A Comprehensive Study on Existence theory and Ulam's stabilities of Impulsive Fractional Langevin Equation

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Abstract

In this paper, a class of impulsive fractional Langevin equation is considered and proceeds to derive a solution formula for this equation, incorporating Mittag-Leffler functions. The solution is obtained through an analysis of linear Langevin equation involving distinct fractional derivatives. We establish the existence and uniqueness results of the solution by employing mathematical tools such as boundedness, continuity, monotonicity, and non-negativity properties of Mittag-Leffler functions and fixed point methods. Furthermore, we establish appropriate conditions and results to discuss Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stability of our proposed model, with the help of fixed point theorem. Finally, the theoretical findings are illustrated through a practical example.

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1 Introduction

Fractional differential equations represent a generalized form of classical differential equations with fractional order derivatives. The field of fractional calculus has matured significantly and finds numerous applications in diverse areas such as porous media, electrochemistry, economics, electromagnetics, physical sciences, and medicine. Notably, fractional differential equations play a crucial role in viscoelasticity, statistical physics, optics, signal processing, control systems, electrical circuits, astronomy etc. Several pivotal articles have contributed theoretical tools for qualitatively analyzing this field, highlighting both the connections and distinctions between classical integral models and fractional differential equations. More recently, there has been growing interest in a subclass known as fuzzy fractional differential equations. Researchers have investigated solvability results for nonlocal problems within fuzzy fractional differential systems, particularly

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under the framework of gh-differentiability in fuzzy metric spaces. These investigations have extended to encompass fuzzy wave equations, and numerous references provide in-depth insights into this intriguing field, see [1, 17, 19, 27, 37, 36, 31, 22, 23, 24, 25].

The Langevin equation, which was originally introduced by Paul Langevin in 1908, has emerged as an exceptionally versatile and indispensable tool for understanding a wide spectrum of phenomena. Its applicability extends across various fields, including physics, engineering, economics, and medicine, where it plays a pivotal role in elucidating complex processes. Whether unraveling the intricate dynamics of particles in a physical system or shedding light on the behavior of systems under stochastic influences, the Langevin equation has proven to be a robust and accurate modeling framework. Its applications are far-reaching, encompassing the realm of defense systems, where it aids in the analysis of unpredictable scenarios, as well as image processing, chemistry, astronomy, and mechanical and electrical engineering, where it provides invaluable insights for problem-solving and optimization. Moreover, the Langevin equation finds its place in the study of Brownian motion, serving as a fundamental tool when describing the effect of random oscillation forces, often characterized as Gaussian noise. Furthermore, in the pursuit of cleaner data and noise reduction, fractional order differential equations emerge as essential allies. These equations offer a sophisticated means of enhancing signal processing and minimizing noise interference in various applications. For a more in-depth exploration of this multifaceted subject, interested readers are encouraged to delve into the extensive body of work presented in references such as [2, 12, 20, 21, 29, 30].

The study of impulsive differential equations has garnered substantial attention from researchers in recent times, owing to their wide-ranging applications in various domains of science and technology. These equations serve as a powerful tool for describing dynamic processes that undergo abrupt changes and discontinuous jumps in their states. Numerous physical systems, such as the motion of a pendulum clock, the impact dynamics of mechanical systems, species preservation through periodic stocking or harvesting, and the functioning of the heart, naturally exhibit impulsive phenomena as part of their behavior. Additionally, impulsive behavior is prevalent in various other scenarios, such as interruptions in cellular neural networks, the operation of dampers with percussive effects, electromechanical systems with relaxational oscillations, and dynamical systems with automatic regulations. The extensive spectrum of applications underscores the significance and widespread relevance of impulsive differential equations in contemporary research. For a comprehensive exploration of this subject, we refer interested readers to sources such as [10, 13, 42, 18, 47, 49, 5, 45, 32]. The remarkable diversity of applications in this field underscores its importance and the notable attention it has received from the research community.

At the University of Wisconsin, Ulam posed a fundamental question in 1940 regarding the stability of functional equations, which has since become a cornerstone in mathematical stability theory. Ulam's question revolved around the conditions under which an additive mapping exists in the vicinity of an approximately additive mapping, as detailed in [38]. In 1941, Hyers made a significant contribution by providing a partial answer to Ulam's query, particularly within the context of Banach spaces, as documented in [14].

Subsequently, this notion of stability became known as Ulam-Hyers stability. Building on this foundation, in 1978, Rassias introduced a remarkable generalization of Ulam-Hyers stability by considering variables, thus expanding the scope of this mathematical framework. For further insights into this intriguing topic and its various developments, we recommend exploring additional references such as [33, 35, 41, 48, 50, 15, 40]. These sources delve into the intricacies and applications of stability theory, offering a comprehensive understanding of its significance in mathematics.

Recently, the existence, uniqueness and different types of fractional differential equations stability of nonlinear implicit fractional differential equations with Caputo fractional derivative have received a considerable attention, see [7, 9, 34, 35, 39].

Wang *et al.* [44], conducted an investigation into the generalized stability of the Ulam–Hyers–Rassias type for a fractional differential equation of the form:

$$\begin{cases} {}^{c}D^{\alpha}_{0,\omega} z(\omega) = f(\omega, z(\omega)), \quad \omega \in (\omega_i, s_i], \quad i = 0, 1, \ldots, m, \quad 0 < \alpha < 1, \\ z(\omega) = g_i(\omega, z(\omega)), \quad \omega \in (s_{i-1}, \omega_i], \quad i = 1, 2, \ldots, m. \end{cases}$$

Zada *et al.* [46], conducted a study to examine the existence and uniqueness of solutions. They employed the Diaz–Margolis fixed point theorem for their analysis. Furthermore, they explored different manifestations of stability within the context of Ulam–Hyers stability. This research focused on a particular class of nonlinear implicit fractional differential equations that feature non-instantaneous integral impulses and nonlinear integral boundary conditions:

$$\begin{cases} {}^{c}D^{\alpha}_{0,\omega} z(\omega) = f(\omega, z(\omega), {}^{c}D^{\alpha}_{0,\omega} z(\omega)), \ \omega \in (\omega_i, s_i], \ i = 0, 1, \ldots, m, \ 0 < \alpha < 1, \ \omega \in (0, 1], \\ z(\omega) = I^{\alpha}_{s_{i-1},\omega_i}(\xi_i(\omega, z(\omega))), \ \omega \in (s_{i-1}, \omega_i], \quad i = 1, 2, \ldots, m, \\ z(0) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^T (T - \varsigma)^{\alpha-1} \eta(\varsigma, z(\varsigma)) d\varsigma. \end{cases}$$

Building on the aforementioned research, this paper delves into an examination of several concepts, including existence, uniqueness, Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stability, for a nonlinear implicit impulsive Langevin equation by incorporates two fractional derivatives:

$$\begin{cases} D^{\vartheta}(D^{\theta} + \kappa) u(\xi) = f(\xi, u(\xi), D^{\vartheta} u(\xi)), \quad \xi \in J' = J - \{\xi_1, \xi_2, \xi_3, \ldots, \xi_m\}, J = [0, T], \\ \Delta u(\xi_k) = u(\xi^+) - u(\xi^-) = I_k(u(\xi)), \\ u(0) = 0, \quad u(\eta_k) = 0, u(1) = 0, \eta_k = (\xi_k, \xi_{k+1}), k = 0, 1, 2 \ldots, m - 1, \end{cases} \tag{1.1}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are a given functions, $0 < \theta, \vartheta < 1$, with $0 < \theta + \vartheta < 1$, and $\kappa > 0, 0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_m < \xi_{m+1} = 1, u(\xi_k^+) = \lim_{\varepsilon \to 0^+} u(\xi_k + \varepsilon)$, $u(\xi_k^-) = \lim_{\varepsilon \to 0^-} u(\xi_k + \varepsilon)$, represent the right and left limits of $u(\xi)$ at $\xi = \xi_k$ the constants I_k denotes the size of the jump.

The paper's second section presents a set of notations, definitions, and supplementary findings. In Section 3, we establish the existence and uniqueness of solutions for the model described in problem (1.1). These results are derived using the Banach contraction principle and Krasnoselski fixed point theorem. In Section 4, our focus shifts to an investigation of various stability concepts, including Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability, all pertaining to the proposed model. Finally, we illustrate our main findings with an example that reinforces our conclusions.

2 Preliminaries

We revisit certain fractional calculus definitions from sources such as [27, 17].

Definition 1 *The fractional integral of order θ with respect to the variable ω for a function f is defined as follows:*

$$I^{\theta}_{0,\omega} f(\omega) = \frac{1}{\Gamma(\theta)} \int_0^{\omega} f(\varsigma)(\omega-\varsigma)^{\theta-1} d\varsigma, \quad \omega > 0,\ \theta > 0.$$

Here, the function $\Gamma(\cdot)$ corresponds to the Gamma function.

Definition 2 *The fractional derivative of order θ for the function f is the Riemann-Liouville fractional derivative:*

$${}^{L}D^{\theta}_{0,\omega} f(\omega) = \frac{1}{\Gamma(n-\theta)} \frac{d^n}{d\omega^n} \int_0^{\omega} \frac{f(\varsigma)}{(\omega-\varsigma)^{\theta+1-n}} d\varsigma, \quad \omega > 0,\ n-1 < \theta < n.$$

Definition 3 *The Caputo derivative of fractional order θ for f is*

$${}^{c}D^{\theta}_{0,\omega} f(\omega) = \frac{1}{\Gamma(n-\theta)} \int_0^{\omega} (\omega-\varsigma)^{n-\theta-1} f^{(n)}(\varsigma) d\varsigma, \qquad \text{where } n = [\theta] + 1.$$

Definition 4 *The classical Caputo derivative of order θ of f is*

$${}^{c}D^{\theta}_{0,\omega} = {}^{L}D^{\theta}_{0,\omega}\left(f(\omega) - \sum_{k=0}^{n-1} \frac{\omega^k}{k!} f^{(k)}(0) \right), \quad \omega > 0,\ n-1 < \theta < n.$$

Remark 2.1 **(a)** *Operator $D^{\theta,\vartheta}$ also can be written as*

$$D^{\theta,\vartheta} f(x) = (I^{\vartheta(1-\theta)} D^{(1-\vartheta)(1-\theta)})) = I^{\vartheta(1-\theta)} D^{\gamma},\ \gamma = \theta + \vartheta - \theta\vartheta.$$

(b) *If $\vartheta = 0$, then $D^{\theta,\vartheta} = D^{\theta,0}$ is called Riemman–Liouville fractional derivative.*

(c) *If $\vartheta = 1$, then $D^{\theta,\vartheta} = I^{1-\theta} D$ is called Caputo fractional derivative.*

Lemma 2.1 *[17] The fractional differential equation ${}^{c}D^{\theta} f(\omega) = 0$ with $\theta > 0$, involving Caputo differential operator ${}^{c}D^{\theta}$ have a solution in the following form:*

$$f(\omega) = c_0 + c_1\omega + c_2\omega^2 + \cdots + c_{m-1}\omega^{m-1},$$

where $\quad c_i \in \mathbb{R}, \quad i = 0, 1, \ldots, m-1 \quad$ *and* $\quad m = [\theta] + 1.$

Lemma 2.2 *[17] For arbitrary $\theta > 0$, we have*

$$I^{\theta}\big({}^{c}D^{\theta}f(\omega)\big) = c_0 + c_1\omega + c_2\omega^2 + \cdots + c_{m-1}\omega^{m-1},$$

where $c_i \in \mathbb{R}, \quad i = 0, 1, \ldots, m-1$ *and* $m = [\theta] + 1$.

