Still Lifes and Oscillators in the Game of Life on the Hyperbolic Plane

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Abstract. The Game of Life gained widespread interest due to its properties and its implications on fields such as biology, mathematics, and computer science. Hyperbolic geometry, on the other hand, is a type of geometry in which tilings are less restrictive than its Euclidean counterpart. Using the hyperbolic plane also provides other benefits, such as solving problems in less time. In this paper, we discovered still lifes and oscillators to add to the existing limited body of knowledge. We also noted the attributes of these patterns and formulated some proofs based on these patterns and their properties.

Keywords: Hyperbolic Game of Life, Still Life, Oscillators

1 INTRODUCTION

John Conway's Game of Life is a cellular automaton that sparked widespread interest since its introduction in a Scientific American article published in 1970 [1,5,8]. It is a zero-player game of patterns that evolve, based on a set of rules from an initial configuration of cells.

Despite following a simple set of rules, the Game of Life is considered to be Turing-complete [12], which means that it can simulate any Turing machine. Given this, it can be used to learn theory and solve computational problems and also to study physical and biological systems [1] in the evolution of ecological communities [3].

The Game of Life is usually studied on the Euclidean plane. However, our study will focus on the hyperbolic plane, where common NP-problems, such as the 3-SAT problem [11], Hyperbolic Minesweeper [10], and the Vertex Cover problem [2] have been solved in polynomial time. Moreover, tilings on the hyperbolic space are much less restrictive. This opens the doors for the discovery of a greater number of patterns that could not exist on the Euclidean plane.

One of the main applications of the Game of Life is the discovery of new patterns and their unique behavior on a particular tiling under a specific set of rules [8]. Currently, there is limited knowledge of patterns of the Game of Life in the hyperbolic plane with Gu [7] being the only contributor. Given this, there

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is a need to discover and prove the existence of more patterns in the hyperbolic space. In the next few sections, we will briefly tackle some important concepts of this study. Following this, we will present the still lifes and oscillators that we have discovered.

2 THE GAME OF LIFE

To understand the Game of Life, we first define the concepts that serve as integral parts of how it works. The game is a popular example of a cellular automaton. Each cell within the grid of a cellular automaton acts as a finite automaton $(Q, \Sigma, \delta, q_0, F)$ that can change its state $q \in Q$ every given time unit. At each discrete time step, the cells will evolve simultaneously following a set of rules, given by the transition function δ. Depending on the type of cellular automata, the input alphabet Σ, start state q_0, and set of accept states F can change. A specific example of a two-dimensional cellular automata is the generalized Game of Life from which Conway's Game of Life can be derived.

Definition 1 (Game of Life). *The Game of Life is a two-dimensional cellular automata played on an infinite square grid, where each square is called a cell. Each cell can have one of two states: dead or alive. A time step is called a* ***generation****. For each generation $t_i : i \in N$, cells evolve simultaneously following the rules of the game. Each cell has a neighborhood of cells, which affects the two rules of the game.*

It is worth noting that our study will consider the Moore's neighborhood.

Definition 2 (Moore's Neighborhood). *Moore's neighborhood is defined as*

$$N^M_{(x_0,y_0)} = \{(x, y) : |x - x_0| \leq 1, |y - y_0| \leq 1\}$$

In other words, the Moore's neighborhood of a given cell consists of all the cells that touch either the sides or the corners of a cell. An example of a Moore's neighborhood is shown in Figure 1, which is composed of a total of eight cells (those labelled with numbers 1 to 8).

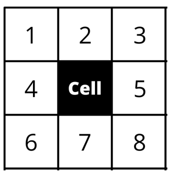


Fig. 1. The Moore's neighborhood of a given cell (colored black) in a square grid on the Euclidean plane

The birthability and survivability rules are the two rules that govern the game. These rules determine what state a cell will have in the next generation.

Definition 3 (Birthability). *The birthability rule states that a dead cell during t_i (the current generation) will be born in t_{i+1} (the next generation) depending on the number/s of live neighbors it currently has. Otherwise, it will remain dead.*

Definition 4 (Survivability). *The survivability rule states that a live cell during t_i will survive in t_{i+1} depending on the number/s of live neighbors it currently has. Otherwise, it will die.*

Definition 5 (Rulestring for Birthability and Survivability). *Birthability and survivability rules can be generalized into a rulestring of form Bx/Sy. In this context, the x substring is the set of numbers that represent the number of live neighbors that will cause a dead cell to be born in the next generation. On the other hand, the y substring is the set of numbers that represent the number of live neighbors that will cause a live cell to survive into the next generation. For both substrings, the set of numbers are subsets of $\{0, 1, 2, \cdots, n\}$ where n is the number of neighbors of the cell [1,9].*

For the Game of Life, each cell acts as a finite automaton $(Q, \Sigma, \delta, q_0, F)$. The set of states $Q = \{0, 1\}$, where a dead cell assumes state 0 and a live cell assumes state 1. The number of live neighbors of a specific cell is represented by the alphabet $\Sigma = \{m : 0 \leq m \leq n\}$, where n is the number of neighbors. The set of accept states is $F = \{0, 1\}$ because a cell can end up either dead or alive. Given a rulestring Bx/Sy, the transition function δ can be defined using the transition diagram in Figure 2.

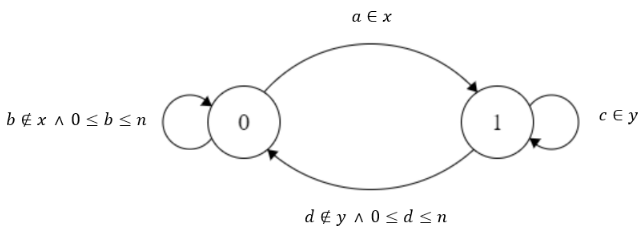


Fig. 2. Transition function δ representing the rulestring Bx/Sy of the Game of Life where n is the number of neighbors of a cell

Conway's Game of Life operates on rulestring $B3/S2, 3$, designed so that initial populations of cells would neither die quickly nor expand indefinitely [1]. Furthermore, this represents the biological systems of overpopulation and underpopulation, wherein both overcrowding of neighbors and isolation from neighbors can cause a cell to die [9]. Figure 3 shows a sample simulation of the game under rulestring $B3/S2, 3$ on during the $0th$ (initial configuration) and $1st$ timesteps.

