Rainbow Antimagic Coloring on Amalgamation of Wheel Graph (W_3)

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Abstract. Suppose G a simple, and finite graph. A labelling of a graph G is a bijection f from $V(G)$ to the set $\{1,2,...,|V(G)|\}$. Rainbow antimagic coloring is a bijection f if for any two edge u and v in path $u-v$, where $w(uv) = f(u) + f(v)$ and $u,v \in V(G)$ and if in each pair of vertices u and v there are no two vertices that have the same weight in one path. A graph is said to be a rainbow antimagic connection if G has a rainbow antimagic labeling. Thus any rainbow antimagic labeling induces a rainbow coloring of G where the edge uv is assigned with the color $w(uv)$. The rainbow antimagic connection number of G, denoted by $rac(G)$, is the smallest number of colors taken over all rainbow colourings induced by rainbow antimagic labelling of G. We have found the bound of the rainbow antimagic connection graph with amalgamation operation of wheel graph (W_3), $3t \leq rac(amal(W_3,v,t)) \leq 3t+2$.

Keywords: rainbow antimagic coloring, amalgamation, wheel graph

1 Introduction

One discipline within mathematics, graph theory, encompasses a multitude of intriguing topics that can continue to be researched and advanced, with some of these being those associated with graph labeling and coloring. A graph is defined as a pair of integer sets (V,E) or a set of joined objects, which are represented by vertices and edges, and are denoted by $G = (V,E)$ [1]. The term "graph labeling" is defined as a function f that maps the elements of a graph (vertices and edges) to a set of positive integers. If the domain of f is the set of vertices (or edges), then f is designated as a vertex (or edge) labeling. If the domain is $V(G) \bigcup E(G)$, then f is designated as a total labeling [2].

In parallel with the growth of mathematical knowledge, the field of labeling has also undergone periodic developments. The topic of labeling was initially introduced by Sadlack (1964), namely the magic labeling, and subsequently developed by Rosa (1970) into the total edge magic labeling. In 1990, the anti-magic labeling was first introduced by Hartsfield and Ringel [3]. An anti-magic labeling is defined as a bijective function f such that the vertex weights of G do not have the same value. The vertex weight of a vertex x, denoted by $w(x)$, is the sum of the labels of each edge associated with x. In other words, $w(x) = \Sigma_e \in E(x) f(E)$ [4].

Another topic of interest in graph theory is coloring. An edge coloring of a graph G is said to be edge-true-coloring if every pair of distinct edges incident on a vertex

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in G receive unequal colors [5]. Moreover, [6] introduced a novel coloring concept that integrates the principles of connected graphs with edge labeling, designated as "rainbow coloring." In graph theory, edge labeling is defined as a mapping from the set of graph edges to the set of numbers $\{1,2,...,k\}$. In rainbow coloring, there is a "rainbow path", which is a path with all its edge labels are not the same. Edge labeling is called rainbow coloring if every pair of vertices can be connected by a "rainbow path" [7]. The rainbow connection number (rc) of a connected graph G is defined as the minimum number of colors required to render the graph G rainbow connected [8].

In 2019, for the first time, the concept of rainbow coloring with antimagic labeling was combined into one new concept, namely rainbow antimagic coloring which is defined when the edge weight$w(e) = g(u) + g(v)$ induces coloring on the edges and there is always a rainbow path between every pair of two vertices [9]. The rainbow antimagic connection number, denoted by $rac(G)$, is defined as the minimum number of distinct colors present in a graph [10].

The scope of discussion is expanding, as is the range of graphs studied in graph theory. The graphs under consideration encompass both simple graphs that already exist and new graphs that result from the operation of existing graphs. There are numerous operations that can be applied to graphs. One such operation, which is employed in this paper, is the vertex amalgamation operation of wheel graphs. The amalgamation operation is a method of generating a new graph from multiple copies of a given graph. In each copy, a vertex is selected and attached to a fixed vertex [11].