Lemma 2.3 *[27] Let $\theta > 0$ and $\vartheta > 0$, $f \in L^1\big([a, b]\big)$.*

Then $I^{\theta}I^{\vartheta}f(\omega) = I^{\theta+\vartheta}f(\omega), {}^{c}D^{\theta}_{0,\omega}({}^{c}D^{\vartheta}_{0,\omega}f(\omega)) = {}^{c}D^{\theta+\vartheta}_{0,\omega}f(\omega)$ *and* $I^{\theta}D^{\theta}_{0,\omega}f(\omega) = f(\omega), \omega \in [a, b]$.

Let $J = [0, T]$, $J_0 = [0, \omega_1]$, $J_1 = (\omega_1, \omega_2]$, $J_2 = (\omega_2, \omega_3], \ldots,$ $J_{i-1} = (\omega_{i-1}, \omega_i]$, $J_i = (\omega_i, T]$,
$J^{'} = J - \{\omega_0, \omega_1, \omega_2, \ldots, \omega_i\}$. Also for convenience use the notation $J_i = (\omega_i, \omega_{i+1}]$.

Theorem 2.2 *Let $\mathcal{M}$ be a closed convex and nonempty subset of a Banach space X. Let A, B be two operators such that*

1. *$\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$ whenever $x, y \in \mathcal{M}$,*
2. *$\mathcal{A}$ is a compact and continuous,*
3. *$\mathcal{B}$ is a contraction mapping.*

Then there exists $z \in M$ such that $z = \mathcal{A}z + \mathcal{B}z$.

Theorem 2.3 *[[4]**(Banach's fixed point theorem)**]. Let B be a Banach space. Then any contraction mapping $N : B \to B$ has a unique fixed point.*

Theorem 2.4 *A general solution u of the problem (1.1) is given by*

$$u(\xi) = \begin{cases} \displaystyle\int_0^{\xi}(\xi - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz \\ \displaystyle - N_m\int_0^{\eta_0}(\eta_0 - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0 - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz, \ \textit{for } \xi \in J_0, \\ \displaystyle\int_0^{\xi}(\xi - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz \\ \displaystyle + M_mI_k(u(\xi)) - M_mN_m\int_0^{\eta_0}(\eta_0 - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0 - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz \\ \displaystyle + M_m\int_0^{\eta_k}(\eta_m - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_m - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz, \textit{for} \xi \in J_k, k = \{1, \ldots, m\}, \end{cases} \tag{2.1}$$

where $M_m = \max\{\frac{\mathbf{E}_{\theta}(-\xi^{\theta}\kappa) - \mathbf{E}_{\theta}(-\eta_m^{\theta}\kappa)}{\mathbf{E}_{\theta}(-\xi_m^{\theta}\kappa) - \mathbf{E}_{\theta}(-\eta_m^{\theta}\kappa)}\} = \frac{\mathbf{E}_{\theta}(-\xi^{\theta}\kappa) - \mathbf{E}_{\theta}(-\xi_m^{\theta}\kappa)}{\mathbf{E}_{\theta}(-\xi_m^{\theta}\kappa) - \mathbf{E}_{\theta}(-\eta_m^{\theta}\kappa)}$ *and* $N_m = \max\{\frac{1 - \mathbf{E}_{\theta}(-\xi^{\theta}\kappa)}{1 - \mathbf{E}_{\theta}(-\eta_0^{\theta}\kappa)}\}$.

Proof *To accomplish our objectives, we initiate our investigation by examining linear Langevin equations that feature two distinct fractional derivatives.*

$$D^{\vartheta}(D^{\theta} + \kappa)u(\xi) = f(\xi), \quad \xi \in J = [0, T], \tag{2.2}$$

by integrating the above equation from zero to ξ*, we have*

$$(D^{\theta}+\kappa)u(\xi)=\frac{1}{\Gamma(\vartheta)}\int_0^{\xi}(\xi-s)^{\vartheta-1}f(s)ds-a_k,\quad k=1,2,3,\ldots,m, \tag{2.3}$$

where a_k *are constants.*

Utilizing the same concepts and methodologies as employed in Wang et al.[41], we arrive at the comprehensive solution for

$$(D^{\theta}+\kappa)u(\xi)=h(\xi) \tag{2.4}$$

is

$$u(\xi)=\Phi(\xi)b_i+\int_0^{\xi}(\xi-s)^{\theta-1}\Psi(\xi-s)h(s)ds, \tag{2.5}$$

where $\Phi(\xi)=\int_0^{\infty}w_{\theta}(\rho)e^{-\xi^{\theta}\rho\kappa}d\theta, \Psi(\xi)=\rho\int_0^{\infty}\rho w_{\theta}(\rho)e^{-\xi^{\theta}\rho\kappa}d\theta$*. Here* w_{θ} *is a one–side probability density function (see Wang [41]) defined on* $(0,\infty)$ *and satisfying* $\int_0^{\infty}\theta^{\nu}w_{\theta}(\rho)d\theta=\frac{\Gamma(1+\nu)}{\Gamma(1+\theta\nu)}, -1<\nu<\infty$*. Meanwhile, the solution of the equation* (2.4) *have been considered in the monograph [17], and it is given by the following expression:*

$$u(\xi)=\mathbf{E}_{\theta}(-\xi^{\theta}\kappa)b_i+\int_0^{\xi}(\xi-s)^{\theta-1}\mathbf{E}_{\theta,\theta}(-(\xi-s)^{\theta}\kappa)h(s)ds, \tag{2.6}$$

where $\mathbf{E}_{\theta}$ *is the classical Mittag-Leffler function:* $\mathbf{E}_{\theta}(z)=\sum_{k-0}^{\infty}\frac{z^k}{\Gamma(k\theta+1)}, z\in\mathbb{R},\theta>0$ *and the function* $\mathbf{E}_{\theta,\theta}$ *is the generalized Mittag-Leffler function:* $\mathbf{E}_{\theta,\theta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\theta+\theta)}, z,\theta\in\mathbb{R},\theta>0$ *combined* (2.5) *and* (2.6)*, we can rewrite* $\Phi(\xi)=\mathbf{E}_{\theta}(-\xi^{\theta}\kappa), \Psi(\xi)=\mathbf{E}_{\theta,\theta}(-\xi^{\theta}\kappa)$*. Note* $\mathbf{E}_{\theta,\theta}(z)=\theta\mathbf{E}'_{\theta}(z)$ *and so*

$$(\xi-s)^{\theta-1}\mathbf{E}_{\theta,\theta}(-(\xi-s)^{\theta}\kappa)=\frac{d}{ds}[\frac{1}{\kappa}\mathbf{E}_{\theta}(-(\xi-s)^{\theta}\kappa)]. \tag{2.7}$$

This yields that $\int_0^{\xi}(\xi-s)^{\theta-1}\mathbf{E}_{\theta,\theta}(-(\xi-s)^{\theta}\kappa)ds=\frac{1}{\kappa}[1-\mathbf{E}_{\theta}(-\xi^{\theta}\kappa)]$*. So the final formula of solution of the equation* (2.2) *should be*

$$\begin{aligned}u(\xi) &= \mathbf{E}_{\theta}(-\xi^{\theta}\kappa)b_i+\int_0^{\xi}(\xi-s)^{\theta-1}\mathbf{E}_{\theta,\theta}(-(\xi-s)^{\theta}\kappa)\Big(\int_0^{s}\frac{(s-z)^{\vartheta-1}}{\Gamma(\vartheta)}f(z)dz-a_k\Big)ds\\ &= \mathbf{E}_{\theta}(-\xi^{\theta}\kappa)b_i-a_k\int_0^{\xi}(\xi-s)^{\theta-1}\mathbf{E}_{\theta,\theta}(-(\xi-s)^{\theta}\kappa)ds\\ &\quad+\int_0^{\xi}\int_0^{s}\frac{(s-z)^{\vartheta-1}}{\Gamma(\vartheta)}(\xi-s)^{\theta-1}\mathbf{E}_{\theta,\theta}(-(\xi-s)^{\theta}\kappa)f(z)dzds\\ &= \mathbf{E}_{\theta}(-\xi^{\theta}\kappa)b_i-\frac{1}{\kappa}[1-\mathbf{E}_{\theta}(-\xi^{\theta}\kappa)]a_k\\ &\quad+\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)f(z)dz,\end{aligned} \tag{2.8}$$

where $\mathbf{E}_{\theta,\theta+\vartheta}$ *is the generalized Mittag–Leffler function:* $\mathbf{E}_{\theta,\theta+\vartheta}=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\theta+\theta+\vartheta)}$.

For $\xi \in J_0$*, by integrating the first equation in* (1.1)*, one can derive the following result*

$$u(\xi) = \mathbf{E}_{\theta}(-\xi^{\theta}\kappa)b_k - \frac{1}{\kappa}[1 - \mathbf{E}_{\theta}(-\xi^{\theta}\kappa)]a_k + \int_0^{\xi} (\xi - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz. \quad (2.9)$$

Using the conditions $u(0) = 0$ *and* $u(\eta_0) = 0$*, we get*

$$a = \frac{\kappa}{[1 - \mathbf{E}_{\theta}(-\eta_0^{\theta}\kappa)]} \int_0^{\eta_0} (\eta_0 - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0 - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz, \qquad b = 0. \quad (2.10)$$

Submitting (2.10) *in* (2.9)*, we obtain*

$$\begin{aligned} u(\xi) &= -N_m \int_0^{\eta_0} (\eta_0 - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0 - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz \\ &\quad + \int_0^{\xi} (\xi - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz. \end{aligned} \quad (2.11)$$

For $\xi \in J_1$*, by integrating the first equation in* (1.1)*, one can derive the following result:*

$$u(\xi) = \mathbf{E}_{\theta}(-\xi^{\theta}\kappa)b_1 - \frac{[1 - \mathbf{E}_{\theta}(-\xi^{\theta}\kappa)]}{\kappa}a_1 + \int_0^{\xi} (\xi - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz \quad (2.12)$$