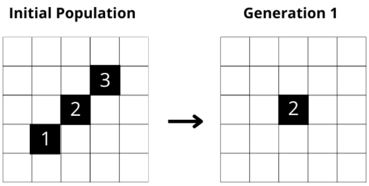


Fig. 3. An example of a game under rulestring $B3/S2,3$ on a square grid in the Euclidean space

From Figure 3, we can see that the initial population includes three live cells (the black cells). To determine which cells will be born or remain alive in the next generation, we consider each of the cells, regardless of its state, and count its live neighbors.

We can verify that each of the dead cells will have at most two live neighbors. However, the birthability rule stipulates that a dead cell will only be born if it has three live neighbors. Thus, no new cells will be born. Looking at the live cells, we can also verify that only cell 2 (the one in the center) will stay alive in the next generation as it is the only cell with two or three live neighbors; this is stipulated in the survivability rule. Hence, the result would be the grid in Generation 1.

3 TILINGS OF THE HYPERBOLIC PLANE

The Game of Life is typically studied on the Euclidean plane, which entails the type of geometry familiar to most. However, this study focuses on the Game of Life in the hyperbolic plane. The Poincaré Disk Model is used to visualize the hyperbolic plane on the Euclidean plane. Throughout this paper, we will only consider regular tilings of the Poincaré Disk. The following definitions were taken from [6] and [13].

Definition 6 (Tilings). *Otherwise known as tessellations, pavings, or mosaics, a tiling of a plane can be described as a family of tiles, composed of polygons, that cover the plane without any gaps or overlaps.*

A ***regular tiling*** *is a special class of tilings wherein only congruent copies of a specific regular polygon are used as tiles. Under this context, tilings are edge-to-edge meaning that for any two arbitrary tiles:*

1. *They are disjoint, meaning that they have no common point, or*
2. *They share exactly one point, which is a vertex on each of the polygons, or*
3. *They share one segment, which is an edge found on each of the polygons.*

Examples of tilings on the Euclidean and hyperbolic space are shown in Figures 4 and 5 respectively.

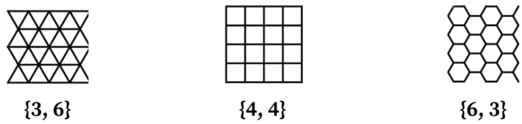


Fig. 4. The three regular tilings of the Euclidean plane

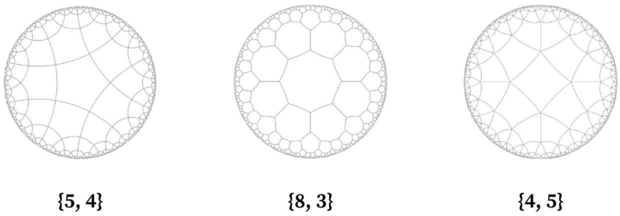


Fig. 5. Some regular tilings of the hyperbolic plane

Definition 7 (Schläfli Symbol). *A Schläfli Symbol of the form $\{p, q\}$ can represent a regular tiling, where p refers to the number of sides of the regular polygon used as the tiling, while q refers to the number of polygons that meet on a single vertex. In the context of the Euclidean plane, a tiling is only considered a regular tiling if and only if it satisfies the equality $(p-2)(q-2) = 4$. On the hyperbolic plane using the Poincaré Disk Model, an analogous condition is expressed by the inequality $(p-2)(q-2) > 4$.*

Note that there are only three regular tilings, $\{3,6\}$, $\{4,4\}$, and $\{6,3\}$, for the Euclidean plane while there are infinitely many possible regular hyperbolic tilings.

4 PATTERNS IN THE GAME OF LIFE

Since the Game of Life was introduced, the discovery of new patterns has been a topic that many people study [8]. Conway identified three objects that often emerge under rulestring $B3/S2,3$ with the square tiling under the Euclidean plane [1,8,9]. These are the Still Lifes, Oscillators, and Moving Objects.

Definition 8 (Still Lifes). *Still lifes are objects that do not change in the next generation. They are classified based on the number of live cells in the pattern.*

Definition 9 (Oscillators). *Oscillators are stationary objects that change but return to their initial state after at least two generations. They can be classified based on their period or the number of generations required for the initial state to repeat again.*

Definition 10 (Moving Objects). *Moving objects move across the grid over several generations.*

4.1 Some Known Patterns in the Euclidean Plane

Some known still lifes and oscillators in the Euclidean plane described in [9] are shown in Figures 6 and 7 respectively.

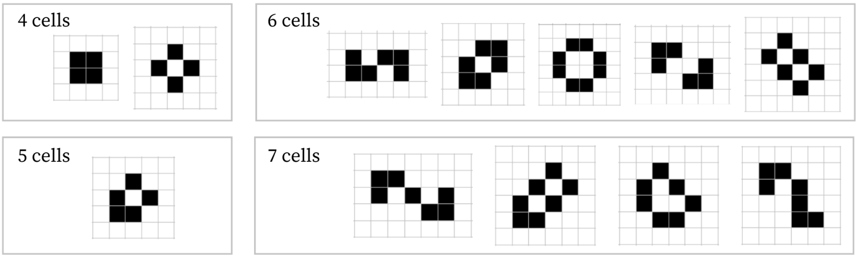


Fig. 6. Still lifes with four cells (block, tub), five cells (boat), six cells (snake, ship, beehive, aircraft carrier, barge), and seven cells (long snake, long boat, loaf, eater 1)

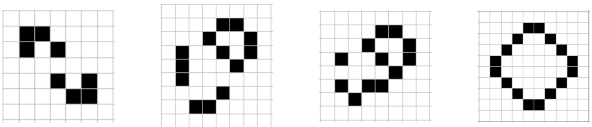


Fig. 7. Left to right: bipole (period 2), jam (period 3), mold (period 4), octagon 2 (period 5)

4.2 Known Patterns in the Hyperbolic Plane

In the paper by Gu [7], several still lifes and oscillators were identified in the hyperbolic Game of Life.

A block can be generalized as a set of cells meeting at a single vertex. It is a still life with rulestring $Bi/S(q-1)$, where $i > 2$ and $q \geq 4$. Under rule $B1$, a still life exists for a tiling $\{3, q\}$. Still lifes also exist under rule $B2$. Some examples of still lifes are shown in Figure 8.

