



Exploring Fixed Point for Contractive Mapping using Partial Pompeiu-Hausdorff Metric and Partial Order Relations

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Abstract. In this paper, we discuss the existence of fixed point for set-valued mappings in partially ordered partial metric spaces. Our theorems improve some result of Sedghi et al (2013). By using the contraction's mapping principle of Aydi et al (2012) and partial ordered's definition on Gregorio (2014), we prove that the existence of fixed points is guaranteed.

Keywords: fixed point theory, contractive mapping, partial pompeiu-hausdorff metric, partial order relations

1 Introduction

In economics and game theory, studying about setvalued mappings' fixed points is very important. This condition shows that the study of set-valued mappings' fixed points is quite applicable. This is one of the reasons why the research on the scope of set-valued mappings' fixed points continues to develop. The pioneer, Nadler in 1969 [1], introduced the theorem about fixed point for set-valued mappings to generalize the contraction principle of Banach [2].

Theorem 1. [1] Suppose that (Ω, d) be a complete metric space and $CB(\Omega) = \{A \subseteq \Omega : A \neq \emptyset, A \text{ closed and bounded}\}$. If $F : \Omega \rightarrow CB(\Omega)$ and there exist $\kappa \in [0, 1)$ such that $H(F(\alpha), F(\beta)) \leq \kappa d(\alpha, \beta)$, for all $\alpha, \beta \in \Omega$, then set-valued \mathcal{F} has a fixed point.

Aydi et al. (2012) adapted the fixed point theorem introduced by Nadler to the context of partial metric spaces [3].

Theorem 2. [3] Let (Ω, p) be a complete partial metric space and $CB^p(\Omega) = \{A \subseteq \Omega : A \neq \emptyset, A \text{ closed and bounded}\}$. If $F : \Omega \rightarrow CB^p(\Omega)$ and there exist $\kappa \in [0, 1)$ such that $H^p(F(\alpha), F(\beta)) \leq \kappa p(\alpha, \beta)$ for all $\alpha, \beta \in \Omega$, then set-valued F has a fixed point.

The discussion about the theorem of set-valued mappings' fixed points has been extended to the context of metric spaces with partial order, as has been done by Beg and Butt [4], who have shown the existence of setvalued mappings' fixed points on partially ordered metric spaces. What has been done by Beg and Butt aims to generalize the theorem of single-valued mappings' fixed point on metric spaces with partial order, as has been done by Ran and Reurings [5].

Theorem 3. [4] Suppose that (Ω, \preceq) is partially ordered set and (Ω, d) is a complete metric space. If a non-decreasing sequence $\alpha_n \rightarrow \alpha$ in metric space Ω , then $\alpha_n \preceq \alpha$, for each n . Let mapping $F : \Omega \rightarrow CB(\Omega)$ satisfy the following conditions:

a. there exist $\kappa \in (0, 1)$ with

$$H(F(\alpha), F(\beta)) \leq \kappa d(\alpha, \beta)$$

for each $\alpha \preceq \beta$

b. if $d(\alpha, \beta) < \varepsilon < 1$ for some $\beta \in F(\alpha)$ therefore we have $\alpha \preceq \beta$,

c. there exist $\alpha_0 \in \Omega$ and $\alpha_1 \in \text{script } F(\alpha_0)$ with $\alpha_0 \preceq \alpha_1$ such that $d(\alpha_0, \alpha_1) < 1$,

then F has a fixed point.

The study about the theorem of fixed points for setvalued mappings on metric spaces with partial order is still developing. Sedhgi et al [6] and Gregorio [7] conducted some of the studies. Using the ordering relation defined by Altun and Imdad [8], Sedhgi proved the theorem of set-valued mappings' fixed point on partial metric spaces with partial order.

