Branes and Representations of DAHA $C^\vee C_1$

Junkang Huang*, Satoshi Nawata**, and Yutai Zhang***

* Department of Physics and Center for Field Theory and Particle Physics, Fudan University, Shanghai, China,
`junkang-huang@outlook.com`, **`snawata@gmail.com`, ***`zyt742770851@gmail.com`

Abstract. In this proceeding, we briefly summarize the results of [1], which investigates the representation theory of the spherical double affine Hecke algebra (DAHA) of type $C^\vee C_1$, using brane quantization. Our analysis establishes a bijective correspondence between compact Lagrangian A-branes in the $\mathrm{SL}(2,\mathbb{C})$ character variety of four-punctured spheres and finite-dimensional representations of the spherical DAHA. This correspondence provides evidence for a derived equivalence between two categories: the A-brane category of the character variety and the representation category of the spherical DAHA. The D_4 root system plays a central role in understanding both the geometric and representation-theoretic structures underlying this correspondence.

Keywords: brane quantization, double affine Hecke algebra, topological string theory

1 Introduction

Brane quantization is a quantization framework that employs the ideas from A-model topological string theory, as introduced by Gukov and Witten [2], which profoundly bridges various frameworks and interdisciplinary fields.

In this approach, one consider the topological A-model on a complexification $\mathfrak{X}$, which is typically an affine hyper-Kähler manifold with a holomorphic symplectic form Ω, of a symplectic manifold M. The real part $\mathrm{Re}\,\Omega$ of Ω restricts to the symplectic form on M, and imaginary part $\mathrm{Im}\,\Omega$ restricts to zero on M. Considering the A-model on $\mathfrak{X}$ with respect to the symplectic form $\omega_{\mathfrak{X}} := \mathrm{Im}\,\Omega$, the brane quantization provides a bridge between the deformation quantization and the geometric quantization. To be more specific, in this framework, the deformation quantization $\mathscr{O}^q(\mathfrak{X})$ of the coordinate ring of $\mathfrak{X}$, with respect to the holomorphic symplectic form Ω, corresponds to the algebra of topological strings from a special coisotropic A-brane $\mathfrak{B}_{\mathrm{cc}}$ (the *canonical coisotropic brane* [3,4]) to itself. And the Hilbert space of the geometric quantization of a symplectic manifold M corresponds to the space of topological strings between $\mathfrak{B}_{\mathrm{cc}}$ and the Lagrangian A-brane $\mathfrak{B}_M$[2,?,6]. Since the algebra of topological strings from $\mathfrak{B}_{\mathrm{cc}}$ to itself naturally acts on the space of topological strings from them to other A-branes, this naturally proposes a functor

$$\mathbf{R}\mathrm{Hom}(-,\mathfrak{B}_{\mathrm{cc}}) : D^b A\text{-}\mathsf{Brane}(\mathfrak{X},\omega_{\mathfrak{X}}) \xrightarrow{\sim} D^b\,\mathsf{Rep}(\mathscr{O}^q(\mathfrak{X}))\ , \tag{1}$$

R. Pichai and J. Walcher (eds.), *Proceedings of the International Conference on Knots, Quivers and Beyond (ICKQB 2025)*, Advances in Physics Research 14,
https://doi.org/10.2991/978-94-6463-789-2_4

which is expected to provide a derived equivalence.

In our work [1], we consider the case where the target space $\mathfrak{X}$ is the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a four-punctured sphere $C_{0,4}$, which is a hyper-Kähler manifold with three complex structures I, J, K. The description as moduli space of flat connections is in complex structure J, which can be explicitly parametrized as an affine cubic equation(4). While in complex structure I, $\mathfrak{X}$ can be interpreted as the moduli space of SU(2) parabolic Higgs bundles on $C_{0,4}$[7]. From a physical perspective, $\mathfrak{X}$ corresponds to the Coulomb branch of a 4d $\mathcal{N}=2$ supersymmetric SU(2) gauge theory (SQCD) with four fundamental hypermultiplets [8] on $S^1 \times \mathbb{R}^3$.

For this space $\mathfrak{X}$, the algebra $\mathscr{O}^q(\mathfrak{X})$ is known[9] to be the spherical subalgebra of the double affine Hecke algebra (DAHA) of type $C^\vee C_1$[10,11,12,13,14,15,16]. This algebra, denoted as $S\ddot{H}_{q,\boldsymbol{t}}$, is parametrized by a deformation parameter q and four complex parameters $\mathbf{t} = (t_1, t_2, t_3, t_4)$ characterizing the monodromies around the four punctures. The appearance of DAHA as the deformation quantization of coordinate rings of character varieties and affine Grassmannians was examined in the pioneering works [17,9,18,19,20]. These perspectives were further examined from a physics standpoint in [21], where brane quantization was employed to study the representation theory of DAHA of type A_1.

The main results of our work [1] can be summarized into two claims. For non-compact A-branes, which are branes of type (A,B,A), we show that

Claim 1 *Under the functor* (1)*, the* (A,B,A)*-branes supported on 24 lines in the affine cubic surface* $\mathfrak{X}$ *correspond to the polynomial representation of* $S\ddot{H}_{q,\boldsymbol{t}}$ *and its images under the* D_4 *Weyl group and the cyclic permutation group corresponding to the triality of* $\mathfrak{so}(8)$.

Claim 2 *For* $\mathfrak{X} = \mathcal{M}_{\mathit{flat}}(C_{0,4}, \mathrm{SL}(2,\mathbb{C}))$*, the functor* (1) *restricts to a derived equivalence of the full subcategory of compact Lagrangian* A*-branes of* $\mathfrak{X}$ *and the category of finite-dimensional* $S\ddot{H}_{q,\boldsymbol{t}}$*-modules.*

This paper is organized as follows to present our work: In Section 2, we introduce the basics about the spherical DAHA $S\ddot{H}_{q,\boldsymbol{t}}$ of type $C^\vee C_1$. We will analyze its symmetries and the connection to the D_4 root system, which is followed by a discussion of the polynomial representation of $S\ddot{H}_{q,\boldsymbol{t}}$ and the Askey-Wilson polynomials. Section 3 provides an investigation of the geometry of the target space $\mathfrak{X}$ in both complex structure I and J. Then we examine the Hitchin fibration in the special case where $\beta = 0 = \gamma$, showing the connection to the affine D_4 root system and identifying the A-branes explicitly. In §4, we explicit match the polynomial representations and the A-brane for the generic fibers and global nilpotent cone.