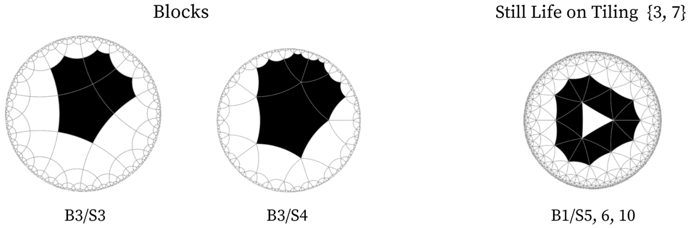


Fig. 8. Some still lifes on tilings $\{3,4\}$, $\{3,5\}$ and $\{3,7\}$

A star on tiling $\{p,3\}$, where p is even, is composed of the central polygon c and every polygon that touches every other edge of c. A star is a still life under rulestring $B2/S1k$, where $k = \frac{p}{2}$. On the other hand, a firework on tiling $\{p,4\}$ is composed of the central polygon and every polygon that touches only one of its vertices. A firework is a still life under rulestring $B2/S1, p$. Some examples of stars and fireworks are shown in Figure 9.

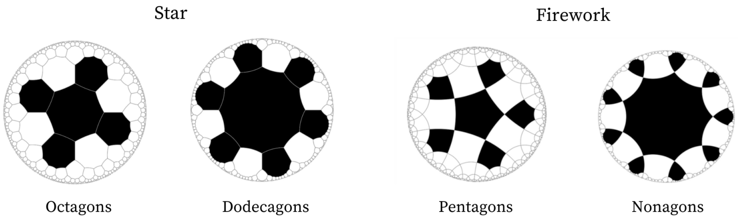


Fig. 9. Stars composed of octagons and dodecagons, and Fireworks composed of pentagons and nonagons

The two-cell blinker and the three-cell spinner are period 2 oscillators on tiling $\{8,3\}$ under rule $B2/S3$.

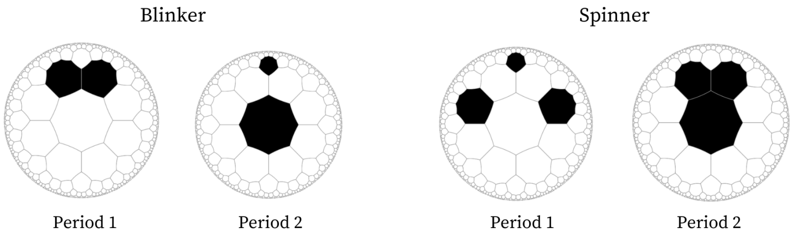


Fig. 10. Blinker and Spinner in the Hyperbolic Game of Life

5 MORE STILL LIFES AND OSCILLATORS

The main results of this paper are additional still lifes and oscillators in the Game of Life on the hyperbolic plane. Before we expound on the discovered patterns and their properties, some definitions must be established for uniformity. These definitions were derived from Gu's paper [7].

Definition 11 (Center). *The center is any tile chosen from the hyperbolic plane.*

In this paper, the designated center (denoted as c) is presumed to be at the center of the Poincaré disk visualization.

Definition 12 (Distance Between Cells). *Suppose that A is a set of cells $a, b \in A$ such that they are connected—they share a common edge or vertex, or are connected via another cell/s. The distance between a and b is the minimum number of cells that connect them.*

The notion of distance will be used to establish the concept of levels.

Definition 13 (Level). *A level refers to the distance (in cells) of an arbitrary cell from the center.*

Thus, the center c will always be at level 0, while other cells may be of level n where $n > 0$. Given these definitions, we will now present the still lifes and oscillators that we discovered.

5.1 Discovered Still Lifes

Definition 14 (Light bulb). *A light bulb is a pattern on tiling $\{n, 4\}$ where n is an odd integer greater than or equal to 5. The live cells in this pattern include c, three neighbors of c that only touch consecutive vertices (say vertices 1,2,3), and the neighbor of c touching the edge directly opposite to vertex 2. The pattern of a light bulb can be generated based on Algorithm 1. On the other hand, an example of a light bulb is shown in Figure 11.*

Algorithm 1 Generating a light bulb

1: Choose c and set it as a live cell
2: Select three neighbors (n_i) of c such that:
   – None of them share an edge with c and
   – $dist(n_1, n_2) = 1$ and $dist(n_2, n_3) = 1$
3: Set these cells as live cells
4: Choose another neighbor of c such that it shares an edge with the c and is located at the "opposite" of n_2
5: Set this cell as a live cell

Proposition 1. *A light bulb is a still life with rulestring $Bw/S1,4$ where $w > 3$.*

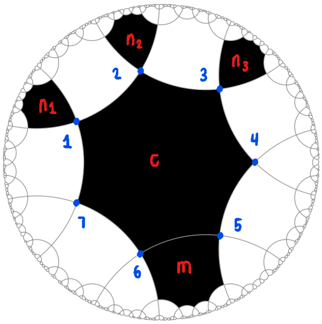


Fig. 11. A light bulb with tiling $\{7, 4\}$ with labeled live cells and vertices

Proof. Suppose we have a light bulb with rulestring $Bw/S1,4$ where $w > 3$. We know that c has four live neighbors. This includes the three n-sided cells that do not share an edge with c and touch c only via vertices $1, 2, 3$. We will call these three cells n_1, n_2 and n_3 (ordered from left to right). The fourth live neighbor is the n-sided cell that shares an edge with c such that the edge is composed of vertices $\frac{n+3}{2}$ and $\frac{n+5}{2}$. We will call this cell m. A visualization of what was previously described is shown in Figure 11. Note that c will stay alive in the next generation due to the survivability rule $S1,4$.

Each of these live neighbors only has one live neighbor—the center tile. All these cells will stay alive in the next generation due to the survivability rule $S1,4$. They are located on level 1.

There are also dead cells at this level, each having one, two, or three live neighbors. Note that there are two cells that are neighbors of c touching the edge with vertices 1 and 2 and the edge with vertices 2 and 3, respectively. These cells have three live neighbors, namely c and the two live cells touching each vertex of the edge it shares with c. The remaining cells that share an edge with c each have at most three live neighbors. Specifically, a cell that shares an edge with c will have either one live neighbor (only c), two live neighbors (c and either n_1, n_3, or m), or three live neighbors (c, either n_1 or n_3, and m)—only when the light bulb has tiling $\{5, 4\}$. All the other unmentioned cells share only a vertex with c. The cells that touch either vertex $\frac{n+3}{2}$ or vertex $\frac{n+5}{2}$ have two live neighbors, namely c and m. The cells touching the remaining vertices only have one live neighbor—the center tile. At level 2, all the dead cells have at most one neighbor, which could be one of the live cells at level 1. Cells in higher levels have no more live neighbors because all live cells are only in levels 0 or 1. All these dead cells will remain dead in the next generation due to the birthability rule. With this, none of the cells will (ever) change its state, making the pattern under this rulestring a still life. Hence, the proposition holds true. $\square$

Definition 15 (Flower). *A flower is a pattern with a tiling of* $\{p, q\}$ *where* c *and every polygon touching its* p *edges are set as live cells. An example of a flower is shown in Figure 12.*

Proposition 2. *A flower is a still life with rulestring* $Bw/S3, p$ *where* $w \notin \{0, 1, 3\}$.