Theorem 4. Let (Ω, p) be a complete partial metric. If $\alpha \preceq \beta$ then $p(\alpha, \alpha) \leq p(\beta, \beta)$, for all $\alpha, \beta \in \Omega$. A function $\phi : \Omega \rightarrow \mathbb{R}$ is below bounded and \leq the ordered introduced by ϕ . Suppose that $F : \Omega \rightarrow 2^\Omega$ be a setvalued mapping, $[\alpha, \infty)$ denote the elements of Ω , namely β such that $\alpha \preceq \beta$, and \mathcal{M} denote the elements of $\alpha \in \Omega$ such that $F(\alpha) \cap [\alpha, \infty)$ is nonempty set. Suppose that

a. Set-valued mapping F is upper semi continuous, that is, for $\alpha_n \in \Omega$ and $\beta_n \in F(\alpha_n)$ with $\alpha_n \rightarrow \alpha_0$ and $\beta_n \rightarrow \beta_0$, implies that $\beta_0 \in F(\alpha_0)$,

b. $M \neq \emptyset$,

c. for each $\alpha \in M$, we have $F(\alpha) \cap M \cap [\alpha, \infty) \neq \emptyset$,

then F has a fixed point α^* . It means there exists sequence (α_n) with $\alpha_{n-1} \preceq \alpha_n$ where $\alpha_n \in \mathcal{F}(\alpha_n)$, for each n , such that $\alpha_n \rightarrow \alpha^*$. If mapping ϕ is lower semicontinuous, therefore $\alpha_n \preceq \alpha^*$ for all n

On the other hand, Gregorio used a relation to give the theorem of fixed point for set-valued mapping in partial metric space with partial order.

Definition 1. Let (Ω, \preceq) be a partially ordered set, and (Ω, p) is complete partial metric space.

Let $F : \Omega \rightarrow CB^p(\Omega)$ be set-valued mapping. Let we consider the definition of relation as follows:

a. $S <^{(I)} J$ if for each $\sigma \in S$, there exists $\tau \in J$ such that $\sigma \preceq \tau$.

b. $S <^{(II)} J$ if for each $\sigma \in S$, there exists $\tau \in J$ such as that $\sigma \preceq \tau$ and $p(\sigma, \tau) \leq H^p(S, \mathcal{T})$.

Moreover, suppose that $\alpha, \beta \in \Omega$ such that $\alpha \preceq \beta$ then F is called:

a. monotone non-decreasing in type (I) when

$$F(\alpha) <^{(I)} F(\beta)$$

b. monotone non-decreasing in type (II) when

$$F(\alpha) <^{(II)} F(\beta)$$

Gregorio's fixed point theorem as follow:

Theorem 5. [7] *Let (Ω, \preceq) be a partially ordered set and (Ω, p) is a complete partial metric space. Suppose that mapping $\mathcal{F} : \Omega \rightarrow CB^p(\Omega)$ be a set-valued mapping with four conditions:*

- a. *there exist $\alpha_0 \in \Omega$ such that $\alpha_0 \in F(\alpha_0)$*
 - b. *set-valued mapping F is monotone non-decreasing sequence of type (II)*
 - c. *if $\alpha_n \rightarrow \alpha$ is a non-decreasing sequence in Ω thus $\alpha_n \preceq \alpha$ for each n*
 - d. *for each $\alpha, \beta \in \Omega$ with $\alpha \preceq \beta$, $H^p(F(\alpha), F(\beta)) \leq \phi(p(\alpha, \beta))p(\alpha, \beta)$, where ϕ is an MT-function,*
- then F has a fixed point.*

Based on this work, we will combine the techniques that are employed by Aydi et al. and Gregorio in improving Sedghi et al.'s fixed point theorem.

2 Preliminaries

For a metric space (Ω, d) , and suppose that

$$CB(\Omega) = \{A \subseteq \Omega : \emptyset \neq A, A \text{ closed and bounded}\}$$

A point $\alpha \in \Omega$ is fixed point of a set-valued mapping $F : \Omega \rightarrow CB(\Omega)$ if $\alpha \in F(\alpha)$.

Furthermore, the Pompeiu-Hausdorff metric is mapping $H : CB(\Omega) \times CB(\Omega) \rightarrow [0, \infty)$ defined by

$$H(A, B) := \max\{d(A, \mathcal{B}), d(B, A)\} \quad (1)$$

with the definition $d(A, B) = \sup\{d(\alpha, B) : \alpha \in A\}$ and $d(\alpha, B) = \inf\{d(\alpha, \beta) : \beta \in B\}$. Furthermore, a pair $(CB(\Omega), H)$ is called by Pompeiu-Hausdorff metric spaces.