2 Polynomial Representations of Double Affine Hecke Algebra of Type $C^\vee C_1$

The spherical DAHA $S\ddot{H}_{q,\boldsymbol{t}}$ is generated by x, y, z satisfying the defining relations

$$\begin{aligned} [x,y]_q &= (q^{-1}-q)z + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\theta_3 \ , \\ [y,z]_q &= (q^{-1}-q)x + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\theta_1 \ , \\ [z,x]_q &= (q^{-1}-q)y + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\theta_2 \ , \end{aligned} \tag{2}$$

where the q-commutator is defined as $[f,g]_q \equiv q^{-\frac{1}{2}}fg - q^{\frac{1}{2}}gf$ and

$$\theta_1 = \overline{t_1}\;\overline{t_2} + \overline{t_3}\;\overline{t_4} \ , \quad \theta_2 = \overline{t_1}\;\overline{t_3} + \overline{t_2}\;\overline{t_4} \ , \quad \theta_3 = \overline{t_1}\;\overline{t_4} + \overline{t_2}\;\overline{t_3} \ , \tag{3}$$

using the notation $\overline{f} \equiv f + f^{-1}$. In addition, the generators x, y, z satisfy a Casimir relation

$$-q^{-\frac{1}{2}}xyz + q^{-1}x^2 + qy^2 + q^{-1}z^2 + q^{-\frac{1}{2}}\theta_1 x + q^{\frac{1}{2}}\theta_2 y + q^{-\frac{1}{2}}\theta_3 z + \theta_4(q) = 0 \ , \tag{4}$$

where $\theta_4(q) = \overline{t_1}^2 + \overline{t_2}^2 + \overline{t_3}^2 + \overline{t_4}^2 + \overline{t_1}\;\overline{t_2}\;\overline{t_3}\;\overline{t_4} - \overline{q} - 2$.

As is evident from (2), the spherical DAHA $S\ddot{H}_{q,\boldsymbol{t}}$ becomes commutative in the "classical" limit $q = 1$, and the Casimir relation (4) reduces to the equation of an affine cubic surface:

$$-xyz + x^2 + y^2 + z^2 + \theta_1 x + \theta_2 y + \theta_3 z + \theta_4(1) = 0 \ . \tag{5}$$

2.1 Symmetries of the Spherical DAHA

The spherical DAHA $S\ddot{H}_{q,\boldsymbol{t}}$ exhibits three key types of symmetries, each of which plays an essential role in its structure.

Weyl group $W(D_4)$*:* The parameters $\theta_1, \theta_2, \theta_3, \theta_4$ can be naturally interpreted as the characters of the vector, spinor, conjugate spinor, and adjoint representations the Lie algebra $\mathfrak{so}(8) = D_4$. The characters are invariant under this Weyl group $W(D_4)$ of $\mathfrak{so}(8)$, and so is the q-deformed version $\theta_4(q)$. Consequently, the spherical DAHA is also invariant under $W(D_4)$.

Braid group B_3*:* The spherical subalgebra $S\ddot{H}_{q,\boldsymbol{t}}$ is invariant under the B_3 action

$$\begin{aligned} \tau_+ :\ & (x,y,z,\theta_1,\theta_2,\theta_3,\theta_4) \mapsto \left(x, q^{-\frac{1}{2}}(xy - q^{-\frac{1}{2}}z - \theta_3), y, \theta_1, \theta_3, \theta_2, \theta_4\right) \ , \\ \tau_- :\ & (x,y,z,\theta_1,\theta_2,\theta_3,\theta_4) \mapsto \left(q^{-\frac{1}{2}}(xy - q^{-\frac{1}{2}}z - \theta_3), y, x, \theta_3, \theta_2, \theta_1, \theta_4\right) \ , \\ \sigma :\ & (x,y,z,\theta_1,\theta_2,\theta_3,\theta_4) \mapsto \left(y, x, q^{-\frac{1}{2}}(xy - q^{-\frac{1}{2}}z - \theta_3), \theta_2, \theta_1, \theta_3, \theta_4\right) \ . \end{aligned} \tag{6}$$

B_3 has a $\mathbb{Z}_3$ cyclic permutation subgroup generated by $(x,y,z,\theta_1,\theta_2,\theta_3,\theta_4) \mapsto (y,z,x,\theta_2,\theta_3,\theta_1,\theta_4)$.

Sign-flip group $\mathbb{Z}_2^{\times 2}$*:* Lastly, there is a symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_2^{\times 2}$ which flips the signs of x, y, z and $\theta_1, \theta_2, \theta_3$. See [1] for the details.

2.2 Polynomial Representations

The spherical DAHA $\ddot{SH}_{q,\boldsymbol{t}}$ acts on $\mathscr{P} \equiv \mathbb{C}_{q,\boldsymbol{t}}[X, X^{-1}]^{\mathbb{Z}_2}$ [22,14,23], which is the space of symmetric Laurent polynomials with coefficients in the ring $\mathbb{C}_{q,\boldsymbol{t}}$ of rational functions in q and $\boldsymbol{t}$. The representation is given by pol : $\ddot{SH}_{q,\boldsymbol{t}} \to \mathrm{End}(\mathscr{P})$ and the actions of the generators are given by:

$$
\begin{aligned}
\mathrm{pol}(x) &= X + X^{-1} \ , \\
\mathrm{pol}(y) &= A(X)(\varpi - 1) + A(X^{-1})(\varpi^{-1} - 1) - q^{\frac{1}{2}} t_1 t_3 - q^{-\frac{1}{2}} (t_1 t_3)^{-1} \ , \\
\mathrm{pol}(z) &= q^{\frac{1}{2}} \left[X A(X) \varpi + X^{-1} A(X^{-1}) \varpi^{-1} \right] \\
&\quad - \frac{X + X^{-1}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} \left[A(X) + A(X^{-1}) + q^{\frac{1}{2}} t_1 t_3 + q^{-\frac{1}{2}} t_1^{-1} t_3^{-1} \right] - \frac{\theta_3}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} \ .
\end{aligned}
\tag{7}
$$

with $A(X)$ a rational function and ϖ the q-shift operator defined by $\varpi \cdot f(X) = f(qX)$. Under the polynomial representation (7), pol(y) acts diagonally on the Askey-Wilson polynomials. We can define the raising and lowering operators in $\ddot{SH}_{q,\boldsymbol{t}}$ with respect to this basis [24] as

$$
R_n = -q^{1-n} x - q t_1 t_3 z + E_n \ , \quad L_n = -q^{n+2} x - \frac{q}{t_1 t_3} z + D_n \ , \tag{8}
$$

where D_n and E_n are two constants. Under the polynomial representation, these operators raise and lower the labels n of the Askey-Wilson polynomials as follows:

$$
\mathrm{pol}(R_n) \cdot P_n = q^{1-n} \left(t_1^2 t_3^2 q^{2n+1} - 1 \right) P_{n+1} \ , \tag{9}
$$

$$
\begin{aligned}
\mathrm{pol}(L_n) \cdot P_n = &-q^{1-n} \left(q^n - t_1^{-1} t_2 t_3^{-1} t_4 \right) \left(q^n - t_1^{-1} t_2 t_3^{-1} t_4^{-1} \right) \left(q^n - t_1^{-1} t_2^{-1} t_3^{-1} t_4 \right) \\
&\times \left(q^n - t_1^{-1} t_2^{-1} t_3^{-1} t_4^{-1} \right) \frac{(q^n - 1)(q^n - t_1^{-2})(q^n - t_3^{-2})(q^n - t_1^{-2} t_3^{-2})}{\left(q^{2n} - t_1^{-2} t_3^{-2} \right)^2 \left(q^{2n-1} - t_1^{-2} t_3^{-2} \right)} P_{n-1} \ .
\end{aligned}
\tag{10}
$$

A common way to construct a finite-dimensional representation is to find null vectors for the raising and lowering operators. The raising operators can never be null, while there are conditions where the lowering operator annihilates some $P_n(x; q, \boldsymbol{t})$, which becomes the lowest weight state of a sub-representation. In this way, a finite-dimensional $\ddot{SH}_{q,\boldsymbol{t}}$-module appears as the quotient $\mathscr{P}/(P_n)$ of the polynomial representation by the ideal (P_n).