Since

$$\begin{aligned} u(\xi^+) &= \mathbf{E}_{\theta}(-\xi^{\theta}\kappa)b_1 - \frac{1}{\kappa}[1 - \mathbf{E}_{\theta}(-\xi^{\theta}\kappa)]a_1 \\ &\quad + \int_0^{\xi} (\xi - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz, \end{aligned}$$

and

$$\begin{aligned} u(\xi^-) &= -N_m \int_0^{\eta_0} (\eta_0 - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0 - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz \\ &\quad + \int_0^{\xi} (\xi - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz, \end{aligned}$$

from $u(\xi_1^+) = u(\xi_1^-) + I_1(u(\eta_1)) = 0$*, it follows*

$$\begin{aligned} \mathbf{E}_{\theta}(-\xi_1^{\theta}\kappa)b_1 - \frac{1}{\kappa}[1 - \mathbf{E}_{\theta}(-\xi_1^{\theta}\kappa)]a_1 &= I_1(u(\xi)) \\ &\quad -N_m \int_0^{\eta_0} (\eta_0 - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0 - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz, \end{aligned}$$

and

$$\mathbf{E}_{\theta}(-\eta_1^{\theta}\kappa)b_1 - \frac{1}{\kappa}[1 - \mathbf{E}_{\theta}(-\eta_1^{\theta}\kappa)]a_1 + \int_0^{\eta_1} (\eta_1 - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_1 - z)^{\theta}\kappa)f(z, u(z), D^{\vartheta}u(z))dz = 0,$$

solving the above equations for unknowns a_1 and b_1 and putting the values in equation (2.12), we can get

$$\begin{aligned} u(\xi) &= \int_0^{\xi} (\xi - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz + \left(\frac{\mathbf{E}_{\theta}(-\xi^{\theta}\kappa) - \mathbf{E}_{\theta}(-\eta_1^{\theta}\kappa)}{\mathbf{E}_{\theta}(-\xi_1^{\theta}\kappa) - \mathbf{E}_{\theta}(-\eta_1^{\theta}\kappa)}\right) I_1(u(\xi)) \\ &\quad - \left(\frac{\mathbf{E}_{\theta}(-\xi^{\theta}\kappa) - \mathbf{E}_{\theta}(-\eta_1^{\theta}\kappa)}{\mathbf{E}_{\theta}(-\xi_1^{\theta}\kappa) - \mathbf{E}_{\theta}(-\eta_1^{\theta}\kappa)}\right) N_m \int_0^{\eta_0} (\eta_0 - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0 - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz \\ &\quad + \left(\frac{\mathbf{E}_{\theta}(-\xi^{\theta}\kappa) - \mathbf{E}_{\theta}(-\xi_1^{\theta}\kappa)}{\mathbf{E}_{\theta}(-\xi_1^{\theta}\kappa) - \mathbf{E}_{\theta}(-\eta_1^{\theta}\kappa)}\right) \int_0^{\eta_1} (\eta_1 - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_1 - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz \end{aligned}$$

$$\begin{aligned} &= \int_0^{\xi} (\xi - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz + M_m I_1(u(\xi)) \\ &\quad - M_m N_m \int_0^{\eta_0} (\eta_0 - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0 - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz \\ &\quad + M_m \int_0^{\eta_1} (\eta_1 - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_1 - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz. \end{aligned}$$

Repeating the above methods on the subinterval $J_k, k = 2, 3, \ldots, m-1$ respectively.

Finally, for $\xi \in J_k$, by integrating both sides of the first equation in (1.1), one can derive the following result

$$\begin{aligned} u(\xi) &= \int_0^{\xi} (\xi - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz + M_m I_k(u(\xi)) \\ &\quad - M_m N_m \int_0^{\eta_0} (\eta_0 - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0 - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz \\ &\quad + M_m \int_0^{\eta_m} (\eta_m - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_m - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz. \end{aligned}$$

Lemma 2.4 *[43], Let $0 < \vartheta, \theta < 1$. The functions $\mathbf{E}_{\theta}, \mathbf{E}_{\theta,\theta}$ and $\mathbf{E}_{\theta,\theta+\vartheta}$ are nonnegative and have the following properties:*

1. *For any $\kappa > 0$ and $\xi \in J$,*

$$\mathbf{E}_{\theta}(-\xi^{\theta}\kappa) \leq 1, \quad \mathbf{E}_{\theta,\theta}(-\xi^{\theta}\kappa) \leq \frac{1}{\Gamma(\theta)}, \quad \mathbf{E}_{\theta,\theta+\vartheta}(-\xi^{\theta}\kappa) \leq \frac{1}{\Gamma(\theta+\vartheta)}.$$

   Moreover, $\mathbf{E}_{\theta}(0) = 1, \quad \mathbf{E}_{\theta,\theta}(0) = \frac{1}{\Gamma(\theta)}, \quad \mathbf{E}_{\theta,\theta+\vartheta}(0) = \frac{1}{\Gamma(\theta+\vartheta)}$.

2. *For any $\kappa > 0$ and $\xi_1, \xi_2 \in J$,*

$$\mathbf{E}_{\theta}(-\xi_2^{\theta}\kappa) \to \mathbf{E}_{\theta}(-\xi_1^{\theta}\kappa) \quad as \quad \xi_2 \to \xi_1,$$

$$\mathbf{E}_{\theta,\theta}(-\xi_2^{\theta}\kappa) \to \mathbf{E}_{\theta,\theta}(-\xi_1^{\theta}\kappa) \quad as \quad \xi_2 \to \xi_1,$$

$$\mathbf{E}_{\theta,\theta+\vartheta}(-\xi_2^{\theta}\kappa) \to \mathbf{E}_{\theta,\theta+\vartheta}(-\xi_1^{\theta}\kappa) \quad as \quad \xi_2 \to \xi_1,$$

Or rather,

$$\|\mathbf{E}_\theta(-\xi_2^\theta\kappa) - \mathbf{E}_\theta(-\xi_1^\theta\kappa)\| = \mathbb{O}(|\xi_2 - \xi_1|^\theta) \quad as \quad \xi_2 \to \xi_1,$$

$$\|\mathbf{E}_{\theta,\theta}(-\xi_2^\theta\kappa) - \mathbf{E}_{\theta,\theta}(-\xi_1^\theta\kappa)\| = \mathbb{O}(|\xi_2 - \xi_1|^\theta) \quad as \quad \xi_2 \to \xi_1,$$

$$\|\mathbf{E}_{\theta,\theta+\vartheta}(-\xi_2^\theta\kappa) - \mathbf{E}_{\theta,\theta+\vartheta}(-\xi_1^\theta\kappa)\| = \mathbb{O}(|\xi_2 - \xi_1|^\theta) \quad as \quad \xi_2 \to \xi_1.$$

3. *For any* $\kappa > 0$ *and* $\xi_1, \xi_2 \in J$ *and* $\xi_1 \leq \xi_2$,

$$\mathbf{E}_\theta(-\xi_2^\theta\kappa) \leq \mathbf{E}_\theta(-\xi_1^\theta\kappa),$$

$$\mathbf{E}_{\theta,\theta}(-\xi_2^\theta\kappa) \leq \mathbf{E}_{\theta,\theta}(-\xi_1^\theta\kappa),$$

$$\mathbf{E}_{\theta,\theta+\vartheta}(-\xi_2^\theta\kappa) \leq \mathbf{E}_{\theta,\theta+\vartheta}(-\xi_1^\theta\kappa).$$

4. *For any* $\kappa > 0$ *and* $\xi_* > 0$,

$$\mathbf{E}_\theta(-\xi_*^\theta\kappa) > 0.$$

3 Existence and uniqueness results

In this section, we focus on establishing the existence and uniqueness of solutions for the problem described by equation (1.1). Recent literature has witnessed numerous works addressing fractional impulsive initial and boundary value problems. However, it is noteworthy that both the research conducted by Omar *et al.* [26] and the work of Zada *et al.* [40] have brought to light certain inaccuracies in previous solutions for specific impulsive fractional differential equations. They accomplished this by introducing counterexamples and developing a comprehensive framework to explore a more accurate solution for such problems. This endeavor is largely inspired by the findings in Wang *et al.* [43].

Before stating and proving the main results, we introduce the following hypotheses.

(H_1) $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is jointly continuous.

(H_2) There exists a function $n(\cdot) \in L^{1/q_1}(J, \mathbb{R}^+)$ such that $|f(\xi, x, y)| \leq n(\xi)$ for all $\xi \in J$ and all $x, y \in R$, where $q_1 \in (0, \theta + \vartheta)$.

(H_3) There exists a function $h(\cdot) \in L^{1/q_2}(J, \mathbb{R}^+)$ such that $|f(\xi, x, y) - f(\xi, w, z)| \leq h(\xi)|x - w| + L_f|y - z|$ for all $\xi \in J$ and all $x, y, w, z \in R$, where $q_2 \in (0, \theta + \vartheta)$.

(H_4) There exists $L_k > 0$, such that $\big|I_k(u) - I_k(v)\big| \leq L_k|u - v|$, for each $v \in J_k$, $k = 1, 2, \ldots, m$, and for all $u, v \in \mathbb{R}$.

Theorem 3.1 *Assume that* $(H_1) - (H_3)$ *hold. If*

$$M_F M_G < 1, \tag{3.1}$$

then the problem (1.1) *has an unique solution, where*

$$M_F = \frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}, \tag{3.2}$$

$M_G = \max\big(\{1+\frac{L_f\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+M_m(L_k+1-N_m)+\frac{L_f\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{L_f\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}\},\{1+\frac{L_f(\xi^{\theta+\vartheta}+\eta_0^{\theta+\vartheta})}{\Gamma(\theta+\vartheta+1)}-N_m\}\big)$.

Proof *Consider an operator* $N : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ *defined by*

$$(Nu)(\xi) = \begin{cases} \int_0^\xi (\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^\theta\kappa)f(z,u(z),D^\vartheta u(z))dz \\ -N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^\theta\kappa)f(z,u(z),D^\vartheta u(z))dz, \text{ for } \xi\in J_0, \\ \int_0^\xi (\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^\theta\kappa)f(z,u(z),D^\vartheta u(z))dz \\ +M_mI_k(u(\xi))-M_mN_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^\theta\kappa)f(z,u(z),D^\vartheta u(z))dz \\ +M_m\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_k-z)^\theta\kappa)f(z,u(z),D^\vartheta u(z))dz, \text{ for } \xi\in J_k. \end{cases} \tag{3.3}$$

To establish that N *is a contraction mapping, we will break down our proof into two distinct steps.*

Step: 1. $N(u) \in PC(J,\mathbb{R})$ *for every* $u \in PC(J,\mathbb{R})$.