Matthews [9], in 1992, introduced a new concept of metric space, we can call it as partial metric space, to generalize the metric space concept. One of the characteristics of this partial metric space is the nonzero self-distance.

Definition 2. *Let Ω be nonempty set. Partial metric on Ω is a mapping $p : \Omega \times \Omega \rightarrow [0, \infty)$ which satisfies following conditions:*

$$(P1) \quad p(\alpha, \beta) = p(\beta, \alpha)$$

$$(P2) \quad \text{If } p(\alpha, \alpha) = p(\alpha, \beta) = p(\beta, \beta) \text{ then } \alpha = \beta,$$

$$(P3) \quad p(\alpha, \alpha) \leq p(\alpha, \beta)$$

$$(P4) \quad p(\alpha, \gamma) + p(\beta, \beta) \leq p(\alpha, \beta) + p(\beta, \gamma),$$

for all $\alpha, \beta, \gamma \in \Omega$. Pair (Ω, p) is called a partial metric space.

Every partial metric p that is defined on Ω generates a T_0 topology τ_p on Ω which has a base in the family of open p -ball

$$\{B_p(\alpha, \varepsilon) : \alpha \in \Omega, \varepsilon > 0\}$$

where

$$B_p(\alpha, \varepsilon) = \{\beta \in \Omega : p(\alpha, \beta) < p(\alpha, \alpha) + \varepsilon\}$$

for every $\alpha \in \Omega$ and $\varepsilon > 0$ [3, 6, 9–12]. A basic illustration of partial metric space is the pair $([0, \infty), p)$, where $p(\alpha, \beta) = \max\{\alpha, \beta\}$ for all $\alpha, \beta \in [0, \infty)$. Some partial metric space's properties in this work are refer to [12–14] and reference therein.

Definition 3. Let (Ω, p) be a partial metric space. A sequence (α_n) is said to be a convergence sequence to a point $\alpha \in \Omega$ iff

$$p(\alpha_n, \alpha) = p(\alpha, \alpha)$$

Definition 4. Let (Ω, p) be a partial metric space. A sequence (α_n) is called Cauchy sequence iff

$$p(\alpha_n, \alpha_m)$$

is finite.

Definition 5. Let (Ω, p) be a partial metric space. If every Cauchy sequence (α_n) converges to point $\alpha \in \Omega$ such that

$$p(\alpha_n, \alpha_m) = p(\alpha, \alpha)$$

then (Ω, p) is known as complete partial metric space.

In 2012, Aydi et al[3] generalize Pompeiu-Hausdorff metric space to partial metric space, so we have p Pompeiu-Hausdorff metric space or partial Pompeiu Hausdorff metric space. Therefore, for a partial metric space (Ω, p) , we have

$CB^p(\Omega) = \{A \subseteq \Omega : A \neq \emptyset, A \text{ closed and bounded}\}$ The p -Pompeiu-Hausdorff metric is mapping

$H^p : CB^p(\Omega) \times CB^p(\Omega) \rightarrow [0, \infty)$ defined

$$H^p(A, B) := \max\{p(A, B), p(\mathcal{B}, A)\} \tag{2}$$

with the definition $p(A, B) = \sup\{p(\alpha, B) : \alpha \in A\}$, and $p(\alpha, B) = \inf\{p(\alpha, \beta) : \beta \in B\}$. Furthermore, the pairs $(CB^p(\Omega), H^p)$ is called p -Pompeiu-Hausdorff (partial Pompeiu-Hausdorff) metric space.

Theorem 6. Suppose that (Ω, p) be partial metric space, we have

- a. $p(A, A) = \sup\{p(\alpha, \alpha) : \alpha \in A\}$,
- b. $p(A, \mathcal{A}) \leq p(A, B)$,
- c. If $p(A, B) = 0$, then $A = B$,
- d. $p(A, B) \leq p(A, C) + p(C, B)$

$$- \inf\{p(\gamma, \gamma) : \gamma \in C\}$$

for each $A, B, C \in CB^p(\Omega)$,

The following properties are refer to [3, 15–17] and reference therein

Theorem 7. Let (Ω, p) be a partial metric space. Suppose that $A \subseteq \Omega, A \neq \emptyset$, then $\alpha \in A$ if and only if

$$p(\alpha, A) = p(\alpha, \alpha)$$

Theorem 8. Let (Ω, p) be a partial metric space, we have

- a. $H^p(A, A) \leq H^p(A, B)$,
 - b. $H^p(A, B) = H^p(B, A)$,
 - c. $H^p(A, B) \leq H^p(A, C) + H^p(C, B)$
 - $\inf\{p(\gamma, \gamma) : \gamma \in C\}$
 - d. If $H^p(A, B) = 0$ then $A = B$,
- for each $A, B, C \in CB^p(\Omega)$.