Therefore, we can study finite-dimensional representations by imposing the condition pol$(L_n) \cdot P_n = 0$. This amounts to the shortening conditions neatly repackaged as

$$
q^n = t_1^{-r_1} t_2^{-r_2} t_3^{-r_3} t_4^{-r_4} =: \boldsymbol{t}^{-r}, \qquad r \in \mathsf{R}(D_4) \cup \{(0,0,0,0)\} \ . \tag{11}
$$

3 Geometry of Coulomb Branch

In complex structure J, the moduli space $\mathcal{M}_{\text{flat}}(C_{0,4}, SL(2,\mathbb{C}))$ is characterized by the monodromy representation of the fundamental group of $C_{0,4}$:

$$\begin{aligned}&\mathcal{M}_{\text{flat}}(C_{0,4}, \mathrm{SL}(2,\mathbb{C})) = \\ &\qquad\qquad \langle M_1, M_2, M_3, M_4 \in \mathrm{SL}(2,\mathbb{C}) \mid M_1M_2M_3M_4 = 1\rangle / \mathrm{SL}(2,\mathbb{C})\ ,\end{aligned} \tag{12}$$

where the quotient by $\mathrm{SL}(2,\mathbb{C})$ is taken with respect to conjugation. To describe the character variety geometrically, we introduce holonomy variables as holomorphic functions on $\mathcal{M}_{\text{flat}}(C_{0,4}, \mathrm{SL}(2,\mathbb{C}))$:

$$\begin{aligned}&x = -\operatorname{Tr}(M_1M_2)\ , \quad y = -\operatorname{Tr}(M_1M_3)\ , \quad z = -\operatorname{Tr}(M_2M_3)\ , \\ &\overline{t_j} = \operatorname{Tr}(M_j)\ , \qquad (j = 1,2,3,4)\ .\end{aligned} \tag{13}$$

By applying trace identities in $SL(2,\mathbb{C})$, the holonomy variables are subject to the cubic surface equation (4). Therefore, in the classical limit, $S\ddot{H}_{q,\boldsymbol{t}}$ corresponds to the coordinate ring of the moduli space. The cubic surface admits a holomorphic symplectic form $\Omega_J = \omega_K + i\omega_I$ of complex structure J given by

$$\Omega_J = -\frac{1}{2\pi}\frac{dx \wedge dy}{\partial f/\partial z} = -\frac{1}{2\pi}\frac{dx \wedge dy}{2z - xy + \theta_3}. \tag{14}$$

By comparing the Poisson brackets of the generators for the coordinate ring x, y, z and the algebraic relation in $S\ddot{H}_{q,\boldsymbol{t}}$, one can show that, under the identification $q = e^{2\pi i\hbar}$, the Poisson brackets can be obtained by

$$\left.\frac{[-,-]}{i\hbar}\right|_{\hbar\to 0} = \{-,-\}\ , \tag{15}$$

where $[x, y] = xy - yx$ is the ordinary commutator. This verifies that the spherical DAHA $S\ddot{H}_{q,\boldsymbol{t}}$ is the deformation quantization of the coordinate ring with respect to the holomorphic symplectic form Ω_J [9].

Next, we will turn to complex structure I, which is convenient in describing the geometry of the space. Denote the four punctures of $C_{0,4}$ as $p_j, j = 1,2,3,4$. The moduli space of Higgs bundles over $C_{0,4}$ is the space of holomorphic rank 2 vector bundle E and the holomorphic one form $\varphi \in H^0(\operatorname{End} E \otimes K_C[D])$ (called Higgs field), subject to the following tame ramification at p_j

$$A = \alpha_j\, d\vartheta + \cdots\ , \quad \varphi = \frac{1}{2}(\beta_j + i\gamma_j)\frac{dz}{z} + \cdots \tag{16}$$

Here, $z = re^{i\vartheta}$ is a local coordinate on a small disk centered at p_j, and the ramification data is a triple $(\alpha_j, \beta_j, \gamma_j) \in \mathfrak{t} \times \mathfrak{t} \times \mathfrak{t}$ where we denote the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{su}(2)$. By expressing them as conjugate to diagonal matrices

$$\operatorname{diag}(\alpha_j, -\alpha_j) \sim \alpha_j \in \mathfrak{t}\ , \quad \operatorname{diag}(\beta_j, -\beta_j) \sim \beta_j \in \mathfrak{t}\ , \quad \operatorname{diag}(\gamma_j, -\gamma_j) \sim \gamma_j \in \mathfrak{t}\ , \tag{17}$$

we will often use the triple $(\alpha_j,\beta_j,\gamma_j)\in\mathfrak{t}^3$ interchangeably with the ramification parameters $(\upalpha_j,\upbeta_j,\upgamma_j)\in\mathbb{R}^3$ in the following discussions, by a slight abuse of notation. These parameters are related with t_j parameter in the character variety via

$$t_j=\exp(-2\pi(\upgamma_j+i\upalpha_j))\ ,\qquad (j=1,2,3,4). \tag{18}$$

Moduli space of Higgs bundle admits the Hitchin fibration over the Hitchin base $\mathcal{B}_H$[7]:

$$\begin{aligned} h:\mathcal{M}_H(C_{0,4},\mathrm{SU}(2))&\to\mathcal{B}_H=H^0(C_{0,4},K_C^{\otimes 2})\ .\\ (E,\varphi)&\mapsto \mathrm{Tr}\big(\varphi^2\big)\ , \end{aligned} \tag{19}$$

where the Hitchin base $\mathcal{B}_H$ is identified with the u-plane that parametrizes the complex structure u of the Seiberg-Witten curve. At special points of the u-plane, a Hitchin fiber degenerates into a singular fiber as classified by Kodaira[25,26]. The geometry of the moduli space undergoes significant changes as the ramification parameters (16) at the four punctures are varied, leading to changes in the types of singular fibers.

The resulting classification can be conveniently summarized by the usage of affine D_4 Dynkin diagram. Each type of singular fiber forms a chain of $\mathbb{CP}^1$ components connected according to the structure of an affine ADE Dynkin diagram (except for Kodaira type II). Consequently, the intersection matrix of a singular fiber matches the corresponding affine Cartan matrix, up to an overall sign difference. Given the ramification parameters $\upgamma_j$, we define the evaluation map as

$$\mathrm{ev}_{\upgamma}:\mathsf{R}(D_4)\to\mathbb{R}\ ;\ r=(r_1,r_2,r_3,r_4)\mapsto\upgamma_1r_1+\upgamma_2r_2+\upgamma_3r_3+\upgamma_4r_4\ . \tag{20}$$

The kernel of this linear map precisely gives the root system associated with the singular fibers.