If $\xi \in J_0$, *then for every* $u \in PC(J,\mathbb{R})$. *and any* $\delta > 0, 0 < \xi < \xi+\delta \le \xi_1$, *by* (H_2), *Lemma 2.4 and Holder inequality, we get*

$$\begin{aligned} & |(N)u(\xi+\delta)-(N)u(\xi)| \\ \le\; & \Big|\int_0^{\xi+\delta}(\xi+\delta-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi+\delta-z)^\theta\kappa)f(z,u(z),D^\vartheta u(z))dz \\ & -\frac{1-\mathbf{E}_\theta(-(\xi+\delta)^\theta\kappa)}{1-\mathbf{E}_\theta(-\eta_0^\theta\kappa)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^\theta\kappa)f(z,u(z),D^\vartheta u(z))dz \\ & -\int_0^\xi(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^\theta\kappa)f(z,u(z),D^\vartheta u(z))dz \\ & +N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^\theta\kappa)f(z,u(z),D^\vartheta u(z))dz\Big| \end{aligned}$$

$$\begin{aligned}
= & \ \Big|\int_0^{\xi+\delta}(\xi+\delta-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi+\delta-z)^{\theta}\kappa)f(z,u(z),D^{\vartheta}u(z))dz \\
& -\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)f(z,u(z),D^{\vartheta}u(z))dz\Big| \\
& +\Big|\frac{\mathbf{E}_{\theta}(-(\xi+\delta)^{\theta}\kappa)-\mathbf{E}_{\theta}(-\xi^{\theta}\kappa)}{1-\mathbf{E}_{\theta}(-\eta_0^{\theta}\kappa)} \\
& \times\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)f(z,u(z),D^{\vartheta}u(z))dz\Big| \\
\leq & \int_0^{\xi}(\xi+\delta-z)^{\theta+\vartheta-1}|\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi+\delta-z)^{\theta}\kappa)-\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)|n(z)dz \\
& +\int_0^{\xi}|(\xi+\delta-z)^{\theta+\vartheta-1}-(\xi-z)^{\theta+\vartheta-1}||\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)|n(z)dz \\
& +\int_{\xi}^{\xi+\delta}|(\xi+\delta-z)^{\theta+\vartheta-1}|\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi+\delta-z)^{\theta}\kappa)|n(z)dz \\
& +\Big|\frac{\mathbf{E}_{\theta}(-(\xi+\delta)^{\theta}\kappa)-\mathbf{E}_{\theta}(-\xi^{\theta}\kappa)}{1-\mathbf{E}_{\theta}(-\eta_0^{\theta}\kappa)}\Big|\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)n(z)dz
\end{aligned}$$

$$\begin{aligned}
\leq & \ \mathbf{O}(\delta^{\theta})\Big(\int_0^{\xi}(\xi+\delta-z)^{\frac{\theta+\vartheta-1}{1-q_1}}dz\Big)^{1-q_1}\Big(\int_0^{\xi}n(z)^{\frac{1}{q_1}}dz\Big)^{q_1} \\
& +\frac{1}{\Gamma(\theta+\vartheta)}\Big(\int_0^{\xi}((\xi-z)^{\theta+\vartheta-1}-(\xi+\delta-z)^{\theta+\vartheta-1})^{\frac{1}{1-q_1}}dz\Big)^{1-q_1}\Big(\int_0^{\xi}n(z)^{\frac{1}{q_1}}dz\Big)^{q_1} \\
& +\frac{1}{\Gamma(\theta+\vartheta)}\Big(\int_{\xi}^{\xi+\delta}(\xi+\delta-z)^{\frac{\theta+\vartheta-1}{1-q_1}}dz\Big)^{1-q_1}\Big(\int_{\xi}^{\xi+\delta}n(z)^{\frac{1}{q_1}}dz\Big)^{q_1} \\
& +\frac{\mathbf{O}(\delta^{\theta})}{1-\mathbf{E}_{\theta}(-\eta_0^{\theta}\kappa)}\frac{1}{\Gamma(\theta+\vartheta)}\Big(\int_0^{\eta_0}(\eta_0-z)^{\frac{\theta+\vartheta-1}{1-q_1}}dz\Big)^{1-q_1}\Big(\int_0^{\eta_0}n(z)^{\frac{1}{q_1}}dz\Big)^{q_1}
\end{aligned}$$

$$\leq \mathbf{O}(\delta^{\theta})\Big(\frac{\|n\|_{L^{\frac{1}{q_1}}(J_0)}}{(1-\mathbf{E}_{\theta}(-\eta_0^{\theta}\kappa))\Gamma(\theta+\vartheta)(1+p_1)^{1-q_1}}+\frac{\|n\|_{L^{\frac{1}{q_1}}(J_0)}}{(1+p_1)^{1-q_1}}\Big)+\frac{2\delta^{(1+p_1)(1-q_1)}\|n\|_{L^{\frac{1}{q_1}}(J_0)}}{\Gamma(\theta+\vartheta)(1+p_1)^{1-q_1}}\to 0 \tag{3.4}$$

as $\delta\to 0$, where we use the facts

- $\int_0^{\xi}(\xi+\delta-z)^{\frac{\theta+\vartheta-1}{1-q_1}}dz\leq\frac{1}{1+p_1}$,
- $\int_0^{\xi}((\xi-z)^{\theta+\vartheta-1}-(\xi+\delta-z)^{\theta+\vartheta-1})^{\frac{1}{1-q_1}}dz\leq\frac{\delta^{1+p_1}}{1+p_1}$,
- $\int_{\xi}^{\xi+\delta}(\xi+\delta-z)^{\frac{\theta+\vartheta-1}{1-q_1}}dz=\frac{\delta^{1+p_1}}{1+p_1}$,
- $|(Fu)(\eta_0)|\leq\frac{1}{\theta+\vartheta}\Big(\int_0^{\eta_0}(\eta_0-z)^{\frac{\theta+\vartheta-1}{1-q_1}}dz\Big)^{1-q_1}\Big(\int_0^{\eta_0}n(z)^{\frac{1}{q_1}}dz\Big)^{q_1}\leq\frac{\|n\|_{L^{\frac{1}{q_1}}(J_0)}}{\Gamma(\theta+\vartheta)(1+p_1)^{1-q_1}}$.

Thus we obtain $N(u) \in C(J_0, \mathbb{R})$.

If $\xi \in J_1$, *then for every* $u \in C(J_1, \mathbb{R})$ *and any* $\delta > 0, \xi_1 < \xi < \xi + \delta \leq \xi_2$, *one can obtain*

$$\begin{aligned} |(N)u(\xi+\delta) - (N)u(\xi)| &\leq \mathbf{O}(\delta^{\theta})\Big(\frac{(1-\mathbf{E}_{\theta}(-\eta_1^{\theta}\kappa))|(Fu)(\eta_1)| + |(T_0u)(\xi_1)| + I_1}{(\mathbf{E}_{\theta}(-\xi_1^{\theta}\kappa)) - (\mathbf{E}_{\theta}(-\eta_1^{\theta}\kappa))} \\ &\quad + \frac{\|n\|_{L^{\frac{1}{q_1}}(J_0)}}{(1+p_1)^{1-q_1}}\Big) + \frac{2\delta^{(1+p_1)(1-q_1)}\|n\|_{L^{\frac{1}{q_1}}(J_0)}}{\Gamma(\theta+\vartheta)(1+p_1)^{1-q_1}} \to 0, \end{aligned}$$

as $\delta > 0$, *where we use the fact* $|(T_0u)(\xi_1)| \leq \frac{1}{1-\mathbf{E}_{\theta}(-\eta_0^{\theta}\kappa)}|(Fu)(|\eta_0)|$ *and* $|(Fu)(\eta_1)| \leq \frac{\|n\|_{L^{\frac{1}{q_1}}(J_0)}}{\Gamma(\theta+\vartheta)(1+p_1)^{1-q_1}}$, *Thus we obtain* $N(u) \in C(J_1, \mathbb{R})$.

With the same argument, one can verify that $N(u) \in C(J_i, \mathbb{R})$, *for every* $u, \in C(J_k, \mathbb{R}), k = 2, \ldots, m$. *From the above fact, we can conclude that* $N(u) \in PC(J, \mathbb{R})$, *for every* $u, \in C(J, \mathbb{R})$.

Step 2. *N is a contraction mapping on* $PC(J, \mathbb{R})$.

If $\xi \in J_0$, *for every arbitrary* $u, \overline{u} \in PC(J_0, \mathbb{R})$, *by* (H_3), *Lemma 2.4 and Holder inequality, we get*

$$\begin{aligned} |N(u) - N(\overline{u})| &\leq \Big|\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)f(z,u(z),D^{\vartheta}u(z))dz \\ &\quad -N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)f(z,u(z),D^{\vartheta}u(z))dz \\ &\quad -\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)f(z,\overline{u}(z),D^{\vartheta}\overline{u}(z))dz \\ &\quad +N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)f(z,\overline{u}(z),D^{\vartheta}\overline{u}(z))dz\Big| \end{aligned}$$

$$\begin{aligned} &\leq \int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z)) - f(z,\overline{u}(z),D^{\vartheta}\overline{u}(z))|dz \\ &\quad -N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z)) - f(z,\overline{u}(z),D^{\vartheta}\overline{u}(z))|dz \\ &\leq \frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}(h(z)|u(z)-\overline{u}(z)| + L_f|D^{\vartheta}u(z) - D^{\vartheta}\overline{u}(z)|)dz \\ &\quad -\frac{N_m}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}(h(z)|u(z)-\overline{u}(z)| + L_f|D^{\vartheta}u(z) - D^{\vartheta}\overline{u}(z)|dz \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\theta+\vartheta)}\bigg(\int_0^{\xi}(\xi-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\bigg)^{1-q_2}\bigg(\int_0^{\xi}h(z)^{\frac{1}{q_2}}dz\bigg)^{q_2}\|u(z)-\overline{u}(z)\| \\
&\quad+\frac{L_f\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}\|u(z)-\overline{u}(z)\|+\frac{L_f\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}|u(z)-\overline{u}(z)| \\
&\quad-\frac{N_m}{\Gamma(\theta+\vartheta)}\bigg(\int_0^{\eta_0}(\eta_0-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\bigg)^{1-q_2}\bigg(\int_0^{\eta_0}h(z)^{\frac{1}{q_2}}dz\bigg)^{q_2}\|u(z)-\overline{u}(z)\| \\
&\le \bigg(\frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}+\frac{L_f(\xi^{\theta+\vartheta}+\eta_0^{\theta+\vartheta})}{\Gamma(\theta+\vartheta+1)}-\frac{N_m\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\bigg)\|u(z)-\overline{u}(z)\| \\
&\le \frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\bigg(1+\frac{L_f(\xi^{\theta+\vartheta}+\eta_0^{\theta+\vartheta})}{\Gamma(\theta+\vartheta+1)}-N_m\bigg)\|u(z)-\overline{u}(z)\|
\end{aligned}
$$