For our results, we use space which refer to the following definition.

Definition 6. A partial order is a binary relation \preceq on Ω which satisfies three conditions:

- a. $\alpha \preceq \alpha$ (reflexivity),
- b. if $\alpha \preceq \beta$ and $\beta \preceq \alpha$ then $\alpha = \beta$ (antisymmetry),
- c. if $\alpha \preceq \beta$ and $\beta \preceq \gamma$ then $\alpha \preceq \gamma$, (transitivity),

for all $\alpha, \beta, \gamma \in \Omega$. The pairs (Ω, \preceq) is called partially ordered set.

Definition 7. [18, 19] If (Ω, \preceq) is partially ordered set and (Ω, d) is metric space then we have partially ordered metric space (Ω, d, \preceq) .

Definition 8. [18, 19] If (Ω, d) in Definition 7 is complete metric space, then triple (Ω, d, \preceq) in Definition 7 is called complete partially ordered metric space.

3 Main Results

From Definition 7 and Definition 8 we can derive the general definition as an extended result to the partial metric spaces.

Definition 9. If (Ω, \preceq) is partially ordered set and (Ω, p) is partial metric space then we have partially ordered partial metric space (Ω, p, \preceq) .

Definition 10. If (Ω, p) in Definition 9 is complete partial metric space, then triple (Ω, p, \preceq) in Definition 9 is called by complete partially ordered partial metric space.

We begin with Theorem 9 that show the fixed point of set-valued mapping in partially ordered partial metric space.

Theorem 9. Let (Ω, p, \preceq) be a complete partially ordered partial metric space. Suppose that mapping $F : \Omega \rightarrow CB^p(\Omega)$, and $[\alpha, \infty) = \{\beta \in \Omega : \alpha \preceq \beta\}$, also $M = \{\alpha \in \Omega : F(\alpha) \cap [\alpha, \infty)\}$ with $M \neq \emptyset$ such that for each $\alpha \in M, F(\alpha) \cap M \cap [\alpha, \infty) \neq \emptyset$. Suppose that four assumptions are satisfied:

- F is upper semi continuous, i.e. sequence $\alpha_n \in \Omega$ and $\beta_n \in F(\alpha_n)$ with $\alpha_n \rightarrow \alpha_0$ and $\beta_n \rightarrow \beta_0$ implies $\beta_0 \in F(\alpha_0)$,
- there exist $\alpha_0 \in \Omega$ and some $\alpha_1 \in F(\alpha_0)$ such that $p(\alpha_0, \alpha_1) < 1$,
- for $A, B \in CB^p(\Omega)$ with $\alpha \in A$ and $\beta \in B$ satisfy $p(\alpha, \beta) \leq H^p(A, B)$
- there exist $\kappa \in [0, 1)$ such that

$$H^p(F(\alpha), F(\beta)) \leq \kappa p(\alpha, \beta)$$

for all $\alpha \preceq \beta$.

then set-valued mapping F has a fixed point α , i.e. there exist sequence (α_n) with $\alpha_{n-1} \preceq \alpha_n$ whenever.

$$\alpha_n \in \mathcal{F}(\alpha_{n-1}), n = 1, 2, 3, \dots$$

Proof. Since $M \neq \emptyset$, then there exist $\alpha_0 \in M$. This means there exist $\alpha_1 \in F(\alpha_0) \cap M$ with $\alpha_0 \preceq \alpha_1$. Since $\alpha_1 \in F(\alpha_0)$, then by assumption 2 we have