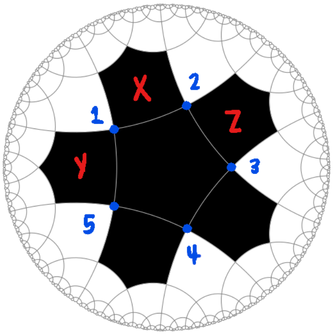


Fig. 12. An example of a flower with tiling $\{5, 4\}$

Proof. Suppose we have a flower with rulestring $Bw/S3, p$ where $w \notin \{0, 1, 3\}$. We know that c has p live neighbors. Each of these p cells has three live neighbors. For a quick proof, suppose we select one live neighbor of c, which we call x, as shown in Figure 12. From here we can see that x shares edge $1, 2$ with the center tile c. Thus, c is automatically one of its live neighbors. Based on our definition, x will have two more live neighbors because there will be one live cell that shares edge $1, 5$ with c (labeled y in the figure)—it is neighbors with x since they meet on vertex 1. The other live cell shares edge $2, 3$ with c (labeled z in the figure)—this time, it is neighbors with x since they meet on vertex 2. This will be generalizable for all p live neighbors of c. Rule $S3, p$ ensures that c and all its live neighbors will stay alive in the next generation.

Note that all p live neighbors are on level 1. On this level, there are also dead cells each having three live neighbors. For a quick proof, we focus on the vertices of c. Since c is a p-sided polygon, it has p vertices. On each of the p vertices, three live cells meet, which consists of c and the two polygons touching the adjacent edges that meet at the vertex. Thus, each dead cell around a vertex has three live neighbors. Note that the dead cells in Level 2 have at most one live neighbor, which would be one of the live cells in level 1. Dead cells in higher levels have no live neighbors because all live cells are only in levels 0 or 1. Bw where $w \notin \{0, 1, 3\}$ ensures that no dead cell will be born in the next generation. With this, none of the cells will (ever) change its state, making the flower under this rulestring a still life. Hence, the proposition holds true. $\square$

Definition 16 (Lace). *A lace is a pattern with a tiling of* $\{4, q\}$ *generated based on Algorithm 2. Examples of the lace are shown in Figure 13.*

Algorithm 2 Generating a lace

Choose c
for each n in N **do** ▷ N is the set of neighbors of c
    if n shares an edge with c **then**
        for each *neighbor* in n **do**
            if *neighbor* shares an edge with n and *neighbor* is not c **then**
                Set *neighbor* as a live cell
            end if
        end for
    end if
end for

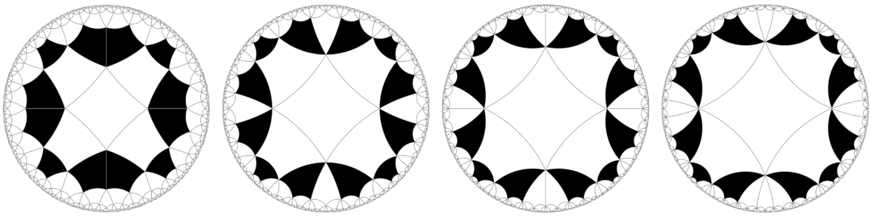


Fig. 13. Examples of laces with tilings of $\{4, 5\}$, $\{4, 6\}$, $\{4, 7\}$ and $\{4, 8\}$ respectively (from left to right)

Proposition 3 (Lace). *A lace is a still life with rulestring* $Bw/S2, v$ *where* $w \notin \{0, 1, 2, 5, 8\}$ *and* $v \notin \{0, 2\}$.

Proof. Suppose that we have a lace with rulestring $Bw/S2, v$ where $w \notin \{0, 1, 2, 5, 8\}$ and $v \notin \{0, 2\}$. We also suppose that A is the set of dead cells. For each $a \in A$, it has x live neighbors. Note that for the dead cells not to be born in the next generation, we need to ensure that for each $a \in A$, $x \notin w$. For our proof, we will check the dead cells on a per-level basis and show that $x \notin w$.

Case 1 (Level 0). We start by checking level 0, which only contains c. Note that c is a dead cell, which can be verified by checking Algorithm 2. The tiling for a lace is $\{4, q\}$ where $p = 4$. Thus, we know that c and all the other cells have four sides and four vertices. We let N_E be the set of neighbors of c that shares an edge with c. In Figure 14(a), $N_E = \{n_{AB}, n_{BC}, n_{CD}, n_{DA}\}$. At each vertex V of c, it has two live neighbors, namely the live neighbor of n_{XV} and n_{VY} that share vertex V. Hence, c has a total of $4 \times 2 = 8$ live neighbors—colored green in Figure 14(a). Since $8 \notin w$, the birthability rule ensures that c will not be born.

Case 2 (Level 1). Now we consider the cells at level 1, which will be split into two subcases. We first check the number of live neighbors for each $n \in N_E$. Note that all these cells are dead cells, which can be verified from Algorithm 2. As an example, we consider cell $n_{AB} \in N_E$ which has the vertices A, B, A'_B, B'_A. Cell n_{AB} has two live neighbors when we consider vertex A. These are the live neighbors of n_{AB} and n_{DA} that share vertex A—colored green in Figure 14(b). Cell n_{AB} also has two live neighbors when we consider vertex B. These are the live neighbors of n_{AB} and n_{BC} that share vertex B—colored yellow in Figure 14(b). Lastly, n_{AB} has one live neighbor—the live neighbor that shares edge $A'B'$ that is colored pink in Figure 14(b). Note that this is generalizable for all $n \in N_E$. Hence, each $n \in N_E$ has $2 + 2 + 1 = 5$ live neighbors. Since $5 \notin w$, the birthability rule ensures that each $n \in N_E$ will not be born.

