W(x_iy_i) = 3n + 1$ for $i$ = even and $i \neq n$ then we get the edge weight $W_14 = \frac{3n + 1}{2}$, $W_15 = W(x_iy_i) = \frac{3n + 1}{2}$ for $n = odd$ then we get the edge weight $W_15 = \frac{3n + 2}{2}$. $W_16 = W(x_iy_i) = \frac{3n + 2}{2}$ for $n = even$ then we get the edge weight $W_16 = \frac{3n + 4}{2}$. From the second step we get that $W_3 \subset W_1$, $W_4 \subset W_1$, $W_5 \subset W_1$, $W_6 \subset W_1$, $W_7 \subset W_1$, $W_8 \subset W_1$, $W_9 \subset W_1$, $W_{10} \subset W_1$, $W_{11} \subset W_1$, $W_{12} \subset W_1$, $W_{13} \subset W_1$, $W_{14} \subset W_1$, $W_{15} \subset W_1$ and $W_{16} \subset W_1$. Based on the set of edge weights, the number of edge weights is $3n$, so $rac(T_jn) \leq 3n$. We know from the upper bound and lower bound that $3n \leq rac(T_jn) \leq 3n$ we get that the rainbow antimagic connection number of tunjung graph is $rac(T_jn) = 3n$.

Based on the edge weight above, the rainbow path of $u-v$ is found which can be seen in table 7 and for illustration of rainbow antimagic coloring of graph $T_jn$ is in Figure 7, with $rac(T_jn) = 3n = 24$.

<table>
<thead>
<tr>
<th>Case</th>
<th>u - v</th>
<th>Rainbow Connection</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x_i \ x_j$</td>
<td>c</td>
<td>$i \neq j, 1 \leq i \leq n, 1 \leq j \leq n$</td>
</tr>
<tr>
<td>2</td>
<td>$y_i \ y_j$</td>
<td>c</td>
<td>$i \neq j, 1 \leq i \leq n, 1 \leq j \leq n$</td>
</tr>
<tr>
<td>3</td>
<td>$z_i \ z_j$</td>
<td>c</td>
<td>$i \neq j, 1 \leq i \leq n, 1 \leq j \leq n$</td>
</tr>
<tr>
<td>4</td>
<td>$x_i \ z_j$</td>
<td>c</td>
<td>$1 \leq i \leq n, 1 \leq j \leq n$</td>
</tr>
<tr>
<td>5</td>
<td>$y_i \ z_j$</td>
<td>c</td>
<td>$1 \leq i \leq n, 1 \leq j \leq n$</td>
</tr>
<tr>
<td>6</td>
<td>$y_i \ z_j$</td>
<td>c</td>
<td>$1 \leq i \leq n, 1 \leq j \leq n$</td>
</tr>
</tbody>
</table>
CONCLUSION

In this paper, we have studied 7 theorems about rainbow coloring and rainbow antimagic coloring of graph. There is 3 theorems about rainbow coloring and 4 theorems about rainbow antimagic coloring of double quadrilateral windmill graph ($DQ_n$), double quadrilateral flower graph ($FDQ_n$), split graph of star ($Spl(K_{1,n})$), and tunjung graph ($T_{jn}$).

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REFERENCES