3.1 A-model on the Coulomb Branch

The primary goal of the analysis is to identify the A-branes inside the target space $\mathfrak{X}$. As is mentioned in the introduction, when the target space is of quaternionic dimension one, there are two types of A-branes: the canonical coisotropic brane $\mathfrak{B}_{\mathrm{cc}}$ and the Lagrangian branes $\mathfrak{B}_{\mathbf{L}}$.

Canonical Coisotropic Brane $\mathfrak{B}_{\mathbf{cc}}$ The canonical coisotropic brane is a holomorphic line bundle over the target space itself, which is parameterized by a single complex parameter $\hbar=|\hbar|e^{i\theta}\in\mathbb{C}^\times$. The curvature F of such a bundle combines with the B-field $B\in H^2(\mathfrak{X},\mathrm{U}(1))$ of the A-model into a gauge invariant combination $F+B$, which is constrained by:

$$\varOmega:=F+B+i\omega_{\mathfrak{X}}=\frac{\varOmega_J}{i\hbar}\ . \tag{21}$$

In the brane quantization set up, we choose $\omega_{\mathfrak{X}}$ as the symplectic form of original symplectic manifold, $\varOmega=\varOmega_J/i\hbar$ as its complexification, the holomorphic symplectic form. Thanks to the hyper-Kähler structure $\omega_{\mathfrak{X}}^{-1}(F+B)=J$, the brane automatically satisfies the condition $(\omega_{\mathfrak{X}}^{-1}(B+F))^2=-1$, which is required for a coisotropic A-brane [3].

With this setup, the algebra of open $(\mathfrak{B}_{\mathrm{cc}},\mathfrak{B}_{\mathrm{cc}})$-strings gives rise to the deformation quantization of the coordinate ring on $\mathfrak{X}$ with respect to Ω_J, which aligned with $S\ddot{H}_{q,\boldsymbol{t}}$.

Lagrangian $\boldsymbol{A}$-branes $\mathfrak{B}_{\mathbf{L}}$ As the target symplectic manifold $(\mathfrak{X},\omega_{\mathfrak{X}})$ is of quaternionic dimension one, A-branes of the other types are all Lagrangian A-branes, namely it has a Lagrangian submanifold $\mathbf{L}$ as its support, endowed with the structure of a flat Spin^c-bundle.

The space of $(\mathfrak{B}_{\mathrm{cc}},\mathfrak{B}_{\mathbf{L}})$-open string arises from the geometric quantization of $\mathbf{L}$, namely the space of holomorphic sections $\mathfrak{B}_{\mathrm{cc}}\otimes\mathfrak{B}_{\mathbf{L}}^{-1}$ over $\mathbf{L}$. When the support is a compact Lagrangian submanifold, this space is conjectured to be related to the finite-dimensional representations of $S\ddot{H}_{q,\boldsymbol{t}}$ as in (1) and one can employ the B-model perspective to compute the dimension of the representation space $\mathrm{Hom}(\mathfrak{B}_{\mathbf{L}},\mathfrak{B}_{\mathrm{cc}})$ using Hirzebruch-Riemann-Roch formula, which in our case simplifies to

$$\dim\mathscr{L}=\int_{\mathbf{L}}\mathrm{ch}(\mathfrak{B}_{\mathrm{cc}})=\int_{\mathbf{L}}\frac{F+B}{2\pi}\ , \tag{22}$$

for a real two-dimensional Lagrangian $\mathbf{L}$. The volume of a Lagrangian submanifold $\mathbf{L}$ is is defined as the integration of the Kähler form ω_I over $\mathbf{L}$

$$\mathrm{vol}_I(\mathbf{L}):=\int_{\mathbf{L}}\frac{\hbar}{2\pi}\mathrm{Im}\Omega=\int_{\mathbf{L}}\frac{\omega_I}{2\pi}\ . \tag{23}$$

3.2 Global Nilpotent Cone

When the ramification parameters α_j are generic while the others are set to zero, $\beta_j=\gamma_j=0$ $(j=1,2,3,4)$, the Hitchin fibration $h:\mathcal{M}_H(C_p,\mathrm{SU}(2))\to\mathcal{B}_H$ in (19) possesses a singular fiber only at the origin of the base, $\mathbf{N}=h^{-1}(0)$, which is referred to as the *global nilpotent cone*. The global nilpotent cone $\mathbf{N}$ is a singular fiber of Kodaira type I_0^* [27], characterized by a configuration of five irreducible components, each of which is topologically $\mathbb{CP}^1$, arranged in the shape of the affine D_4 Dynkin diagram as in Figure 1:

$$\mathbf{N}=\mathbf{V}\cup\bigcup_{j=1}^{4}\mathbf{D}_j\ . \tag{24}$$

At generic values of α_j, each irreducible component of the global nilpotent cone serves as a generator of the second integral homology group, $H_2(\mathcal{M}_H,\mathbb{Z})\cong\mathbb{Z}^{\oplus 5}$. Using the basis $\{[\mathbf{D}_1],[\mathbf{D}_2],[\mathbf{D}_3],[\mathbf{D}_4],[\mathbf{V}]\}$, the intersection pairing between these homology classes is represented by the Cartan matrix of the affine D_4 Dynkin diagram and the second homology group $H_2(\mathcal{M}_H,\mathbb{Z})$ can be identified

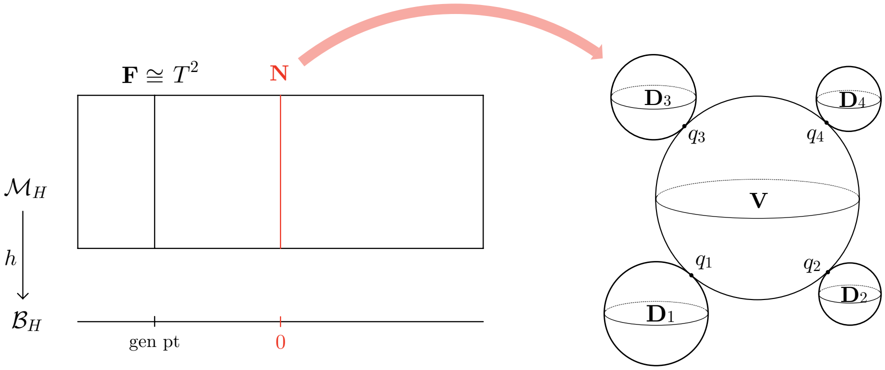

Fig. 1. The schematic figure of the Hitchin fibration $\mathcal{M}_H \to \mathcal{B}_H$ when the ramification parameters α_j are generic, and the others are zero $\beta_j = \gamma_j = 0$, $(j = 1, 2, 3, 4)$. A generic fiber $\mathbf{F}$ is topologically a two-torus, and the global nilpotent cone $\mathbf{N}$ at the origin of the Hitchin base $\mathcal{B}_H$ is a singular fiber of Kodaira type I_0^*, which is illustrated on the right.

with the root lattice of the affine D_4 Lie algebra. In this identification, the irreducible components correspond to the simple roots of the affine D_4 root system. Furthermore, these components also satisfy a fiber-class relation in $H_2(\mathcal{M}_H, \mathbb{Z})$:

$$[\mathbf{F}] = 2[\mathbf{V}] + \sum_{j=1}^{4} [\mathbf{D}_j]\ , \tag{25}$$

where $[\mathbf{F}]$ denotes the homology class of a generic fiber of the Hitchin fibration [27].