$$p(\alpha_0, \alpha_1) < 1 \tag{3}$$

By assumption 4, there exist $\kappa \in [0, 1)$ such that

$$H^p(F(\alpha_0), F(\alpha_1)) \leq \kappa p(\alpha_0, \alpha_1)$$

for all $\alpha_0 \preceq \alpha_1$. Thus by inequality (3), we obtain

$$H^p(F(\alpha_0), F(\alpha_1)) \leq \kappa \cdot 1 = \kappa$$

Since $\alpha_1 \in F(\alpha_0) \cap M$, this means $\alpha_1 \in M$. This condition guarantee that there exist $\alpha_2 \in F(\alpha_1) \cap M$ with $\alpha_1 \preceq \alpha_2$. Thus $\alpha_2 \in F(\alpha_1)$ and by assumption 3 we have

$$p(\alpha_1, \alpha_2) \leq H^p(F(\alpha_0), F(\alpha_1)) \leq \kappa$$

So, we obtain

$$p(\alpha_1, \alpha_2) < \kappa \tag{4}$$

Furthermore, due to assumption 4, there exist $\kappa \in (0, 1)$ such that

$$H^p(F(\alpha_1), F(\alpha_2)) \leq \kappa p(\alpha_1, \alpha_2)$$

for all $\alpha_1 \preceq \alpha_2$. Then by inequality (4) we have

$$H^p(F(\alpha_1), F(\alpha_2)) \leq \kappa^2$$

Lets consider that $\alpha_2 \in F(\alpha_1) \cap M$ so $\alpha_2 \in M$, then there exist $\alpha_3 \in F(\alpha_2) \cap M$ with $\alpha_2 \preceq \alpha_3$. Therefore $\alpha_3 \in F(\alpha_2)$, and due to assumption 3, we have

$$p(\alpha_2, \alpha_3) \leq H^p(F(\alpha_1), F(\alpha_2)) < \kappa^2$$

By continuing this process, we have $\alpha_n \in F(\alpha_{n-1}) \cap M$ with $\alpha_{n-1} \preceq \alpha_n$. Thus $\alpha_n \in M$ thus there exist

$$\alpha_{n+1} \in F(\alpha_n) \cap M$$

with $\alpha_n \preceq \alpha_{n+1}$. Since $\alpha_{n+1} \in F(\alpha_n)$ then

$$p(\alpha_n, \alpha_{n+1}) \leq H^p(F(\alpha_{n-1}), F(\alpha_n)) < \kappa^n$$

Next, we show that (α_n) is Cauchy sequence in Ω . For $m, n \in \mathbb{N}$ with $m \geq n$, we have

$$\begin{aligned} p(\alpha_n, \alpha_m) &\leq p(\alpha_n, \alpha_{n+1}) + p(\alpha_{n+1}, \alpha_m) \\ &\quad - p(\alpha_{n+1}, \alpha_{n+1}) \\ &\leq p(\alpha_n, \alpha_{n+1}) + p(\alpha_{n+1}, \alpha_m) \\ &\leq \kappa^n + p(\alpha_{n+1}, \alpha_m) \end{aligned}$$

Similarly, for $p(\alpha_{n+1}, \alpha_m)$ we have

$$\begin{aligned} p(\alpha_{n+1}, \alpha_m) &\leq p(\alpha_{n+1}, \alpha_{n+2}) + p(\alpha_{n+2}, \alpha_m) \\ &\quad - p(\alpha_{n+2}, \alpha_{n+2}) \\ &\leq p(\alpha_{n+1}, \alpha_{n+2}) + p(\alpha_{n+2}, \alpha_m) \\ &\leq \kappa^n + p(\alpha_{n+2}, \alpha_m) \end{aligned}$$

Continuing in this way, we obtain

$$\begin{aligned} p(\alpha_n, \alpha_m) &\leq \kappa^n + \kappa^{n+1} + \kappa^{n+2} + \dots + \kappa^{m-1} \\ &= \kappa^n (1 + \kappa + \kappa^2 + \dots + \kappa^{m-n-1}) \\ &= \kappa^n (1 + \kappa + \kappa^2 + \dots + \kappa^{m-n-1}) \\ &= \kappa^n \frac{(1 - \kappa^{m-1})}{1 - \kappa} \\ &\leq \frac{\kappa^n}{1 - \kappa} \end{aligned}$$

Since $\kappa \in [0, 1)$, then $\lim_{n, m \rightarrow \infty} p(\alpha_n, \alpha_m) = 0$. This implies that (α_n) is a Cauchy sequence in Ω . Considering that partial metric space Ω is complete then we know that (α_n) is converges sequences, namely $\alpha_n \rightarrow \alpha$ whenever $n \rightarrow \infty$. So, we obtain

$$p(\alpha_n, \alpha_m) = p(\alpha_n, \alpha) = p(\alpha, \alpha)$$

Since F is semi continuous then $\alpha \in F(\alpha)$. This means α is fixed point of F . Similarly, we can establish the another condition as in the Theorem 10 below.