The theorems that support the completion of this research include :

Theorem 1. *Suppose G is a nontrivial connected graph of size m, then* $rc(G) = m$ *if and only if G is a tree [6].*

Theorem 2. *Suppose G is an arbitrary connected graph. Suppose* $rc(G)$ *and* $\Delta(G)$ *have rainbow connectedness of G and maximum degree of G and maximum degree of G respectively, such that* $rac(G) \geq max\{rc(G), \Delta(G)\}$ *[12].*

2 Method

The methodology employed in this study is library research, which entails an examination of pertinent journals and scientific articles on rainbow antimagic connection numbers. The objective is to identify the methodologies employed in addressing related issues. The subsequent sections delineate the steps involved in this research :

i. Draw the amalgamation of wheel graph (w_3)
ii. Determine the pattern of rainbow connection number and rainbow antimagic connection number of the graph in step one to obtain a new theorem
iii. Prove the theorems of rainbow connection number and rainbow antimagic connection number based on the graph drawing generated in step one.
iv. Formulate conclusions.

3 Main Results

3.1 Rainbow Connection Number

Theorem 3. *If $G \cong Amal(W_3, v, t)$, t is an integer with $t \geq 2$, then*

$$diam(Amal((W_3, v, t)) = 2 \tag{1}$$

$$rc(Amal(W_3, v, t)) = t \tag{2}$$

Proof. From the theorem of Chartrand et al (2008) it is known that $rc(G) \geq diam(G)$. If $rc(G) = diam(G)$, it suffices to show that there exists a rainbow path with coloring $c : E(G) \longrightarrow \{1, 2, \ldots, rc(G)\}$. On the other hand, if $rc(G) > diam(G)$ must be proved by contradiction. Therefore, the proof of Theorem 3 is divided into two proof cases. As for the case of $t \geq 3$ is proved by contradiction, and for the case of $t = 2$, it can be proved directly by showing the path with coloring $c : E(G) \longrightarrow \{1, 2, \ldots, diam(G)\}$.

Construct the rainbow coloring function c for graph $Amal(W_3, v, t)$ as follows :

$$c(ab_i) = c(ax_i) = c(ay_i) = c(b_i x_i) = c(b_i y_i) = c(x_i y_i) = i$$

Case 1. $t = 2$, $diam(G) = 2$

Since $rc(G) = diam(G)$, Theorem 3 is simply proved by showing that there is a rainbow path with coloring $c : E(G) \longrightarrow \{1, 2, \ldots, diam(G)\}$. The illustration of rainbow coloring on the graph resulting from the wheel graph amalgamation operation ($Amal(W_3, v, 2)$) can be seen in Figure 1.

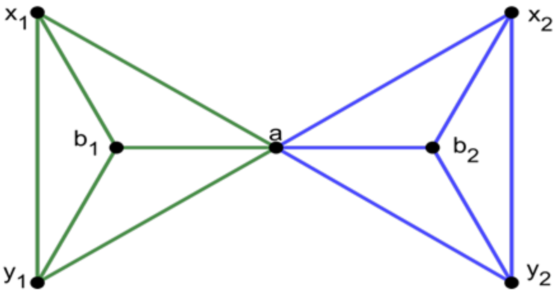

Fig. 1. Rainbow Coloring of $Amal(W_3, v, 2)$

Case 2. $t \geq 3$, $diam(G) = 2$

Graph $Amal(W_3, v, t)$ is a graph that has one main hook vertex which is the connecting vertex between one duplicate graph and another duplicate. Based on the results of the rainbow coloring function construction c shows that the graph $Amal(W_3, v, t)$ has a different color for each duplicate. Furthermore, the results of the construction of the rainbow coloring function c show that edge in the graph $Amal(W_3, v, t)$ has a different color for every I-th side because in Figure 1 there is always a path that is not rainbow when colored as much as its diameter, so to make the graph $Amal(W_3, v, t)$ connected by rainbow path with the minimum color, all duplicates of the graph must be given a different color as much as t.