As a characteristic of the Hitchin fibration, due to its complete integrability, the period of $\omega_I/2\pi$ over a general fiber $\mathbf{F}$ is one, so $\mathrm{vol}_I(\mathbf{F}) = 1$, and the homology relation(25) imposes the constraint

$$2\mathrm{vol}_I(\mathbf{V}) + \sum_{j=1}^{4} \mathrm{vol}_I(\mathbf{D}_j) = 1\ . \tag{26}$$

The relation between the ramification parameters α_j and the volumes of the irreducible components in the global nilpotent cone is subtle and involves the wall-crossing phenomenon. When an irreducible component $\mathbf{V}$ or $\mathbf{D}_j$ shrinks to zero, the Higgs bundle is no longer stable, and the Hitchin moduli space develops a du Val singularity. These singularities correspond to special points in the Kähler moduli space, which are located at codimension-one loci, referred to as walls. Du Val singularities appear precisely at the zeros of the Seiberg-Witten discriminant, which can be neatly summarized as

$$\boldsymbol{t}^r = 1\ , \quad \forall r \in \mathsf{R}(D_4)\ . \tag{27}$$

These walls align with the set of reflection hyperplanes for the affine D_4 weight lattice at level one [28]. In an affine root system, the reflection of an affine weight λ at level one with respect to an affine root $\dot{r} = r - n\delta$

$$s_{\dot{r}}(\lambda) = s_r\left(\lambda - \frac{n}{2}r\right) \tag{28}$$

generating the affine Weyl group $\dot{W}(D_4)$. Therefore, a chamber surrounded by the walls corresponds to a Weyl alcove of type D_4.

A du Val singularity arises when a two-cycle shrinks to zero volume, whose condition is linear in the Kähler parameters α_j. This implies that the volume of a compact two-cycle depends linearly on α_j. Then the configurations of the walls in (27) uniquely determine the volumes of the cycles. Given a point α_j of the Kähler moduli space, the volumes of the two-cycles $\mathbf{V}$ and $\mathbf{D}_j$ $(j = 1, 2, 3, 4)$ are equal to twice the distance from this point to the corresponding walls.

It turns out that, there are 24 chambers in total generically. In the subsequent analysis, we focus on the chamber defined by the following constraints (which we refer to as the center chamber):

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 1 \ , \qquad -\alpha_1 - \alpha_2 + \alpha_3 + \alpha_4 \geq 0 \ , \\ -\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 \geq 0 \ , \qquad -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 \geq 0 \ , \end{aligned} \tag{29}$$

together with the restriction $\alpha_1 \in [0, \frac{1}{4}]$. From the identification between the volume functions and the basis of the affine root system, we deduce the explicit volume functions in this chamber:

$$\begin{aligned} \mathrm{vol}_I(\mathbf{D}_j) &= \Big(1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \ - \alpha_1 - \alpha_2 + \alpha_3 + \alpha_4, \\ &\qquad - \alpha_1 + \alpha_2 - \alpha_3 + \alpha_4, \ - \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4\Big), \\ \mathrm{vol}_I(\mathbf{V}) &= 2\alpha_1 \ , \end{aligned} \tag{30}$$

where the normalization in (26) is appropriately applied.

In fact, we can define the simple roots of the D_4 root system with a suitable choice of positive roots as

$$\{e^1, e^2, e^3, e^4\} = \{(-1, -1, 1, 1), (-1, 1, -1, 1), (-1, 1, 1, -1), (2, 0, 0, 0)\} \ , \tag{31}$$

Then, the highest root is expressed by $\theta = e^1 + e^2 + e^3 + 2e^4 = (1, 1, 1, 1)$. Using these roots, the volume functions can be concisely written as

$$\begin{aligned} &\Big(\mathrm{vol}_I(\mathbf{D}_1), \mathrm{vol}_I(\mathbf{D}_2), \mathrm{vol}_I(\mathbf{D}_3), \mathrm{vol}_I(\mathbf{D}_4), \mathrm{vol}_I(\mathbf{V})\Big) = \\ &\qquad\qquad \Big(1 - \theta \cdot \alpha, \ e^1 \cdot \alpha, \ e^2 \cdot \alpha, \ e^3 \cdot \alpha \ , e^4 \cdot \alpha\Big) \ , \end{aligned} \tag{32}$$

where $r \cdot \alpha = \sum_{j=1}^{4} r_j \alpha_j$ is the Euclidean inner product. The homology class of each irreducible component of the I_0^* singular fiber corresponds to an affine D_4 root as

$$[\mathbf{D}_1] \leftrightarrow e^0 = \delta - \theta \ , \quad [\mathbf{D}_2] \leftrightarrow e^1 \ , \quad [\mathbf{D}_3] \leftrightarrow e^2 \ , \quad [\mathbf{D}_4] \leftrightarrow e^3 \ , \quad [\mathbf{V}] \leftrightarrow e^4 \ , \tag{33}$$

where δ is the imaginary root and the case $\delta \cdot \alpha = 1$ is assumed for the volume function. Using the relation of the second homology classes, the homology class $[\mathbf{F}]$ of a generic Hitchin fiber indeed corresponds to the imaginary root δ.

4 Matching Branes and Representations

In this section, we will show the main result of this project: the explicit correspondence between A-branes and the representations of the $S\ddot{H}_{q,\boldsymbol{t}}$. In particular, we will match the A-brane conditions and the shortening conditions for the generic fibers and global nilpotent cone.

4.1 Non-compact Branes and Infinite-dimensional Representations

To identify the A-brane associated with the polynomial representation in (7), we consider the classical limit $q \to 1$. In this limit, the lowering operator L_n becomes independent of n, so we denote $L^{(c)} = L_n|_{q\to 1}$. In the classical limit, certain operators act as scalar multiplications:

$$\begin{aligned} \mathrm{pol}(y) &= -t_1t_3 - t_1^{-1}t_3^{-1}, \\ -\mathrm{pol}(L^{(c)}) = 0 &= \mathrm{pol}(x) + \frac{\mathrm{pol}(z)}{t_1t_3} + t_1^{-1}t_2^{-1} + t_1^{-1}t_2 + t_3^{-1}t_4^{-1} + t_3^{-1}t_4. \end{aligned} \tag{34}$$

The second equation holds because the lowering operator becomes null in the classical limit $q \to 1$, as demonstrated in (10). Geometrically, this describes the support of the brane $\mathfrak{B}_{\mathbf{P}}$, for the polynomial representation:

$$\mathbf{P} = \{y = -t_1t_3 - t_1^{-1}t_3^{-1}, z = -t_1t_3x - t_1t_4^{-1} - t_1t_4 - t_2^{-1}t_3 - t_2t_3\} \tag{35}$$

The restriction of Ω_J on $\mathbf{P}$ vanishes. Thus $\mathbf{P}$ is Lagrangian with respect to the symplectic forms ω_I and ω_K. This establishes that $\mathfrak{B}_{\mathbf{P}}$ is an (A,B,A)-brane associated with the polynomial representation.