Theorem 10. Suppose that (Ω, p, \preceq) be a complete partially ordered partial metric space.

Let $F : \Omega \rightarrow CB^p(\Omega)$, $(-\infty, \alpha] = \{\beta \in \Omega : \beta \preceq \alpha\}$ and $M = \{\alpha \in \Omega : F(\alpha) \cap (-\infty, \alpha]\}$ with $M \neq \emptyset$ such that for every $\alpha \in M$, we have $\mathcal{F}(\alpha) \cap M \cap (-\infty, \alpha] \neq \emptyset$.

Suppose that four condition are satisfied:

- F is upper semi continuous, i.e, sequence $\alpha_n \in \Omega$ and $\beta_n \in F(\alpha_n)$ with $\alpha_n \rightarrow \alpha_0$ and $\beta_n \rightarrow \beta_0$ implies $\beta_0 \in F(\alpha_0)$,
- there exist $\alpha_0 \in \Omega$ and some $\alpha_1 \in F(\alpha_0)$ such that $p(\alpha_0, \alpha_1) < 1$,
- for $A, B \in CB^p(\Omega)$ with $\alpha \in A$ and $\beta \in B$ we have $p(\alpha, \beta) \leq H^p(A, B)$,
- there exist $\kappa \in [0, 1)$ such that

$$H^p(F(\alpha), F(\beta)) \leq \kappa p(\alpha, \beta)$$

for all $\beta \preceq \alpha$,

then F has fixed point α and there exist sequence (α_n) with $\alpha_n \preceq \alpha_{n-1}$ with $\alpha_n \in \mathcal{F}(\alpha_{n-1})$, $n = 1, 2, 3, \dots$ such that $\alpha_n \rightarrow \alpha$.

Proof. The proof is following Theorem 9's proof.

If we replace the first conditions in

Theorem 9 by this condition:"if any sequence in Ω , $\alpha_n \rightarrow \alpha$ whose concecutive terms are comparable then $\alpha_n \preceq \alpha$ for all n ", while other conditions are still met, we will have conclusion that F has fixed point. This result is described in the Theorem 11 below.

Theorem 11. Suppose that (Ω, p, \preceq) be a complete partially ordered partial metric space. Suppose that $F : \Omega \rightarrow CB^p(\Omega)$, $[\alpha, \infty) = \{\beta \in \Omega : \alpha \preceq \beta\}$, and $M = \{\alpha \in \Omega : F(\alpha) \cap [\alpha, \infty)\}$ with $M \neq \emptyset$ such that for each $\alpha \in M$, $F(\alpha) \cap M \cap [\alpha, \infty) \neq \emptyset$. Suppose that four conditions are followed:

- if $\alpha_n \rightarrow \alpha$ where $\alpha, \alpha_n \in \Omega$ for all n whose have comparable concecutive terms then $\alpha_n \preceq \alpha$ for all n ,
- there exist $\alpha_0 \in \Omega$ and some $\alpha_1 \in F(\alpha_0)$ such that $p(\alpha_0, \alpha_1) < 1$,
- for $A, B \in CB^p(\Omega)$ with $\alpha \in A$ and $\beta \in B$ we have $p(\alpha, \beta) \leq H^p(A, B)$,
- there exist $\kappa \in [0, 1)$ such that

$$H^p(F(\alpha), F(\beta)) \leq \kappa p(\alpha, \beta)$$

for all $\alpha \preceq \beta$,

then α is fixed point of \mathcal{F} .