$$
|N(u)-N(\overline{u})|\le M_FM_G\|u(z)-\overline{u}(z)\|. \tag{3.5}
$$

If $\xi\in J_k$, for arbitrary $u,\overline{u}\in C(J_k,\mathbb{R})$, we get

$$
\begin{aligned}
&|N(u)-N(\overline{u})| \\
\le\ & |\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))-f(z,\overline{u}(z),D^{\vartheta}\overline{u}(z))|dz \\
&-M_mN_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))-f(z,\overline{u}(z),D^{\vartheta}\overline{u}(z))|dz \\
&+M_m\int_0^{\eta_m}(\eta_m-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_m-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))-f(z,\overline{u}(z),D^{\vartheta}\overline{u}(z))|dz \\
&+M_m\|I_k(u(\xi))-I_k(\overline{u}(\xi))\| \\
\le\ & \frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}(h(z)|u(z)-\overline{u}(z)|+L_f|D^{\vartheta}u(z)-D^{\vartheta}\overline{u}(z)|)dz \\
&-\frac{M_mN_m}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}(h(z)|u(z)-\overline{u}(z)|+L_f|D^{\vartheta}u(z)-D^{\vartheta}\overline{u}(z)|)dz \\
&+\frac{M_m}{\Gamma(\theta+\vartheta)}\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}(h(z)|u(z)-\overline{u}(z)|+L_f|D^{\vartheta}u(z)-D^{\vartheta}\overline{u}(z)|)dz \\
&+M_m\|I_k(u(\xi))-I_k(\overline{u}(\xi))\|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{1}{\Gamma(\theta+\vartheta)}\bigg(\int_0^{\xi}(\xi-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\bigg)^{1-q_2}\bigg(\int_0^{\xi}h(z)^{\frac{1}{q_2}}dz\bigg)^{q_2}\|u(\xi)-\overline{u}(\xi)\| \\
&+\frac{L_f\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}\|u(\xi)-\overline{u}(\xi)\|+M_m Ł_k\|(u(\xi))-(\overline{u}(\xi))\| \\
&-\frac{M_mN_m}{\Gamma(\theta+\vartheta)}\bigg(\int_0^{\eta_0}(\eta_0-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\bigg)^{1-q_2}\bigg(\int_0^{\eta_0}h(z)^{\frac{1}{q_2}}dz\bigg)^{q_2}\|u(\xi)-\overline{u}(\xi)\| \\
&+M_m\frac{1}{\Gamma(\theta+\vartheta)}\bigg(\int_0^{\eta_k}(\eta_k-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\bigg)^{1-q_2}\bigg(\int_0^{\eta_k}h(z)^{\frac{1}{q_2}}dz\bigg)^{q_2}\|u(\xi)-\overline{u}(\xi)\| \\
&+\frac{L_f\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}|u(z)-\overline{u}(z)|+\frac{L_f\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}|u(z)-\overline{u}(z)| \\
\leq\ & \frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\|u(\xi)-\overline{u}(\xi)\|+\frac{L_f\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}\|u(\xi)-\overline{u}(\xi)\|+M_mŁ_k\|(u(\xi))-(\overline{u}(\xi))\| \\
&+\frac{L_f\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}\|u(\xi)-\overline{u}(\xi)\|-\frac{M_mN_m\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\|u(\xi)-\overline{u}(\xi)\| \\
&+\frac{L_f\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}\|u(\xi)-\overline{u}(\xi)\|+\frac{M_m\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\|u(\xi)-\overline{u}(\xi)\| \\
\leq\ & \bigg(\frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}+\frac{L_f\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+M_mŁ_k+\frac{L_f\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}-\frac{M_mN_m\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}} \\
&+\frac{L_f\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{M_m\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\bigg)\|u(\xi)-\overline{u}(\xi)\| \\
\leq\ & \frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\big(\frac{L_f\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{L_f\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{L_f\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+M_m-M_mN_m \\
&+M_mŁ_k+1\big)\|u(\xi)-\overline{u}(\xi)\| \\
\leq\ & M_FM_G\|u(\xi)-\overline{u}(\xi)\|,
\end{aligned}
$$

Due to the condition (3.1), *N has a unique fixed point on $PC(J,\mathbb{R})$ by Banach contraction mapping principle.*

Theorem 3.2 *Assume the conditions $(H_1)-(H_3)$ hold. If $M_FM_G<1$, then the problem* (1.1) *has at least a solution on $PC(J,\mathbb{R})$.*

Proof *Setting $\mathbf{B}_r = u \in PC(J,\mathbb{R}) : \|u\|_{PC} \leq r$, where $r \geq M_FM_G$, and M_G, M_F are finite positive constants defined by*
$M_m = \max\{\frac{\mathbf{E}_\theta(-\xi^\theta\kappa)-\mathbf{E}_\theta(-\eta_k^\theta\kappa)}{\mathbf{E}_\theta(-\xi_m^\theta\kappa)-\mathbf{E}_\theta(-\eta_k^\theta\kappa)}\} = \frac{\mathbf{E}_\theta(-\xi^\theta\kappa)-\mathbf{E}_\theta(-\xi_m^\theta\kappa)}{\mathbf{E}_\theta(-\xi_m^\theta\kappa)-\mathbf{E}_\theta(-\eta_k^\theta\kappa)}$ *and* $N_m = \max\{\frac{1-\mathbf{E}_\theta(-\xi^\theta\kappa)}{1-\mathbf{E}_\theta(-\eta_0^\theta\kappa)}\}$.

Step 1. *For every $u \in \mathbf{B}_r$ and if $\xi \in J_0$, by (H_3), Lemma 2.4 and Holder inequality, we get*

$$
\begin{aligned}
|u(\xi)| &\leq \Big|\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)f(z,u(z),D^{\vartheta}u(z))dz \\
&\quad -N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)f(z,u(z),D^{\vartheta}u(z))dz \\
&\leq \int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))|dz \\
&\quad -N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))|dz \\
&\leq \frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}h(z)dz-N_m\frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}h(z)dz \\
&\leq \frac{1}{\Gamma(\theta+\vartheta)}\big(\int_0^{\xi}(\xi-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\big)^{1-q_2}\big(\int_0^{\xi}h(z)^{\frac{1}{q_2}}dz\big)^{q_2} \\
&\quad -\frac{N_m}{\Gamma(\theta+\vartheta)}\big(\int_0^{\eta_0}(\eta_0-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\big)^{1-q_2}\big(\int_0^{\eta_0}h(z)^{\frac{1}{q_2}}dz\big)^{q_2} \\
&\leq \frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}-N_m\frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}} \\
&\leq M_F-N_mM_F.
\end{aligned}
$$

For every $u\in\mathbf{B}_r$ *and if* $\xi\in J_k$*, after a similar computation, we obtain*

$$
\begin{aligned}
|u(\xi)| &\leq \Big|\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))|dz+M_mI_k(|u(\xi)|) \\
&\quad -M_mN_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))|dz \\
&\quad +M_m\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_k-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))|dz \\
&\leq \frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}n(z)dz+M_m\|I_k(u(\xi))\| \\
&\quad -M_mN_m\frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}n(z)dz+\frac{M_m}{\Gamma(\theta+\vartheta)}\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}n(z)dz \\
&\leq \frac{1}{\Gamma(\theta+\vartheta)}\left(\int_0^{\xi}(\xi-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\xi}h(z)^{\frac{1}{q_2}}dz\right)^{q_2}+M_mE_k \\
&\quad -\frac{M_mN_m}{\Gamma(\theta+\vartheta)}\left(\int_0^{\eta_0}(\eta_0-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\eta_0}h(z)^{\frac{1}{q_2}}dz\right)^{q_2} \\
&\quad +\frac{M_m}{\Gamma(\theta+\vartheta)}\left(\int_0^{\eta_k}(\eta_k-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\eta_k}h(z)^{\frac{1}{q_2}}dz\right)^{q_2}
\end{aligned}
$$

$$\begin{aligned} &\leq \frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}} + M_m Ł_k - \frac{M_m N_m \|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}} + \frac{M_m \|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}} \\ &\leq \frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}} \big(1 + M_m Ł_k - M_m N_m + M_m\big) \\ &\leq M_F M_G. \end{aligned}$$

Where, $M_G = \max(1 + M_m Ł_k - M_m N_m + M_m)$. Due to the definition of the ball $\mathbf{B}_r$, we must have $u \in \mathbf{B}_r$ for any $\xi \in J_k$.

Step 2. *N is a contraction mapping on $\mathbf{B}_r$. By equation (3.5) we have, $|N(u) - N(\overline{u})| \leq M_F M_G \|u(\xi) - \overline{u}(\xi)\|$. The assumption $M_F M_G < 1$ implies that N is a contraction mapping.*

Step 3. *N is a completely continuous operator on $\mathbf{B}_r \in J_k, k = 0, 1, 2, \ldots, m$. Similar to Theorem 3.1, one can easy to verify that N is continuous and is uniformly bounded. Thus, N is a completely continuous operator on $\mathbf{B}_r \in J_k, k = 0, 1, 2, \ldots, m$.*

4 Ulam–Hyers stability analysis

Let $\varepsilon > 0$ and $\varphi : J \to \mathbb{R}^+$ be a continuous function. Consider

$$\begin{cases} D^\vartheta(D^\theta + \kappa)u(\xi) - f(\upsilon, u(\xi), D^\vartheta u(\xi))| \leq \varepsilon, & \xi \in J_k, \quad k = 1, 2, \ldots, m, \\ |\Delta\ u(\xi_k) - I_k(u(\xi_k))| \leq \varepsilon, & k = 1, 2, \ldots, m, \end{cases} \tag{4.1}$$

$$\begin{cases} D^\vartheta(D^\theta + \kappa)u(\xi) - f(\xi, u(\xi), D^\vartheta u(\xi))| \leq \varphi(\xi), & \xi \in J_k, \quad k = 1, 2, \ldots, m, \\ |\Delta\ u(\xi_k) - I_k(u(\xi_k))| \leq \psi, & k = 1, 2, \ldots, m, \end{cases} \tag{4.2}$$

and

$$\begin{cases} D^\vartheta(D^\theta + \kappa)u(\xi) - f(\xi, u(\xi), D^\vartheta u(\xi))| \leq \varepsilon\varphi(\xi), & \xi \in J_k, \quad k = 1, 2, \ldots, m, \\ |\Delta\ u(\xi_k) - I_k(u(\xi_k))| \leq \varepsilon\psi, & k = 1, 2, \ldots, m. \end{cases} \tag{4.3}$$

Definition 5 *The problem described by equation (1.1) exhibits Ulam-Hyers stability if there exists a real number denoted as $C_{f,i,q,\sigma}$, such that for any given small value of $\varepsilon > 0$, and for any solution u within the interval $[0, T]$ that satisfies the inequality defined by equation (4.1), there exists a corresponding solution ν within the same interval $[0, T]$, which fulfills the conditions outlined by problem (1.1) such that*

$$|u(\xi) - \nu(\xi)| \leq C_{f,i,q,\sigma}\ \varepsilon \quad \xi \in J. \tag{4.4}$$

Definition 6 *The problem described by equation (1.1) is considered to be generally Ulam-Hyers stable if there exists a function $\phi_{f,i,q,\sigma}$ defined on the interval $[0, T]$ with the properties $\phi_{f,i,q,\sigma}(0) = 0$ and a positive value $\varepsilon > 0$, such that for any solution u*

within the interval $[0,T]$ *that satisfies the inequality defined by equation* (4.1)*, there exists a corresponding solution* ν *within the same interval* $[0,T]$ *that satisfies the conditions specified by problem* (1.1) *such that*

$$|u(\xi)-\nu(\xi)| \leq \phi_{f,i,q,\sigma}\ \varepsilon \quad \xi \in J. \tag{4.5}$$