We can construct new polynomial representations by applying the Weyl group action $W(D_4)$ and the cyclic group $\mathbb{Z}_3$ to $\mathbf{P}^1$. Since both group actions act linearly on the coordinates (x,y,z), they map the line $\mathbf{P}$ to other lines. As a result, 24 distinct lines can be generated from $\mathbf{P}$.

A notable feature of these lines is that they all lie in planes where one of the coordinates x, y, or z remains constant. Thus, the slope of each line can be described by a single complex number. To formalize this, we define the slope as follows: if x is constant along the line, the slope is given by $\frac{dy}{dz}$; if y is constant, the slope is $\frac{dz}{dx}$; and if z is constant, the slope is $\frac{dx}{dy}$.

Furthermore, we denote these slopes by $\mathbb{S}_x$, $\mathbb{S}_y$, and $\mathbb{S}_z$, corresponding to lines in planes where x, y, or z is constant, respectively. Interestingly, these sets have a natural interpretation in terms of SO(8) representation theory:

$$\mathbb{S}_x = \{-\boldsymbol{t}^w \mid w \in \mathsf{P}(\mathbf{8}_V)\}, \quad \mathbb{S}_y = \{-\boldsymbol{t}^w \mid w \in \mathsf{P}(\mathbf{8}_S)\}, \quad \mathbb{S}_z = \{-\boldsymbol{t}^w \mid w \in \mathsf{P}(\mathbf{8}_C)\}, \tag{36}$$

---

[1] The action of braid group will generate more (A,B,A)-branes, see [1] for the details.

where $\boldsymbol{t}^w \equiv t_1^{w_1} t_2^{w_2} t_3^{w_3} t_4^{w_4}$, and $\mathsf{P}(\mathbf{8}_V)$, $\mathsf{P}(\mathbf{8}_S)$, and $\mathsf{P}(\mathbf{8}_C)$ are the weights of the SO(8) vector, spinor, and cospinor representations, respectively. These weights correspond to the shortest weights in the D_4 weight lattice.

Consequently, the 24 non-zero roots of the D_4 root system are organized into three distinct sets, each in one-to-one correspondence with the weights of the SO(8) vector, spinor, and cospinor representations. Furthermore, a similar argument based on the pullback of Ω_J confirms that the branes corresponding to these 24 lines are (A, B, A)-branes in the cubic surface, verifying the Claim 1.

As a result, the lines and their corresponding infinite-dimensional representations can be labeled using the shortest weights $w \in \mathsf{P}(D_4)$. We denote the lines as $\mathbf{P}_w$, the polynomial representations as $\mathscr{P}^w$ and the associated Askey-Wilson polynomials as P_n^w.

The relation between lines and the shortest weights provides a useful criterion to determine whether a finite-dimensional representation, as given in (11), can be obtained by truncating an infinite-dimensional representation supported on a line [1]:

Proposition 1. *The finite-dimensional representation, labeled by a root $r \in \mathsf{R}(D_4)$, can be obtained by truncating the polynomial representation $\mathscr{P}^w$ if and only if $\langle r, w\rangle > 0$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product.*

4.2 Compact Branes and Finite-dimensional Representations

We make explicit verification of the Claim 2 in this section. For simplicity, assume $\hbar$ is real. In this case, a generic fiber $\mathbf{F}$ is Lagrangian with respect to $\mathrm{Im}\Omega = \frac{\omega_K}{\hbar}$, and thus $\mathbf{F}$ can serve as the support of an A-brane $\mathfrak{B}_{\mathbf{F}}$. Thus, we consider a brane $\mathfrak{B}_{\mathbf{F}}$ supported on a generic fiber and the corresponding representation $\mathscr{F}$. The dimension formula (22) and the unit volume of a generic fiber implies

$$m := \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{F}}, \mathfrak{B}_{\mathrm{cc}}) = \int_{\mathbf{F}} \frac{\omega_I}{2\pi\hbar} = \frac{1}{\hbar}. \tag{37}$$

Consequently, the A-brane $\mathfrak{B}_{\mathbf{F}}$ can exist if and only if $\frac{1}{\hbar} = m \in \mathbb{Z}_{>0}$. By $q = e^{2\pi i\hbar}$, the dimension formula implies $q^m = 1$, which is precisely the shortening condition (11) with $r = (0,0,0,0)$. Under this shortening condition, we obtain the finite-dimensional $S\ddot{H}_{q,\boldsymbol{t}}$-module $\mathscr{F}_m = \mathscr{P}/(P_m)$ with

$$\begin{aligned} P_m(X, q = e^{\frac{2\pi i}{m}}, \boldsymbol{t}) &= X^m + X^{-m} + F_m(\boldsymbol{t}), \\ F_m(\boldsymbol{t}) &= \frac{(t_3^m t_4^m + t_3^m/t_4^m)(t_1^{2m} - 1) + (t_1^m t_2^m + t_1^m/t_2^m)(t_3^{2m} - 1)}{(t_1 t_3)^{2m} - 1}. \end{aligned} \tag{38}$$

For a non-trivial example, let us consider the singular fiber I_0^*, i.e. the global nilpotent cone that appears when $\beta_j = 0 = \gamma_j$, as considered in §3.2. As illustrated in Figure 1, the irreducible components of the I_0^* singular fiber consist of both the moduli space $\mathbf{V}$ of SU(2)-bundles on $C_{0,4}$ and the exceptional divisors $\mathbf{D}_j$ $(j = 1, 2, 3, 4)$.

Object Matching By combining the dimension formula (22) with the volume formula for $\mathfrak{B}_{\mathbf{V}}$ given in (30), we find that the dimension of the corresponding representation is:

$$k := \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{V}}, \mathfrak{B}_{\mathrm{cc}}) = \int_{\mathbf{V}} \frac{F+B}{2\pi} = \int_{\mathbf{V}} \frac{\omega_I}{2\pi\hbar} = \frac{2\alpha_1}{\hbar}. \tag{39}$$

The A-brane $\mathfrak{B}_{\mathbf{V}}$ can exist if and only if k is a positive integer. Using (18) and noting that $q = e^{2\pi i\hbar}$, this requirement translates to the condition $q^k = t_1^{-2}$, which corresponds to the shortening condition (11), with the D_4 root $r = (2,0,0,0)$. When the shortening condition is satisfied, the corresponding k-dimensional representation, denoted by $\mathscr{V}_k$, can be explicitly constructed as the quotient $\mathscr{V}_k = \mathscr{P}/(P_k)$.