We let N_O be the set of neighbors of c that share a vertex with c but do not share an edge with any cells in N_E—colored yellow in Figure 14(c). Since each $n \in N_O$ shares a vertex V with c, it has two live neighbors. Since $2 \notin w$, the birthability rule ensures that each $n \in N_E$ will not be born.

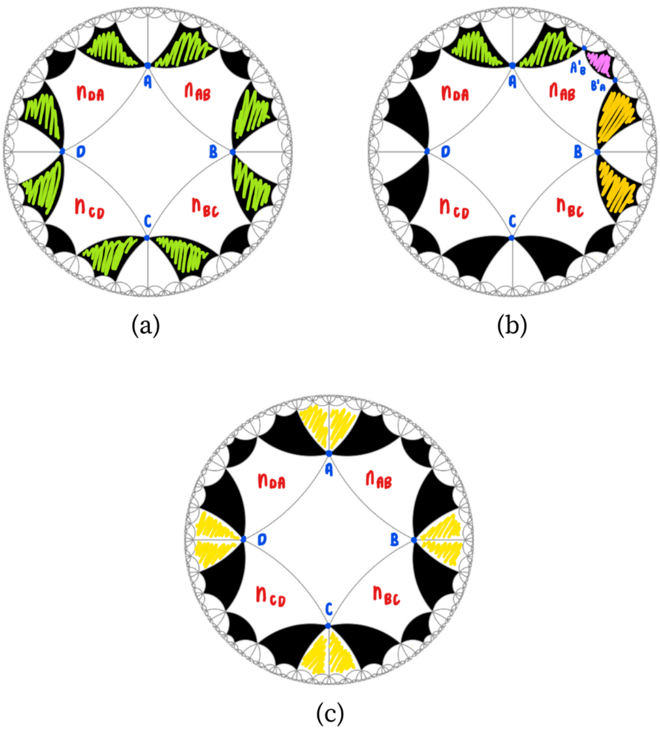


Fig. 14. Neighbors of dead cells on levels 0 (a) and 1 (b & c)

Case 3 (Level 2). Now we consider the cells at level 2, which will be split into three subcases.

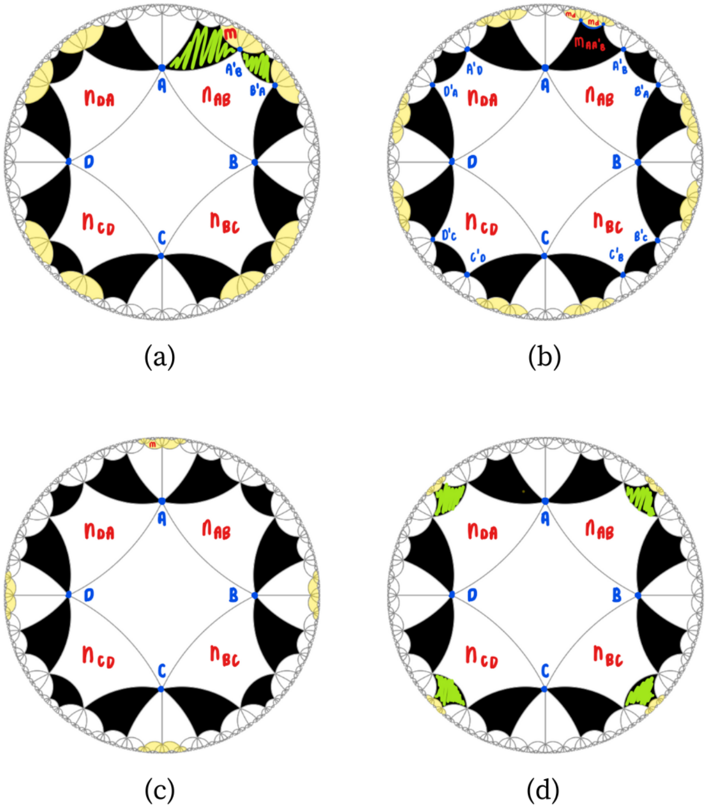


Fig. 15. Neighbors of dead cells on levels 2 (a, b, & c) and 3 (d)

We let M_O be the set of neighbors of each cell $n \in N_E$ that shares a vertex with n but does not share an edge with any cells in N_E and that is not a neighbor of c—colored yellow in Figure 15(a). As an example, we consider cell $n_{AB} \in N_E$ which has the vertices A, B, A'_B, B'_A. Any $m \in M_O$ which shares vertex A'_B with n_{AB} has two live neighbors—the live neighbors of n_{AB} that share vertex A'_B—colored green in Figure 15(a). Note that this is generalizable for all $m \in M_O$. Since $2 \notin w$, the birthability rule ensures that each $m \in M_O$ will not be born.

We let M_E be the set of the eight live neighbors of c—colored green in Figure 14(a). Additionally, we let M_D be the set of neighbors of cells in M_E that are not neighbors of any cell in $N_E \cup \{c\}$—colored yellow in Figure 15(b).

Suppose that we take cell $m_d \in M_D$ that is a neighbor of $m_{AA'_B} \in M_E$ that shares an edge AA'_B with $n_{AB} \in N_E$ either:

- shares an edge with $m_{AA'_B}$, which is the opposite edge of AA'_B of n_{AB}, or

– shares a vertex with $m_{AA'_B}$ which is either adjacent to vertex A but not A'_B or adjacent to A'_B but not A

In whichever case, m_d has only one live neighbor, $m_{AA'_B}$. Note that a similar proof can be used to show that for all $m_d \in M_D$, m_d has one live neighbor. Since $1 \notin w$, the birthability rule ensures that each $m_d \in M_D$ will not be born.

Recall that M_E is the set of the eight live neighbors of c—colored green in Figure 14(a) and N_O is the set of neighbors of c that share a vertex with c but do not share an edge with any cells in N_E— colored yellow in Figure 14(c). We let M_F be the set of neighbors of cells in N_O that are not neighbors of any cell in $M_E \cup \{c\}$—colored yellow in Figure 15(c). Since every $m \in M_F$ is not neighbors with any of the live cells in M_E and every $n \in N_O$ is a dead cell, each $m \in M_F$ has no live neighbors. Since $0 \notin w$, the birthability rule ensures that each $m \in M_F$ will not be born.