Remark 4.1 *Keep in mind that Definition 5* $\Rightarrow$ *Definition 6.*

Definition 7 *The stability of the problem defined by equation* (1.1) *under the Ulam-Hyers-Rassias framework with respect to the functions* (φ,ψ) *is established when there exists a positive constant denoted as* $C_{f,i,q,\sigma,\varphi}$*, such that for any given positive value of* $\varepsilon>0$*, and for any solution* u *within the interval* $[0,T]$ *that satisfies the inequality defined by equation* (4.3)*, there exists a corresponding solution* ν *within the same interval* $[0,T]$ *that satisfies the conditions specified by problem* (1.1) *with*

$$|u(\xi)-\nu(\xi)| \leq C_{f,i,q,\sigma,\varphi}\varepsilon(\varphi(\xi)+\psi)\ \varepsilon \quad \xi \in J. \tag{4.6}$$

Definition 8 *The problem defined by equation* (1.1) *is said to exhibit generalized Ulam-Hyers-Rassias stability with respect to the functions* (φ,ψ) *if there exists a positive constant denoted as* $C_{f,i,q,\sigma,\varphi}$*, such that for any solution* u *within the interval* $[0,T]$ *that satisfies the inequality defined by equation* (4.2)*, there exists a corresponding solution* ν *within the same interval* $[0,T]$ *that satisfies the conditions specified by problem* (1.1) *with*

$$|u(\xi)-\nu(\xi)| \leq C_{f,i,q,\sigma,\varphi}(\varphi(\xi)+\psi)\ \varepsilon \quad \xi \in J. \tag{4.7}$$

Remark 4.2 *It should be noted that Definition 7 implies Definition 8.*

Remark 4.3 *A function* $u\in[0,T]$ *is a solution of the inequality* (4.1) $\Leftrightarrow$ *there exists a function* $g\in[0,T]$ *and a sequence* $g_k, k=1,2,\ldots,m,$ *depending on* g*, such that*

(a) $|g(\xi)|\leq\varepsilon,\ |g_k|\leq\varepsilon \quad \xi\in J_k,\ k=1,2,\ldots,m,$

(b) $D^{\vartheta}(D^{\theta}+\kappa)u(\xi)=f(\xi,u(\xi),D^{\vartheta}u(\xi))+g(\xi), \quad \xi\in J_k,\ k=1,2,\ldots,m,$

(c) $\Delta\ u(\xi_k)=I_k(u(\xi_k))+g_k, \quad \xi\in J_k,\ k=1,2,\ldots,m.$

Remark 4.4 *A function* $u\in[0,T]$ *satisfies* (4.2) $\Leftrightarrow$ *there exists* $g\in[0,T]$ *and a sequence* $g_i, k=1,2,\ldots,m,$ *depending on* g*, such that*

(a) $|g(\xi)|\leq\varphi(\xi),\ |g_k|\leq\psi \quad \xi\in J_k,\ k=1,2,\ldots,m,$

(b) $D^{\vartheta}(D^{\theta}+\kappa)u(\xi)=f(\xi,u(\xi),D^{\vartheta}u(\xi))+g(\xi), \quad \xi\in J_k,\ k=1,2,\ldots,m,$

(c) $\Delta\ u(\xi_k)=I_k(u(\xi_k))+g_k, \quad \xi\in J_k,\ k=1,2,\ldots,m.$

Remark 4.5 *A function* $u\in[0,T]$ *satisfies* (4.2) $\Leftrightarrow$ *there exists* $g\in[0,T]$ *and a sequence* $g_k, k=1,2,\ldots,m,$ *depending on* g*, such that*

(a) $|g(\xi)| \leq \varepsilon\varphi(\xi),\ |g_k| \leq \varepsilon\psi \quad \xi \in J_k,\ k = 1, 2, \ldots, m,$

(b) $D^{\vartheta}(D^{\theta} + \kappa)u(\xi) = f(\xi, u(\xi), D^{\vartheta}u(\xi)) + g(\xi), \quad \xi \in J_k,\ k = 1, 2, \ldots, m,$

(c) $\Delta\ u(\xi_k) = I_k(u(\xi_k)) + g_k, \quad \xi \in J_k,\ k = 1, 2, \ldots, m.$

Theorem 4.6 *If the assumptions* $(H_1) - (H_4)$ *and the inequality* (3.1) *hold, then model* (1.1) *is Ulam–Hyers stable and consequently generalized Ulam–Hyers stable.*

Proof *Let* $\nu \in [0, T]$ *satisfies* (4.1) *and let* u *be the only one solution of*

$$\begin{cases} D^{\vartheta}(D^{\theta} + \kappa)u(\xi) = f(\xi, u(\xi), D^{\vartheta}u(\xi)), \quad \xi \in J' = J - \{\xi_1, \xi_2, \xi_3, \ldots, \xi_m\}, J = [0, T], \\ \Delta u(\xi_k) = u(\xi^+) - u(\xi^-) = I_k u(\xi), \\ u(0) = 0, \quad u(\eta_k) = 0, u(1) = 0, \eta_k = (\xi_k, \xi_{k+1}), k = 0, 1, 2, \ldots, m-1. \end{cases}$$

By Theorem 2.11, we have for each $\xi \in J_k$

$$u(\xi) = \begin{cases} \int_0^{\xi} (\xi - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz \\ - N_m \int_0^{\eta_0} (\eta_0 - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0 - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz, \ \text{for } \xi \in J_0, \\ \int_0^{\xi} (\xi - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\xi - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz \\ + M_m I_k(u(\xi)) - M_m N_m \int_0^{\eta_0} (\eta_0 - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0 - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz \\ + M_m \int_0^{\eta_k} (\eta_k - z)^{\theta+\vartheta-1} \mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_k - z)^{\theta}\kappa) f(z, u(z), D^{\vartheta}u(z))dz, \ \text{for } \xi \in J_k. \end{cases}$$

Since ν *satisfies inequality* (4.1), *so by Remark 4.3, we get*

$$\begin{cases} D^{\vartheta}(D^{\theta} + \kappa)\nu(\xi) = f(\xi, \nu(\xi), D^{\vartheta}\nu(\xi)) + g_k, \quad \xi \in J' = J - \{\xi_1, \xi_2, \xi_3, \ldots, \xi_m\}, J = [0, T], \\ \Delta\nu(\xi_k) = \nu(\xi^+) - \nu(\xi^-) = I_k\nu(\xi) + g_k, \\ \nu(0) = 0, \quad \nu(\eta_k) = 0, \nu(1) = 0, \eta_k = (\xi_k, \xi_{k+1}), k = 0, 1, 2 \ldots, m-1. \end{cases} \tag{4.8}$$

Obviously the solution of (4.8)*, will be*

$$\nu(\xi)=\begin{cases}\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)f(z,\nu(z),D^{\vartheta}\nu(z))dz\\ +\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)g(z)dz\\ -N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)f(z,\nu(z),D^{\vartheta}\nu(z))dz\\ -N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)g(z)dz,\ for\ \xi\in J_0,\\ \int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)f(z,\nu(z),D^{\vartheta}\nu(z))dz\\ +\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)g_k(z)dz+M_mI_m\nu(\xi)+g_k(\xi)\\ -M_mN_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)f(z,\nu(z),D^{\vartheta}\nu(z))dz\\ +M_m\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_k-z)^{\theta}\kappa)f(z,\nu(z),D^{\vartheta}\nu(z))dz\\ -M_mN_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)g_k(z)dz\\ +M_m\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_k-z)^{\theta}\kappa)g_k(z)dz,\ for\ \xi\in J_k,k=\{1,2,3,\dots,m\}.\end{cases}$$

Therefore, for each $\xi\in J_0$*, we have the following*

$$\begin{aligned}|u(\xi)-\nu(\xi)| \leq\ & |\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)f(z,u(z),D^{\vartheta}u(z))dz\\ &-N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)f(z,u(z),D^{\vartheta}u(z))dz\\ &-\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)f(z,\nu(z),D^{\vartheta}\nu(z))dz\\ &+N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)f(z,\nu(z),D^{\vartheta}\nu(z))dz|\\ &+\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)g(z)dz\\ &-N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)g(z)dz\end{aligned}$$

$$\begin{aligned}
\leq\ & \int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))-f(z,\nu(z),D^{\vartheta}\nu(z))|dz \\
& -N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))-f(z,\nu(z),D^{\vartheta}\nu(z))|dz \\
& +\epsilon\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)dz-N_m\epsilon\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)dz \\
\leq\ & \frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}(h(z)|u(z)-\nu(z)|+L_f|D^{\vartheta}u(z)-D^{\vartheta}\nu(z)|)dz \\
& -N_m\frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}(h(z)|u(z)-\nu(z)|+L_f|D^{\vartheta}u(z)-D^{\vartheta}\nu(z)|dz \\
& +\frac{\epsilon}{\Gamma(\theta+\vartheta)}\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}dz-\frac{N_m\epsilon}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}dz \\
\leq\ & \frac{1}{\Gamma(\theta+\vartheta)}\left(\int_0^{\xi}(\xi-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\xi}h(z)^{\frac{1}{q_2}}dz\right)^{q_2}\|u(\xi)-\nu(\xi)\| \\
& +\frac{L_f\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}\|u(\xi)-\nu(\xi)\| \\
& -N_m\frac{1}{\Gamma(\theta+\vartheta)}\left(\int_0^{\eta_0}(\eta_0-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\eta_0}h(z)^{\frac{1}{q_2}}dz\right)^{q_2}\|u(\xi)-\nu(\xi)\| \\
& +\frac{L_f\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}|u(z)-\nu(z)|-\frac{\epsilon\xi^{\theta+\vartheta}}{(\theta+\vartheta)\Gamma(\theta+\vartheta)}+\frac{N_m\epsilon\eta_0^{\theta+\vartheta}}{(\theta+\vartheta)\Gamma(\theta+\vartheta)} \\
\leq\ & \frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\|u(\xi)-\nu(\xi)\|+\frac{L_f\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}\|u(\xi)-\nu(\xi)\| \\
& -\frac{N_m\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\|u(\xi)-\nu(\xi)\|+\frac{L_f\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}\|u(\xi)-\nu(\xi)\| \\
& -\frac{\epsilon\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{N_m\epsilon\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)} \\
\leq\ & M_F\|u(\xi)-\nu(\xi)\|+\frac{L_f}{\Gamma(\theta+\vartheta+1)}(\xi^{\theta+\vartheta}+\eta_0^{\theta+\vartheta})\|u(\xi)-\nu(\xi)\|-M_0M_F\|u(\xi)-\nu(\xi)\| \\
& -\frac{\epsilon\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{N_m\epsilon\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)} \\
\leq\ & (M_F-M_0M_F)\|u(\xi)-\nu(\xi)\|+\frac{L_f(\xi^{\theta+\vartheta}+\eta_0^{\theta+\vartheta})}{\Gamma(\theta+\vartheta+1)}\|u(\xi)-\nu(\xi)\| \\
& -\frac{\epsilon\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{N_m\epsilon\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}
\end{aligned}$$

$$|u(\xi)-\nu(\xi)|\leq\left(\frac{N_m\eta_0^{\theta+\vartheta}-\xi^{\theta+\vartheta}}{(M_F-M_0M_F+2L_f)\Gamma(\theta+\vartheta+1)}\right)\epsilon. \tag{4.9}$$