If we assume that $rc(G) = diam(G)$, then there exists c which is a rainbow coloring $c : (G) \longrightarrow \{1, 2, ..., diam(G)\}$. Note that if $ax_i = i$ is colored $1, 2, ..., diam(G)$, then there will be an edge $ax_i = ax_t$ which causes $ax_i - ax_t$ not to be rainbow. Since the graph $Amal(W_3, v, t))$ is not colored with $diam(G)$ color, the supposition is incorrect and it is proven that $rc(G) = t$.

An illustration of the rainbow coloring of the graph resulting from the amalgamation operation of the wheel graph $(Amal(W_3, v, t))$ can be seen in Figure 2.

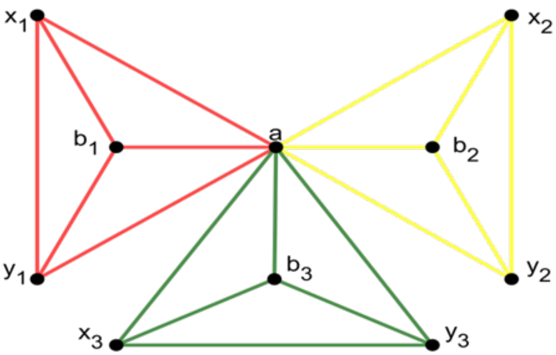

Fig. 2. Rainbow Coloring of $amal(W_3, v, 3)$

It will be shown that for any two vertices x and y on the graph $Amal(W_3,v,t)$ there is always a rainbow path with the conditions shown in the following table :

Table 1. Rainbow Path of $amal(W_3,v,t)$

No	u	v	Condition	Path
1.	b_i	b_j	$i=[1,t-1], j=[i+1,t]$	b_iab_j
2.	x_i	x_j	$i=[1,t-1], j=[i+1,t]$	x_iax_j
3.	y_i	y_j	$i=[1,t-1], j=[i+1,t]$	y_iay_j
4.	b_i	x_j	$i=[1,t-1], j=[i+1,t]$	b_iax_j
5.	b_i	y_j	$i=[1,t-1], j=[i+1,t]$	b_iay_j
6.	x_i	y_j	$i=[1,t-1], j=[i+1,t]$	x_iay_j
7.	x_i	b_j	$i=1, j=[2,t]$	x_iab_j
8.	y_i	b_j	$i=1, j=[2,t]$	y_iab_j
9.	y_i	x_j	$i=1, j=[2,t]$	y_iax_j

3.2 Rainbow Antimagic Connection Number

Theorem 4. *For every integer $n=3$, rainbow antimagic connection number on graph $amal(W_3,v,t)$ is found to be*

$$3t \leq rac(amal(W_3,v,t)) \leq 3t+2 \tag{3}$$

Proof. Amalgamation of Wheel Graph ($amal(W_3,v,t)$) is a graph with vertex set $V(amal(W_3,v,t)) = \{a\}\bigcup\{b_i|i\in\{1,2,\ldots,t\}\}\bigcup\{x_i|i\in 1,2,\ldots,t\}\bigcup\{y_i|i\in\{1,2,\ldots,t\}\}$ and the edge set $E(amal(W_3,v,t)) = \{ab_i i\in\{1,2,\ldots,t\}\}\bigcup\{ax_i i\in\{1,2,3,\ldots,t\}\}\bigcup\{a_iy_i|i\in\{1,2,3,\ldots,t\}\}\bigcup\{b_ix_i i\in\{1,2,3,\ldots,t\}\}\bigcup\{b_iy_i i\in\{1,2,3,\ldots,t\}\}\bigcup\{x_iy_i i\in\{1,2,\ldots,t\}\}$. The cardinality of vertex set of amalgamation of wheel graph is $|V(amal(W_3,v,t))| = 3t+1$ and cardinality of edge set of amalgamation of wheel graph ($amal(W_3,v,t)$) are and $|E(amal(W_3,v,t))| = 6t$, respectively.