Case 4 (Level 3). Now we consider cells at level 3. which will be split into two subcases. We let M_G be the set of neighbors of each cell $n \in N_E$ that shares only an edge with n and does not share a vertex with c—colored green in Figure 15(d). This set is composed of the only live cells in level 2. We then let M_H be the set of neighbors of each cell $m \in M_G$ that is not a member of N_E and is not a neighbor of any $n \in N_E$. Note that all of these cells have one live neighbor—that is, the cell $m \in M_G$. Since $1 \notin w$, the birthability rule ensures that none of these cells will be born.

All cells at level 3 not in M_G have no live neighbors. Since $0 \notin w$, the birthability rule ensures that none of these cells will be born.

Case 5 (Level 4 and onwards). All cells from this level onwards have no live neighbors since all live cells are in levels 1 and 2 only. Since $0 \notin w$, the birthability rule ensures that none of these cells will be born.

Suppose that B is the set of live cells. For each $b \in B$, it has y live neighbors. Note that for the live cells to survive in the next generation, we need to ensure that for each $b \in B$, $y = 2$ or $y \in v$. For our proof, we will check the live cells and show that either $y = 2$ or $y \in v$.

Recall that M_E is the set of the eight live neighbors of c—colored green in Figure 14(a). Also, recall that M_G is the set of neighbors of each cell $n \in N_E$ that shares only an edge with n and does not share a vertex with c—colored green in Figure 15(d). Lastly, recall that N_E is the set of neighbors $\{n_{AB}, n_{BC}, n_{CD}, n_{DA}\}$ of c that shares an edge with c. We let M_I be $M_E \cup M_G$. Note that M_I contains all live cells on the lace. We can split this into 3 cases.

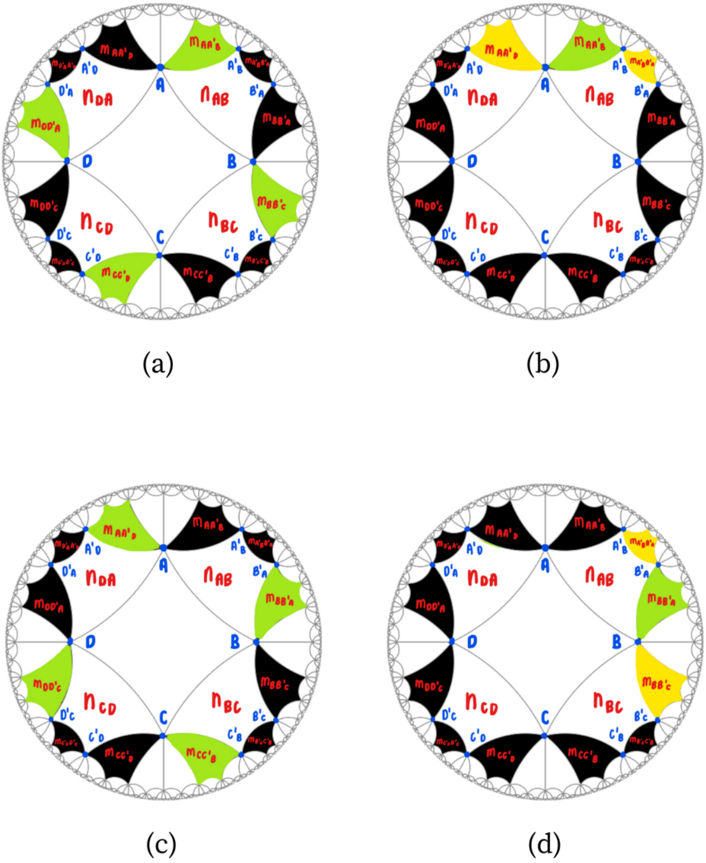


Fig. 16. Neighbors of live cells for cases 1 (a & b) and 2 (c & d)

First, we consider each $m_{XX'_Y} \in M_I$—colored green in Figure 16(a). Each $m_{XX'_Y} \in M_I$ that shares an edge XX'_Y with $n_{XY} \in N_E$ has two live neighbors.

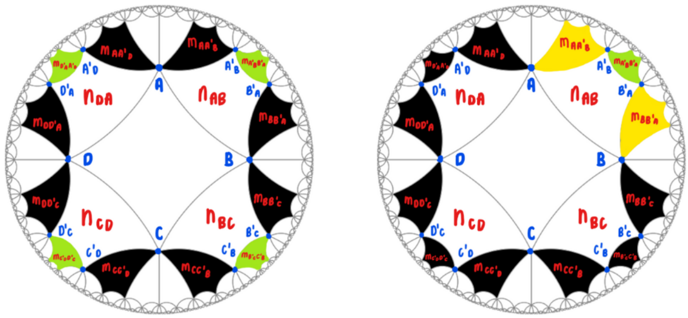
1. $m_{X'_Y Y'_X}$ which shares vertex X'_Y with $m_{XX'_Y}$ and edge $X'_Y Y'_X$ with n_{XY}, and
2. $m_{XX'_Z}$ which shares vertex X with $m_{XX'_Y}$ and edge XX'_Z with n_{ZX}

An example of the two live neighbors (colored yellow) of $m_{AA'_B}$ is shown in Figure 16(b).

Second, we consider each $m_{YY'_X} \in M_I$—colored green in Figure 16(c). Each $m_{YY'_X} \in M_I$ that shares an edge YY'_X with $n_{XY} \in N_E$ has two live neighbors.

1. $m_{X'_Y Y'_X}$ which shares vertex Y'_X with $m_{YY'_X}$ and edge $X'_Y Y'_X$ with n_{XY}, and
2. $m_{YY'_Z}$ which shares vertex Y with $m_{YY'_X}$ and edge YY'_Z with n_{YZ}

An example of the two live neighbors (colored yellow) of $m_{BB'_A}$ is shown in Figure 16(d).



(a) (b)

Fig. 17. Neighbors of live cells for case 3 (a & b)

Third, we consider each $m_{X'_Y Y'_X} \in M_I$—colored green in Figure 17(a). Each $m_{X'_Y Y'_X} \in M_I$ that shares an edge $X'_Y Y'_X$ with $n_{XY} \in N_E$ has two live neighbors.

1. $m_{XX'_Y}$ which shares vertex X'_Y with $m_{X'_Y Y'_X}$ and edge XX'_Y with n_{XY}, and
2. $m_{YY'_X}$ which shares vertex Y'_X with $m_{X'_Y Y'_X}$ and edge YY'_X with n_{XY}

An example of the two live neighbors (colored yellow) of $m_{A'_B B'_A}$ is shown in Figure 17(b).