If $\xi \in J_k$, *for arbitrary* $u, v \in C(J_k, \mathbb{R})$, *we get*

$$\begin{aligned}
&|u(\xi)-\nu(\xi)| \\
\leq\ & \int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))-f(z,\nu(z),D^{\vartheta}\nu(z))|dz \\
&-M_mN_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))-f(z,\nu(z),D^{\vartheta}\nu(z))|dz \\
&+M_m\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_k-z)^{\theta}\kappa)|f(z,u(z),D^{\vartheta}u(z))-f(z,\nu(z),D^{\vartheta}\nu(z))|dz \\
&+\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^{\theta}\kappa)|g_k(\xi)|dz+M_m\|I_k(u(\xi))-I_k(\nu(\xi))\| \\
&-M_mN_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^{\theta}\kappa)|g_k(\xi)|dz \\
&+M_m\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_k-z)^{\theta}\kappa)|g_k(\xi)|dz+M_mg_k(\xi) \\
\leq\ & \frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}(h(z)\|u(\xi)-\nu(\xi)\|+L_f|D^{\vartheta}u(z)-D^{\vartheta}\nu(z)|)dz \\
&+M_m\L_k\|(u(\xi))-(\nu(\xi))\|-\frac{M_mN_m}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}(h(z)\|u(\xi)-\nu(\xi)\| \\
&+L_f|D^{\vartheta}u(z)-D^{\vartheta}\nu(z)|)dz+\frac{M_m}{\Gamma(\theta+\vartheta)}\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}(h(z)\|u(\xi)-\nu(\xi)\| \\
&+L_f|D^{\vartheta}u(z)-D^{\vartheta}\nu(z)|)dz+M_m\epsilon+\frac{\epsilon}{\Gamma(\theta+\vartheta)}\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}dz \\
&-\frac{M_mN_m\epsilon}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}dz+\frac{\epsilon M_m}{\Gamma(\theta+\vartheta)}\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}dz \\
\leq\ & \frac{1}{\Gamma(\theta+\vartheta)}\left(\int_0^{\xi}(\xi-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\xi}h(z)^{\frac{1}{q_2}}dz\right)^{q_2}\|u(\xi)-\nu(\xi)\| \\
&+\frac{L_f\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}\|u(\xi)-\nu(\xi)\|+M_m\L_k\|(u(\xi))-(\nu(\xi))\|+\frac{L_f\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}|u(\xi)-\nu(\xi)| \\
&-\frac{M_mN_m}{\Gamma(\theta+\vartheta)}\left(\int_0^{\eta_0}(\eta_0-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\eta_0}h(z)^{\frac{1}{q_2}}dz\right)^{q_2}\|u(\xi)-\nu(\xi)\| \\
&+\frac{M_m}{\Gamma(\theta+\vartheta)}\left(\int_0^{\eta_k}(\eta_k-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\eta_k}h(z)^{\frac{1}{q_2}}dz\right)^{q_2}\|u(\xi)-\nu(\xi)\| \\
&+\frac{\epsilon\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}-\frac{M_mN_m\epsilon\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{M_m\epsilon\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{L_f\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}|u(\xi)-\nu(\xi)|+M_m\epsilon
\end{aligned}$$

$$\leq \frac{\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\|u(\xi)-\nu(\xi)\| + \frac{L_f}{\Gamma(\theta+\vartheta+1)}(\xi^{\theta+\vartheta}+\eta_0^{\theta+\vartheta}+\eta_k^{\theta+\vartheta})\|u(\xi)-\nu(\xi)\|$$
$$+M_m L_k\|u(\xi)-\nu(\xi)\| - \frac{M_m N_m\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\|u(\xi)-\nu(\xi)\| + \frac{\epsilon\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}$$
$$+\frac{M_m\|h\|_{L^{\frac{1}{q_2}}(J)}}{\Gamma(\theta+\vartheta)(1+p_2)^{1-q_2}}\|u(\xi)-\nu(\xi)\| - \frac{M_m N_m\epsilon\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)} + \frac{M_m\epsilon\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)} + M_m\epsilon$$

$$|u(\xi)-\nu(\xi)| \leq \left(M_F - M_m N_m M_F + M_m M_F + M_m L_k + \frac{L_f(\xi^{\theta+\vartheta}+\eta_0^{\theta+\vartheta}+\eta_k^{\theta+\vartheta})}{\Gamma(\theta+\vartheta+1)}\right)\|u(\xi)-\nu(\xi)\|$$
$$+\frac{\epsilon\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)} - \frac{M_m N_m\epsilon\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)} + \frac{M_m\epsilon\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)} + M_m\epsilon.$$

Thus

$$|u(\xi)-\nu(\xi)| \leq \epsilon C_{f,g,\theta,\vartheta}.$$

Where

$$C_{f,g,\theta,\vartheta} = \frac{\frac{\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)} - \frac{M_m N_m\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)} + \frac{M_m\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)} + M_m}{1-(M_F+3L_f+M_m I_m)}.$$

So equation (1.1) is Ulam–Hyers stable and if we set $\phi(\varepsilon)=\varepsilon C_{f,g,\theta_1,\theta_2}$, $\phi(0)=0$, then equation (1.1) is generalized Ulam–Hyers stable.

Theorem 4.7 *If the assumptions $(H_1)-(H_4)$ and the inequality (3.1) are satisfied, then the problem (1.1) is Ulam–Hyers–Rassias stable with respect to (φ,ψ), consequently generalized Ulam–Hyers–Rassias stable.*

Proof *Let $u\in[0,T]$ be a solution of the inequality (4.3) and let u be the only one solution of the problem (1.1).*

From Theorem 4.6, $\forall\, \upsilon\in J_k$, we get Therefore, for each $\xi\in J_0$, we have the following

$$|u(\xi)-\nu(\xi)| \leq |\int_0^\xi(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^\theta\kappa)f(z,u(z),D^\vartheta u(z))dz$$
$$-N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^\theta\kappa)f(z,u(z),D^\vartheta u(z))dz$$
$$-\int_0^\xi(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^\theta\kappa)f(z,\nu(z),D^\vartheta\nu(z))dz$$
$$+N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^\theta\kappa)f(z,\nu(z),D^\vartheta\nu(z))dz|$$
$$+\int_0^\xi(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^\theta\kappa)g(z)dz$$
$$-N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^\theta\kappa)g(z)dz$$

$$
\begin{aligned}
\leq\ & \int_0^\xi (\xi - z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^\theta\kappa)|f(z,u(z),D^\vartheta u(z)) - f(z,\nu(z),D^\vartheta\nu(z))|dz \\
& -N_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^\theta\kappa)|f(z,u(z),D^\vartheta u(z)) - f(z,\nu(z),D^\vartheta\nu(z))|dz \\
& +\epsilon\int_0^\xi(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^\theta\kappa)\varphi(z)dz \\
& -N_m\epsilon\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^\theta\kappa)\varphi(z)dz \\
\leq\ & \frac{1}{\Gamma(\theta+\vartheta)}\int_0^\xi(\xi-z)^{\theta+\vartheta-1}(h(z)\|u(\xi)-\nu(\xi)\| + L_f|D^\vartheta u(z) - D^\vartheta\nu(z)|)dz \\
& -N_m\frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}(h(z)\|u(\xi)-\nu(\xi)\| + L_f|D^\vartheta u(z) - D^\vartheta\nu(z)|dz \\
& +\frac{\epsilon}{\Gamma(\theta+\vartheta)}\int_0^\xi(\xi-z)^{\theta+\vartheta-1}\varphi(z)dz - \frac{N_m\epsilon}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\varphi(z)dz \\
\leq\ & \frac{1}{\Gamma(\theta+\vartheta)}\left(\int_0^\xi(\xi-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^\xi h(z)^{\frac{1}{q_2}}dz\right)^{q_2}\|u(\xi)-\nu(\xi)\| + L_f\|u(\xi)-\nu(\xi)\| \\
& -N_m\frac{1}{\Gamma(\theta+\vartheta)}\left(\int_0^{\eta_0}(\eta_0-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\eta_0} h(z)^{\frac{1}{q_2}}dz\right)^{q_2}\|u(\xi)-\nu(\xi)\| \\
& +L_f\|u(\xi)-\nu(\xi)\| - \frac{\epsilon\xi^{\theta+\vartheta}\kappa_\varphi\varphi(z)}{(\theta+\vartheta)\Gamma(\theta+\vartheta)} + \frac{N_m\epsilon\eta_0^{\theta+\vartheta}\kappa_\varphi\varphi(z)}{(\theta+\vartheta)\Gamma(\theta+\vartheta)} \\
\leq\ & (M_F - M_0M_F)\|u(\xi)-\nu(\xi)\| + 2L_f\|u(\xi)-\nu(\xi)\| - \frac{\epsilon\xi^{\theta+\vartheta}\kappa_\varphi\varphi(\xi)}{\Gamma(\theta+\vartheta+1)} + \frac{N_m\epsilon\eta_0^{\theta+\vartheta}\kappa_\varphi\varphi(\xi)}{\Gamma(\theta+\vartheta+1)}
\end{aligned}
$$

$$
|u(\xi)-\nu(\xi)| \leq \left(\frac{N_m\eta_0^{\theta+\vartheta} - \xi^{\theta+\vartheta}}{(M_F - M_0M_F + 2L_f)\Gamma(\theta+\vartheta+1)}\right)\epsilon\kappa_\varphi\varphi(\xi). \tag{4.10}
$$