Using the same approach, one can identify an $S\ddot{H}_{q,\boldsymbol{t}}$-module $\mathscr{D}_{\ell_j}^{(j)}$ corresponding to a brane $\mathfrak{B}_{\mathbf{D}_j}$ supported on each exceptional divisor $\mathbf{D}_j$. The dimension of the morphism space is given by:

$$\ell_j := \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{D}_j}, \mathfrak{B}_{\mathrm{cc}}) = \int_{\mathbf{D}_j} \frac{F+B}{2\pi} = \int_{\mathbf{D}_j} \frac{\omega_I}{2\pi\hbar} = \frac{\mathrm{vol}(\mathbf{D}_j)}{\hbar}. \tag{40}$$

The results are summarized in Table 1, based on the volume formulas provided in (30).

finite-dim rep	shortening condition	A-brane	A-brane condition
$\mathscr{F}_m^{(x_m)}$	$q^m = 1$	$\mathfrak{B}_{\mathbf{F}}^{(x_m)}$	$m = \frac{1}{\hbar}$
$\mathscr{V}_k$	$q^k = t_1^{-2}$	$\mathfrak{B}_{\mathbf{V}}$	$k = \frac{2\alpha_1}{\hbar}$
$\mathscr{D}_{\ell_1}^{(1)}$	$q^{\ell_1} = t_1 t_2 t_3 t_4$	$\mathfrak{B}_{\mathbf{D}_1}$	$\ell_1 = \frac{1}{\hbar} - \frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{\hbar}$
$\mathscr{D}_{\ell_2}^{(2)}$	$q^{\ell_2} = t_1 t_2 t_3^{-1} t_4^{-1}$	$\mathfrak{B}_{\mathbf{D}_2}$	$\ell_2 = \frac{-\alpha_1-\alpha_2+\alpha_3+\alpha_4}{\hbar}$
$\mathscr{D}_{\ell_3}^{(3)}$	$q^{\ell_3} = t_1 t_2^{-1} t_3 t_4^{-1}$	$\mathfrak{B}_{\mathbf{D}_3}$	$\ell_3 = \frac{-\alpha_1+\alpha_2-\alpha_3+\alpha_4}{\hbar}$
$\mathscr{D}_{\ell_4}^{(4)}$	$q^{\ell_4} = t_1 t_2^{-1} t_3^{-1} t_4$	$\mathfrak{B}_{\mathbf{D}_4}$	$\ell_4 = \frac{-\alpha_1+\alpha_2+\alpha_3-\alpha_4}{\hbar}$

Table 1. A summary of finite-dimensional $S\ddot{H}_{q,\boldsymbol{t}}$-modules with their shortening conditions and the corresponding A-brane configurations at the I_0^* singular fiber, under the assumption $|q| = 1$.

Beyond a single irreducible component of the I_0^* singular fiber, one can consider a brane supported on a union of several components, denoted by $\mathbf{N}_r$ with r denotes $-a_0\theta + \sum_{i=1}^4 a_i e^i$, which is Lagrangian with respect to $\omega_{\mathfrak{X}} = \omega_K/\hbar$. The homology class of $\mathbf{N}_r$ is represented with the standard basis by

$$[\mathbf{N}_r] = a_0[\mathbf{D}_1] + a_1[\mathbf{D}_2] + a_2[\mathbf{D}_3] + a_3[\mathbf{D}_4] + a_4[\mathbf{V}], \tag{41}$$

Here, e^i are the simple roots, and θ is the highest root in the D_4 root system (31). Then, it is straightforward to show that $r \in \mathsf{R}(D_4)$ if and only if a cycle

$\mathbf{N}_r$ has a self-intersection number minus two: $[\mathbf{N}_r]\cdot[\mathbf{N}_r] = -2$. Using the same approach, we determine the A-brane condition for $\mathbf{N}_r$. Applying the dimension formula (22) and the volume formula (32), the A-brane condition is given by

$$m := \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{N}_r}, \mathfrak{B}_{\mathrm{cc}}) = \int_{\mathbf{N}_r} \frac{\omega_I}{2\pi\hbar} = \frac{a_0 + \left(\sum_{i=1}^{4} a_i e^i - a_0\theta\right)\cdot \alpha}{\hbar}. \tag{42}$$

The corresponding shortening condition is $q^m = \boldsymbol{t}^{-r}$ with $\boldsymbol{t}^r = t_1^{r_1} t_2^{r_2} t_3^{r_3} t_4^{r_4}$. Thus, we write the corresponding $\ddot{SH}_{q,\boldsymbol{t}}$-module as $\mathscr{N}_m^{r,w} = \mathscr{P}^w/(P_m^w)$ where $\langle r, w\rangle > 0$ due to Proposition 1. As a result, if we consider all the cycles with self-intersection number minus two, all 24 roots in the D_4 root system $\mathsf{R}(D_4)$ are exhausted. In this way, the A-brane conditions in (42) recover all the shortening conditions classified in (11).

Morphism Matching In the category of A-branes, a morphism between two A-branes becomes non-trivial when they intersect, forming a bound state [29,30]. Let $\mathfrak{B}_{\mathbf{N}_r}$ and $\mathfrak{B}_{\mathbf{N}_{r'}}$ be A-branes supported on components of the I_0^* singular fiber. The A-brane conditions are given by:

$$m = \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{N}_r}, \mathfrak{B}_{\mathrm{cc}}), \quad m' = \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{N}_{r'}}, \mathfrak{B}_{\mathrm{cc}}). \tag{43}$$

We denote the corresponding $\ddot{SH}_{q,\boldsymbol{t}}$-modules by $\mathscr{N}_m^{r,w}$ and $\mathscr{N}_{m'}^{r',w'}$ where the shortening conditions are given by $q^m = \boldsymbol{t}^{-r}$ and $q^{m'} = \boldsymbol{t}^{-r'}$, respectively. Then we have the following proposition [1]:

Proposition 2. *If* $\mathfrak{B}_{\mathbf{N}_r}$ *and* $\mathfrak{B}_{\mathbf{N}_{r'}}$ *intersect at a point* q*, then the morphism space is one-dimensional*

$$\mathrm{Hom}(\mathfrak{B}_{\mathbf{N}_r}, \mathfrak{B}_{\mathbf{N}_{r'}}) = \mathbb{C}\langle q\rangle \ . \tag{44}$$

Then, the roots $r, r' \in \mathsf{R}(D_4)$ *satisfy* $\langle r, r'\rangle = -2$*. Moreover, there exists a weight* w *such that* $\langle w, r\rangle > 0$*,* $\langle s_r(w), r'\rangle > 0$*. In this setting, there exists a short exact sequence of finite-dimensional* $\ddot{SH}_{q,\boldsymbol{t}}$*-modules*

$$0 \longrightarrow \mathscr{N}_{m'}^{r',s_r(w)} \longrightarrow \mathscr{N}_{m+m'}^{r+r',w} \longrightarrow \mathscr{N}_m^{r,w} \longrightarrow 0 \ . \tag{45}$$

This exact sequence is uniquely determined up to isomorphism, independent of the choice of w*. Thus,* $\mathrm{Ext}^1(\mathscr{N}_m^{r,w}, \mathscr{N}_{m'}^{r',s_r(w)})$ *is one-dimensional.*

As an explicit example, let us consider the morphism space $\mathrm{Hom}^*(\mathfrak{B}_{\mathbf{V}}, \mathfrak{B}_{\mathbf{D}_1})$. Since $\mathbf{D}_1$ and $\mathbf{V}$ intersect at a single point q_1 (see Figure 1), the geometric perspective predicts that the morphism space is one-dimensional. The A-brane $\mathfrak{B}_{\mathbf{N}_r}$, representing their bound state, is supported on $\mathbf{N}_r = \mathbf{V} \cup \mathbf{D}_1$, with the corresponding root $r = -\theta + e_4 = (1, -1, -1, -1)$. The A-brane condition for $\mathfrak{B}_{\mathbf{N}_r}$ is evaluated as:

$$m = \dim \mathrm{Hom}(\mathfrak{B}_{\mathbf{N}_r}, \mathfrak{B}_{\mathrm{cc}}) = \int_{\mathbf{N}_r} \frac{F+B}{2\pi} = \frac{1}{\hbar} - \frac{-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{\hbar}, \tag{46}$$

which translates to the shortening condition $q^m = t_1^{-1}t_2t_3t_4$.