In whichever case, $m \in M_I$ has two live neighbors. Since the number of live neighbors for any live cell is equal to 2, the survivability rule ensures that each $m \in M_I$ will survive. With this, none of the cells will change its state, making the lace under this rulestring a still life. Hence, the proposition holds true. □

Definition 17 (Donut). *A donut is a pattern with a tiling of $\{p, q\}$ where every polygon sharing an edge with c are set as live cells. Some examples of the donut are shown in Figure 18.*

Proposition 4. *A donut is a still life with rulestring $Bw/S2, v$ where $w > 2$ and $w \neq p$ and $v \notin \{2\}$.*

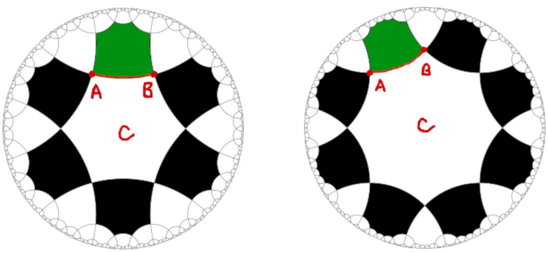


Fig. 18. Donuts with tilings $\{6, 4\}$ (left) and $\{8, 4\}$ (right)

Proof. Suppose that we have a donut with rulestring $Bw/S2,v$ where $w > 2$ and $w \neq p$ and $v \notin \{2\}$. Cell c has p live neighbors that will collectively be referred to as N. All the live cells will be at level 1, with each cell having two live neighbors. To demonstrate, we select one arbitrary cell in N that shares edge AB with c. Refer to Figure 18 for a visualization. Note that edge AB has endpoints A and B. Each endpoint can be used to formulate two other edges where each edge is shared with c. We can deduce that the two cells sharing these two edges are live cells due to the conditions we established previously. This is generalizable to all the other cells in N as well.

On the other hand, all the dead cells in higher levels will have 0 or 1 live neighbor, while the dead cell c in level 0 will have p. Thus, for every timestep, the birthability rule will ensure that no dead cell will ever become a live state. The survivability rule will then ensure that each of the live cells will remain alive even after the timestep. Nevertheless, a substring v can be added to make the property more generalizable. Hence, the proposition holds true. □

After discovering these four still lifes, we made observations regarding their properties and conceptualized a generalized proof for them.

Proposition 5. *Any pattern can be a still life under rulestring Bx/Sy under the following conditions:*

1. *Given the set of dead cells denoted by D, let $A = \{a$ such that a is the number of live neighbors for each d in $D\}$. Then, let $x \subseteq \{b$ such that $b \notin A\}$*
2. *Given the set of live cells denoted by set L, let $y \supseteq \{c$ such that c is the number of live neighbors for each l in $L\}$.*

Proof. Suppose we have a pattern under rulestring Bx/Sy where x and y satisfy the conditions above. Then, rule Bx ensures that no dead cell will be born because for every $b \in x$, no dead cell is surrounded by b live neighbors. Rule Sy ensures that all live cells will survive because for every live cell $l \in L$, there is a $c \in y$ such that l is surrounded by c live neighbors. Thus, the proposition holds true. □

Example 1. Suppose we have a flower in a tiling $\{p, q\}$. Then, $A = \{0, 1, 3\}$. Thus, $x \in \{2, k$ such that $k > 3\}$. And also, $y = \{3, p\}$. Given this, a flower is a still life under rulestring $B4/S3, p$ (but also $B2/S3, p$, $B2, 4/S3, p$, etc.)

5.2 Discovered Oscillators

In this subsection, we will discuss the oscillators that we have discovered. However, we only present conjectures for the oscillators.

Definition 18 (Sunburst). *A sunburst is a pattern with a tiling of $\{p, 4\}$ where every polygon touching only one vertex of c are set as live cells.*

***Conjecture* 1 (*Pinwheel*).** *In a tiling of* $\{p, 4\}$ *with rulestring* $B2/Sv$ *where* $v = \{x|2 < x \leq n\}$ *and* n = *number of neighbors of a cell, there is an oscillator of period 2, which we shall call a pinwheel, whose generations alternate between a sunburst and a donut. An example of the pinwheel with tiling* $\{5, 4\}$ *is shown in Figure 19.*

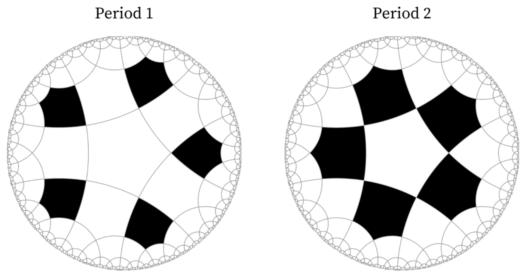


Fig. 19. A pinwheel with the tiling $\{5, 4\}$ under the rulestring $B2/S3$

Definition 19 (Leaves Pattern). *The leaves pattern has a tiling of* $\{p, 4\}$. *The live cells in this pattern are the* $p - 1$ *neighbors of* c *each touching one of the* $p - 1$ *consecutive edges of* c.

Definition 20 (Trunk). *The trunk is a pattern with a tiling of* $\{p, 4\}$. *This can be generated based on Algorithm 3.*

Algorithm 3 Generating a trunk

Choose c
count $\leftarrow 0$
for each n in N **do** ▷ Let N be the neighbors of c
    if n shares only a vertex with c and *count* $< p - 2$ **then**
        Set n as a live cell
        count $=$ *count* $+ 1$
    end if
end for
Find a neighbor of c such that it shares an edge with it and has no live neighbors
Set this cell as a live cell

***Conjecture* 2 (*Tree*).** *In a tiling of* $\{p, 4\}$ *with rulestring* $B2/Sv$ *where* $v = \{x|2 < x \leq n\}$ *and* n = *number of neighbors of a cell, there is an oscillator of period 2, which we shall call a tree, whose generations alternate between the* *leaves and the trunk. An example of the tree with tiling* $\{6, 4\}$ *is shown in Figure 20.*

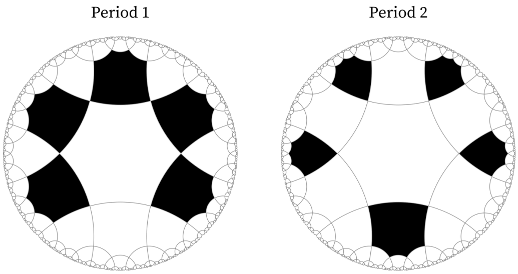


Fig. 20. A tree with the tiling $\{6, 4\}$ under the rulestring $B2/S3$

Definition 21 (Big load). *The big load is a pattern with tiling* $\{n, 4\}$*, where* n *is an integer greater than or equal to 5, generated based on Algorithm 4.*