Proof. On the similar line with the proof of Theorem 9, we only have to check that $\alpha \in F(\alpha)$. Let $\varepsilon > 0$. Since $\alpha_n \rightarrow \alpha$ then there is $N \in \mathbb{N}$ such that for $n \geq N$

$$p(\alpha_n, \alpha) - p(\alpha, \alpha) < \frac{\varepsilon}{\kappa} \quad (5)$$

where $\kappa \in [0, 1)$. Similarly,

$$p(\alpha_n, \alpha) \rightarrow 0 \quad (6)$$

where $n \rightarrow \infty$. It means $p(\alpha, \alpha) = 0$. Therefore, for κ above, by condition 4 we have and

$$\begin{aligned} H^p(F(\alpha_n), F(\alpha)) &\leq \kappa p(\alpha_n, \alpha) \\ H^p(F(\alpha), F(\alpha)) &\leq \kappa p(\alpha, \alpha) \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} H^p(F(\alpha_n), F(\alpha)) - H^p(F(\alpha), F(\alpha)) \\ \leq \kappa(p(\alpha_n, \alpha) - p(\alpha, \alpha)) \end{aligned}$$

By (5) we have

$$H^p(F(\alpha_n), F(\alpha)) - H^p(F(\alpha), F(\alpha)) < \varepsilon$$

This conditions means $F(\alpha_n) \rightarrow F(\alpha)$.

Since $\alpha_{n+1} \in \mathcal{F}(\alpha_n)$ then

$$\begin{aligned} p(\alpha_{n+1}, F(\alpha)) &\leq p(F(\alpha_n), F(\alpha)) \\ &\leq H^p(F(\alpha_n), F(\alpha)) \end{aligned}$$

Since $F(\alpha_n) \rightarrow F(\alpha)$ then $H^p(F(\alpha_n), F(\alpha)) \rightarrow 0$, it implies

$$p(\alpha_{n+1}, \mathcal{F}(\alpha)) \rightarrow 0 \tag{7}$$

for $n \rightarrow \infty$. Furthermore, by using the properties of partial metric we have

$$\begin{aligned} p(\alpha, F(\alpha)) &\leq p(\alpha, \alpha_{n+1}) + p(\alpha_{n+1}, F(\alpha)) \\ &\quad - p(\alpha_{n+1}, \alpha_{n+1}) \\ &\leq p(\alpha, \alpha_{n+1}) + p(\alpha_{n+1}, F(\alpha)) \end{aligned}$$

Then by (6) and (7) we obtain $p(\alpha, F(\alpha)) = 0$. Therefore we have $p(\alpha, F(\alpha)) = p(\alpha, \alpha)$, so it means $\alpha \in \overline{F(\alpha)}$.

Since $F(\alpha) \in CB^p(\alpha)$ then $\alpha \in \overline{F(\alpha)} = F(\alpha)$. So, we conclude that α is fixed point of F . In the same way, we show that the result in the Theorem 12 is proven. Theorem 12 below can be proven in the same way as Theorem 11

Theorem 12. *Let (Ω, p, \preceq) be a complete partially ordered partial metric space.*

Let $F : \Omega \rightarrow CB^p(\Omega)$, $(-\infty, \alpha] = \{\beta \in \Omega : \beta \preceq \alpha\}$ and $M = \{\alpha \in \Omega : F(\alpha) \cap (-\infty, \alpha]\}$ with $M \neq \emptyset$ such that for every $\alpha \in M, F(\alpha) \cap M \cap (-\infty, \alpha] \neq \emptyset$. Suppose that four condition are satisfied:

- a. if $\alpha_n \rightarrow \alpha$ where $\alpha, \alpha_n \in \Omega$ for all n whose have comparable consecutive terms then $\alpha \preceq \alpha_n$ for all n ,
- b. there exist $\alpha_0 \in \Omega$ and some $\alpha_1 \in F(\alpha_0)$ such that $p(\alpha_0, \alpha_1) < 1$,
- c. for $A, B \in CB^p(\Omega)$ with $\alpha \in A$ and $\beta \in B$ we have $p(\alpha, \beta) \leq H^p(A, B)$,
- d. there exist $\kappa \in [0, 1)$ such that

$$H^p(F(\alpha), F(\beta)) \leq \kappa p(\alpha, \beta)$$

for all $\beta \preceq \alpha$,

then α is fixed point of \mathcal{F} .

Proof.The proof of this theorems follows on the similar lines as Theorem 11

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