Without loss of generality, we can take the weight to be $w = (1, 0, -1, 0)$. Consider the polynomial representation $\mathscr{P}^w$ where the action of lowering operators L_m^w contains a factor $(q^m - t_1^{-2})(q^m - t_1^{-1}t_2t_3t_4)$, which can be verified from (10). Suppose that we impose the two shortening conditions simultaneously

$$q^k = t_1^{-2} \ , \quad \text{and} \quad q^{k+\ell} = t_1^{-1}t_2t_3t_4 \ , \tag{47}$$

then we obtain the following short exact sequence from $\mathscr{P}^w$:

$$0 \longrightarrow \mathscr{D}_\ell^{(1)} \longrightarrow \mathscr{N}_{k+\ell}^{r,w} \longrightarrow \mathscr{V}_k \longrightarrow 0 \ . \tag{48}$$

This represents a non-trivial element in $\mathrm{Ext}^1(\mathscr{V}_k, \mathscr{D}_\ell^{(1)})$.

5 Conclulsion

Following the prediction from brane quantization, we establish a correspondence between non-compact/compact A-branes in the $\mathrm{SL}(2,\mathbb{C})$-character variety of a fourth-punctured sphere and the infinite-dimensional/finite-dimensional representations of the spherical DAHA $S\ddot{H}_{q,\boldsymbol{t}}$. This correspondence provides explicit and compelling evidence for the proposed equivalence (1) between the category of A-branes and the representation category of $S\ddot{H}_{q,\boldsymbol{t}}$.

References

1. J. Huang, S. Nawata, Y. Zhang and S. Zhuang, *Branes and Representations of DAHA $C^\vee C_1$: affine braid group action on category*, arXiv:2412.19647 [hep-th].
2. S. Gukov and E. Witten, *Branes and quantization*, *Adv. Theor. Math. Phys.* **13** (2009) 1[arXiv:0809.0305].
3. A. Kapustin and D. Orlov, *Remarks on A branes, mirror symmetry, and the Fukaya category*, *J. Geom. Phys.* **48** (2003) 84[arXiv:hep-th/0109098].
4. A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, *Communications in Number Theory and Physics* **1** (2007) 1–236[arXiv:0604151].
5. S. Gukov, *Quantization via mirror symmetry*, *Japanese Journal of Mathematics* **6** (2011) 65–119[arXiv:1011.2218 [hep-th]].
6. D. Gaiotto and E. Witten, *Probing quantization via branes*, *Surveys Diff. Geom.* **24** (2019) 293–402[arXiv:2107.12251].
7. N. Hitchin, *The self-duality equations on a Riemann surface*, *Proc. London Math. Soc.* **55** (1987) 59–126.
8. N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, *Nucl. Phys.* **B431** (1994) 484–550[arXiv:hep-th/9408099].
9. A. Oblomkov, *Double affine Hecke algebras of rank 1 and affine cubic surfaces*, *International Mathematics Research Notices* **2004** (2004) 877–912[arXiv:math/0306393].

10. I. Cherednik, *Double affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald's operators*, *International Mathematics Research Notices* **1992** (1992) 171–180.
11. I. Cherednik, *Double affine Hecke algebras and Macdonald's conjectures*, *Annals of mathematics* **141** (1995) 191–216.
12. I. Cherednik, *Double affine Hecke algebras.* Cambridge University Press, 2005.
13. J. van Diejen, *Self-dual Koornwinder-Macdonald polynomials*, *Invent. Math.* **126** (1996) 319–339[arXiv:q-alg/9507033].
14. S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, *Annals of mathematics* **150** (1999) 267–282[arXiv:q-alg/9710032].
15. S. Sahi, *Some properties of Koornwinder polynomials*, *Contemp. Math.* **254** (2000) 395–411.
16. J. V. Stokman, *Koornwinder polynomials and affine Hecke algebras*, *Int. Math. Res. Not.* **19** (2000) 1005–1042[arXiv:math/0002090].
17. A. Oblomkov, *Double affine Hecke algebras and Calogero-Moser spaces*, *Representation Theory of the American Mathematical Society* **8** (2004) 243–266[arXiv:math/0303190].
18. E. Vasserot, *Induced and simple modules of double affine Hecke algebras*, *Duke Mathematical Journal* **126** (2005) 251–323[arXiv:math/0207127].
19. R. Bezrukavnikov, M. Finkelberg and I. Mirkovic, *Equivariant homology and K-theory of affine Grassmannians and Toda lattices*, *Compositio Mathematica* **141** (2005) 746–768[math/0306413].
20. M. Varagnolo and E. Vasserot, *Double affine Hecke algebras and affine flag manifolds, I*, in *Affine flag manifolds and principal bundles*, pp. 233–289. Springer, 2010 arXiv:0911.5328 [math.RT].
21. S. Gukov, P. Koroteev, S. Nawata, D. Pei and I. Saberi, *Branes and DAHA representations*, *SpringerBriefs in Mathematical Physics* (2023) [arXiv:2206.03565 [hep-th]].
22. M. Noumi and J. V. Stokman, *Askey–Wilson polynomials: an affine Hecke algebra approach*, *Laredo Lectures on Orthogonal Polynomials and Special Functions* **111** (2004) C144[arXiv:math/0001033].
23. J. V. Stokman, *Difference Fourier transforms for nonreduced root systems*, *Selecta Mathematica* **9** (2003) 409–494[arXiv:math/0111221].
24. S. Sahi, *Raising and lowering operators for Askey-Wilson polynomials*, *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* **3** (2007) 002[arXiv:math/0701134].
25. K. Kodaira, *On the structure of compact complex analytic surfaces, I*, *American Journal of Mathematics* **86** (1964) 751–798.
26. K. Kodaira, *On the structure of compact complex analytic surfaces, II*, *American Journal of Mathematics* **88** (1966) 682–721.
27. S. Gukov, *Gauge theory and knot homologies*, *Fortsch. Phys.* **55** (2007) 473–490[arXiv:0706.2369 [hep-th]].
28. K. Iwasaki, *Finite branch solutions to Painlevé VI around a fixed singular point*, *Advances in Mathematics* **217** (2007) 1889–1934[arXiv:0704.0679 [math.AG]].
29. A. Floer, *Morse theory for Lagrangian intersections*, *Journal of differential geometry* **28** (1988) 513–547.
30. A. Floer, *Witten's complex and infinite-dimensional Morse theory*, *Journal of differential geometry* **30** (1989) 207–221.