Algorithm 4 Generating a big load

1: Choose c
2: Select $p - 3$ cells in N such that they share an edge with c (They must be adjacent) ▷ Let N be the neighbors of c
3: Set these cells as live cells
4: Identify two cells in N such that they share only a vertex with c and has no live neighbors
5: Set these cells as live cells

Definition 22 (Small load). *The small load is a pattern with tiling* $\{n, 4\}$*, where* n *is an integer greater than or equal to 5, generated based on Algorithm 5.*

Algorithm 5 Generating a small load

1: Choose c
2: Select $p - 4$ cells in N such that they share only a vertex with c (These cells must be adjacent) ▷ Let N be the neighbors of c
3: Set these cells as live cells
4: Identify three cells in N such that they share an edge with c and has no live neighbors
5: Set these cells as live cells

***Conjecture* 3 (*Loading spinner*).** *In a tiling of $\{n, 4\}$ where n is an odd integer greater than or equal to 5 with rulestring $B2/Sv$ where $v = \{x|2 < x \leq n\}$ and n = number of neighbors of a cell, there is an oscillator of period 2, which we shall call a loading spinner, whose generations alternate between the small load and the big load. An example of the loading spinner with tiling $\{9, 4\}$ is shown in Figure 21.*

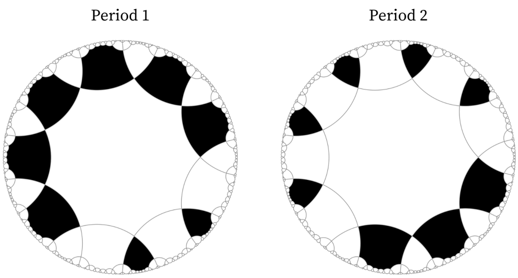


Fig. 21. A loading spinner with the tiling $\{9, 4\}$ under the rulestring $B2/S3$ with a period of 2

***Conjecture* 4 (*Firecracker*).** *In a tiling of $\{p, 4\}$ with rulestring $B2/Sv$ where $v = \{x|2 < x \leq n\}$ and n = number of neighbors of a cell, there is an oscillator of period 2, which we shall call a firecracker, whose generations alternate between the firework and the flower. An example of the tree with tiling $\{5, 4\}$ is shown in Figure 22.*

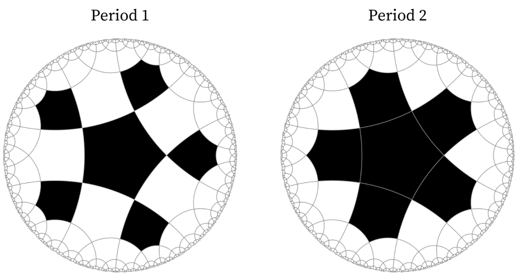


Fig. 22. A firecracker with the tiling $\{5, 4\}$ under the rulestring $B3/Sp$ with a period of 2

After discovering four oscillators, we have made one observation regarding their properties and conceptualized a generalized proof for them.

Proposition 6. *At least one oscillator can be formed for any tiling* $\{p, q\}$ *with rulestring* $Bp/S2, 3, v$ *where* $v \notin \{0, 2, 3, p\}$.

Proof. Suppose that we have a tiling and rulestring that follows the conditions stated above. We then select one cell to be the center c. Note that the c will always share edges with p neighbors, which will collectively be referred to as N. An oscillator can be created by setting all $n \in N$ as live cells while retaining the dead state of c. Some plausible patterns will be like the donuts shown in Figure 18.

Under this configuration, all the live cells will be at level 1, with each cell having two live neighbors. On the other hand, c will have a total of p live neighbors. Thus, for the next timestep, the birthability rule Bp will cause c to be born, while the survivability rule $S2, 3, v$ ensures that each of the live cells will remain alive in the next timestep. In the following timestep, c will die due to the survivability rule as v cannot be p, but all the other live cells will survive due to the survivability rule. Essentially, the state of c will alternate between alive and dead for each generation while the other cells retain their initial state. Hence, the proposition holds true. □

5.3 Still Lifes and Oscillators on $\{p, 4\}$ under $B2/S3, p$

The current library of patterns of the Euclidean Game of Life, composed of still lifes, oscillators, and moving objects, all exist on tiling $\{4, 4\}$ under Conway's rulestring $B3/S2, 3$. Unlike its counterpart on the Euclidean space, the library of patterns of the Hyperbolic Game of Life varies in terms of both tiling and rulestring [7], the sole contributor to the small existing library of patterns, recommended future researchers to look into the discovery of a rulestring and tiling combination where there exists still lifes, oscillators, and moving objects. While we did not focus on the discovery of moving objects, we were able to discover that a flower and a pinwheel are a still life and an oscillator, respectively, on tiling $\{p, 4\}$ under rulestring $B2/S3, p$. From this, we were able to formulate Conjecture 5.

***Conjecture* 5.** *On tiling* $\{p, 4\}$ *under rulestring* $B2/S3, p$, *there exists both a still life (the flower) and an oscillator (the pinwheel).*

6 CONCLUSION

In this paper, we discovered still lifes and period 2 oscillators and wrote conjectures and proofs about them. Some observations on their properties were also noted in this study. This aside, we also discovered a tiling and a rulestring combination where both a still life and an oscillator exist. Such is the flower and pinwheel on the tiling $\{p, 4\}$ under the rulestring $B2/S3, p$. This resembles the

Game of Life on the Euclidean plane where both still lifes and oscillators on tiling $\{4,4\}$ under the rulestring $B3/S2,3$ exist.

7 FUTURE WORKS

While our study was able to add to the existing body of knowledge for still lifes and oscillators on the hyperbolic plane, there are still several areas that could be explored. Thus, for future researchers, we have provided some recommendations on how our study can be improved:

- Discover other oscillators aside from those having a period of two
- Identify more generalizable properties of still lifes and oscillators on the hyperbolic space
- Identify moving objects on the hyperbolic space, including one that exists on tiling $\{p,4\}$ under the rulestring $B2/S3,p$
- Identify unique properties of patterns of the Game of Life on the hyperbolic space that are different from those in the Euclidean space

8 ACKNOWLEDGEMENT

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