If $\xi \in J_k$, for arbitrary $u, v \in C(J_k, \mathbb{R})$, we get

$$
\begin{aligned}
& |u(\xi)-\nu(\xi)| \\
\leq\ & \Bigg|\int_0^\xi(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^\theta\kappa)|f(z,u(z),D^\vartheta u(z)) - f(z,\nu(z),D^\vartheta\nu(z))|dz \\
& -M_mN_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^\theta\kappa)|f(z,u(z),D^\vartheta u(z)) - f(z,\nu(z),D^\vartheta\nu(z))|dz \\
& +M_m\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_k-z)^\theta\kappa)|f(z,u(z),D^\vartheta u(z)) - f(z,\nu(z),D^\vartheta\nu(z))|dz \\
& +\int_0^\xi(\xi-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\xi-z)^\theta\kappa)|g_k(\xi)|dz + M_m\|I_k(u(z)) - I_k(\nu(z))\| \\
& -M_mN_m\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_0-z)^\theta\kappa)|g_k(\xi)|dz \\
& +M_m\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}\mathbf{E}_{\theta,\theta+\vartheta}(-(\eta_k-z)^\theta\kappa)|g_k(\xi)|dz + M_mg_k(\xi)
\end{aligned}
$$

$$\begin{aligned}
\leq\ & \frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}(h(z)\|u(\xi)-\nu(\xi)\|+L_f|D^{\vartheta}u(z)-D^{\vartheta}\nu(z)|)dz\\
&-\frac{M_mN_m}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}(h(z)\|u(\xi)-\nu(\xi)\|+L_f|D^{\vartheta}u(z)-D^{\vartheta}\nu(z)|)dz\\
&+M_m\frac{1}{\Gamma(\theta+\vartheta)}\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}(h(z)\|u(\xi)-\nu(\xi)\|+L_f|D^{\vartheta}u(z)-D^{\vartheta}\nu(z)|)dz\\
&+\frac{\epsilon}{\Gamma(\theta+\vartheta)}\int_0^{\xi}(\xi-z)^{\theta+\vartheta-1}\varphi(z)dz+M_m\|I_k(u(z))-I_k(\nu(z))\|\\
&-\frac{M_mN_m\epsilon}{\Gamma(\theta+\vartheta)}\int_0^{\eta_0}(\eta_0-z)^{\theta+\vartheta-1}\varphi(z)dz+\frac{M_m\epsilon}{\Gamma(\theta+\vartheta)}\int_0^{\eta_k}(\eta_k-z)^{\theta+\vartheta-1}\varphi(z)dz+M_m\epsilon\\
\leq\ & \frac{1}{\Gamma(\theta+\vartheta)}\left(\int_0^{\xi}(\xi-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\xi}h(z)^{\frac{1}{q_2}}dz\right)^{q_2}\|u(\xi)-\nu(\xi)\|+L_f\|u(\xi)-\nu(\xi)\|\\
&+M_m\|I_k(u(z))-I_k(\nu(z))\|+2L_f\|u(\xi)-\nu(\xi)\|\\
&-\frac{M_mN_m}{\Gamma(\theta+\vartheta)}\left(\int_0^{\eta_0}(\eta_0-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\eta_0}h(z)^{\frac{1}{q_2}}dz\right)^{q_2}\|u(\xi)-\nu(\xi)\|\\
&+\frac{M_m}{\Gamma(\theta+\vartheta)}\left(\int_0^{\eta_k}(\eta_k-z)^{\frac{\theta+\vartheta-1}{1-q_2}}dz\right)^{1-q_2}\left(\int_0^{\eta_k}h(z)^{\frac{1}{q_2}}dz\right)^{q_2}\|u(\xi)-\nu(\xi)\|\\
&+\frac{\epsilon\xi^{\theta+\vartheta}\kappa_{\varphi}\varphi(\xi)}{\Gamma(\theta+\vartheta+1)}-\frac{M_mN_m\epsilon\eta_0^{\theta+\vartheta}\kappa_{\varphi}\varphi(\xi)}{\Gamma(\theta+\vartheta+1)}+\frac{M_m\epsilon\eta_k^{\theta+\vartheta}\kappa_{\varphi}\varphi(\xi)}{\Gamma(\theta+\vartheta+1)}+M_m\epsilon\\
\leq\ & M_F\|u(\xi)-\nu(\xi)\|+L_f\|u(\xi)-\nu(\xi)\|+M_mI_m\|u(\xi)-\nu(\xi)\|-M_mN_mM_F\|u(\xi)-\nu(\xi)\|\\
&+L_f\|u(\xi)-\nu(\xi)\|+M_mM_F\|u(\xi)-\nu(\xi)\|+L_f\|u(\xi)-\nu(\xi)\|\\
&+\left(\frac{\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}-\frac{M_mN_m\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{M_m\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+M_m\right)\epsilon\kappa_{\varphi}\varphi(\xi)\\
\leq\ & \frac{\frac{\epsilon\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}-\frac{M_mN_m\epsilon\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{M_m\epsilon\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+M_m\epsilon}{1-(M_F+3L_f+M_mI_m)}\kappa_{\varphi}\varphi(\xi).
\end{aligned}$$

Thus

$$|u(\xi)-\nu(\xi)|\leq C_{f,g,\theta,\vartheta}\epsilon\kappa_{\varphi}\varphi(\xi).$$

Where

$$C_{f,g,\theta,\vartheta}=\frac{\frac{\xi^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}-\frac{M_mN_m\eta_0^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+\frac{M_m\eta_k^{\theta+\vartheta}}{\Gamma(\theta+\vartheta+1)}+M_m}{1-(M_F+3L_f+M_mI_m)}.$$

Example 4.8 *Let us consider the following impulsive fractional Langevin equations*

$$\begin{cases}D^{\frac{1}{3}}\left(D^{\frac{1}{2}}+1\right)u(\xi)=\frac{|\xi+D^{\frac{1}{2}}u(\xi)|}{100(1+e^{\xi})(1+|u(\xi)|)}, & \xi\in[0,1]\setminus\left\{\frac{1}{3}\right\},\\ \Delta z\left(\frac{1}{3}\right)=0.1, & \\ z(0)=0,\quad z\left(\frac{1}{4}\right)=0,\quad z(1)=0. & \end{cases}\tag{4.11}$$

Set $\theta = \frac{1}{2}$, $\vartheta = \frac{1}{3}$, $\kappa = 1$, $\xi_1 = \frac{1}{3}$, $I_1 = 0.1$, $\eta_0 = \frac{1}{4}$, $J = [0,1]$, $m = 1$, *and* $f(\xi, u(\xi), D^{\theta}u(\xi)) = \frac{|\xi + D^{\frac{1}{2}}u(\xi)|}{100(1+e^{\xi})(1+|z|)}$, $(\xi, z) \in J \times [0, \infty)$. *For any* $z_1, z_2 \in \mathbb{R}$ *and* $\xi \in J$, *we have* $|f(\xi, z_1) - f(\xi, z_2)| \leq \frac{1}{200}|z_1 - z_2|$, *which implies that* f *satisfies the Lipschitz condition with* $h(\cdot) := \frac{1}{200} \in L^2(J, \mathbb{R})$. *Moreover, for all* $z \in \mathbb{R}$ *and each* $\xi \in J$, *we have* $|f(\xi, u(\xi))| \leq \frac{1}{200}$, *which implies that* f *satisfies the growth condition with* $n(\cdot) := \frac{1}{200} \in L^2(J, \mathbb{R})$. *For* $\xi \in J$ *and* $q_2 = \frac{1}{2}$, *it is easy to compute* $p_2 = -\frac{1}{3}$, *and* $M_F = 0.0054$, $M_G = 0.954$.

Next, we provide two potential methods to compute the Mittag-Leffler functions E_{α}, *which will aid in verifying the condition* (3.1) *in the above main theorems.*

It follows the series formula for $E_{\theta}(-z)$ *where* $z \geq 0$. *Using Mathematica, we can calculate:* $E_{\theta}(-\eta_0\kappa) = 0.6157$, $E_{\theta}(-\xi_1\kappa) = 0.5781$, *and* $E_{\theta}(-\kappa) = 0.4276$. *Notably, it's evident that* $M_F M_g = 0.0051516$, *which is less than 1.*

Hence, all the assumptions in Theorems 3.1 and 3.2 are satisfied, allowing us to apply our results to the problem (4.11) *whose solution is given by:*

$$u(\xi) = \begin{cases} \int_0^{\xi} (\xi - z)^{\frac{1}{2}+\frac{1}{3}-1} \mathbf{E}_{\frac{1}{2},\frac{1}{2}+\frac{1}{3}}(-(\xi - z)^{\frac{1}{2}}) f(z, u(z), D^{\frac{1}{3}}u(z))dz \\ - N_m \int_0^{\eta_0} (\eta_0 - z)^{\frac{1}{2}+\frac{1}{3}-1} \mathbf{E}_{\frac{1}{2},\frac{1}{2}+\frac{1}{3}}(-(\eta_0 - z)^{\frac{1}{2}}) f(z, u(z), D^{\frac{1}{3}}u(z))dz, \ \textit{for } \xi \in J_0, \\ \int_0^{\xi} (\xi - z)^{\frac{1}{2}+\frac{1}{3}-1} \mathbf{E}_{\frac{1}{2},\frac{1}{2}+\frac{1}{3}}(-(\xi - z)^{\frac{1}{2}}) f(z, u(z), D^{\frac{1}{3}}u(z))dz \\ + M_m I_k(u(\xi)) - M_m N_m \int_0^{\eta_0} (\eta_0 - z)^{\frac{1}{2}+\frac{1}{3}-1} \mathbf{E}_{\frac{1}{2},\frac{1}{2}+\frac{1}{3}}(-(\eta_0 - z)^{\frac{1}{2}}) f(z, u(z), D^{\frac{1}{3}}u(z))dz \\ + M_m \int_0^{\eta_k} (\eta_m - z)^{\frac{1}{2}+\frac{1}{3}-1} \mathbf{E}_{\frac{1}{2},\frac{1}{2}+\frac{1}{3}}(-(\eta_m - z)^{\frac{1}{2}}) f(z, u(z), D^{\frac{1}{3}}u(z))dz, \ \textit{for } \xi \in J_k, k = \{1, \ldots, m\}, \end{cases}$$

Thus, by Theorem 3.1, (4.11) *has a unique solution. Further the conditions of Theorem 4.6 are satisfied so the solution of* (4.11) *is Ulam–Hyers stable and generalized Ulam–Hyers stable. Further it can be easily verified that the conditions of Theorem 4.7 hold and thus* (4.11) *is Ulam–Hyers–Rassias stable and consequently generalized Ulam–Hyers–Rassias stable.*

5 Conclusion

In conclusion, this paper delves into the study of a specific class of impulsive fractional Langevin equations. It achieves this by deriving solution formulas that incorporate Mittag-Leffler functions and impulsive terms. The solutions are a result of a comprehensive analysis of linear Langevin equations, which involve distinct fractional derivatives. The paper further establishes the existence of solutions through the application of various mathematical tools, such as the boundedness, continuity, monotonicity, and non-negativity properties of Mittag-Leffler functions, while employing fixed point methods.

Moreover, the study sets forth the necessary conditions and outcomes to discuss the existence, uniqueness, as well as different forms of Ulam–Hyers and Ulam–Hyers–Rassias stability, all concerning the proposed model. These results are established with the aid of a fixed point theorem. To solidify the theoretical findings, the paper provides practical illustration through a concrete example, thereby demonstrating the practical relevance and applicability of the developed mathematical framework.

Competing interest

The authors declare that they have no competing interest regarding this research work.

Author's contributions

All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

Data Availability

Not applicable

Acknowledgments

Not applicable

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