To prove the rainbow antimagic coloring of the graph $amal(W_3,v,t)$, the lower bound of the graph must be shown and then the upper bound. Based on the theorem of Dafik et al., 2021 that $rac(G) \geq max(\Delta(G), rc(G))$ obtained $rac(amal(W_3,v,t)) \geq max(\Delta(G), rc(G)) = \Delta(G)$ which is $3t$, then the rainbow antimagic connection number of graph $amal(W_3,v,t) \geq 3t$. Furthermore, to determine the upper bound, the following vertex formula is defined by defining the function$f : V(amal(W_3,v,t)) \Longrightarrow 1,2,3,\ldots,3t+1$:

$$\begin{aligned} f(a) &= 3t+1 \\ f(b_i) &= 3t+1-i \\ f(x_i) &= i \\ f(y_i) &= 2t+1-i \end{aligned}$$

The edge weights of the aforementioned vertex labeling are as follows :

$$\begin{aligned} w(ab_i) &= 6t+2-i \\ w(ax_i) &= 3t+1+i \\ w(ay_i) &= 5t+2-i \\ w(b_ix_i) &= 3t+1 \\ w(b_iy_i) &= 5t+2-2i \\ w(x_iy_i) &= 2t+1 \end{aligned}$$

Based on the construction of vertex antimagic labeling and edge weight function above, $3t+2$ colors are obtained in coloring the graph $amal(W_3,v,t)$, therefore from the lower bound and upper bound, rainow antimagic connection number of graph $amal(W_3,v,t))$ is $3t \leq rac(amal(W_3,v,t)) \leq 3t+2$ proved.

The following is an illustration of anti-magic rainbow coloring on the graph $amal(W_3,v,t)$.

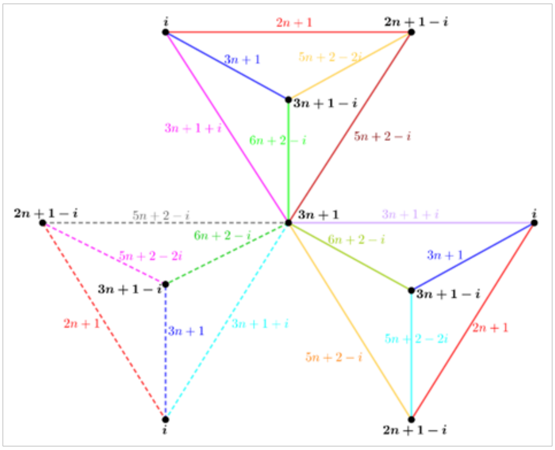

Fig. 3. Rainbow antimagic coloring of graph $amal(W_3,v,t)$

The following table presents the rainbow path of graph $amal(W_3,v,t)$:

4 Conclusion

Based on the description in the discussion, the theorem of rainbow connection number (rc) and rainbow antimagic connection number (rac) of amalgamation of wheel graph is obtained:

1. Theorem of rainbow connection number of $Amal(W_3,v,t)$: If $G \cong Amal(W_3,v,t)$, t is an integer with $t \geq 2$,then

$$diam(Amal(W_3,v,t) = 2$$
$$rc(Amal(W_3v,t)) = t$$

Table 2. Rainbow Antimagic Path of $amal(W_3, v, t)$

No	u	v	Condition	Path
1.	b_i	b_j	$i, j = [1, t]$, dan $i \neq j$	b_i, a, b_j
2.	b_i	x_j	$i, j = [1, t]$, dan $i \neq j$	b_i, a, x_j
3.	b_i	y_j	$i, j = [1, t]$, dan $i \neq j$	b_i, a, y_j
4.	x_i	x_j	$i, j = [1, t]$, dan $i \neq j$	x_i, a, x_j
5.	x_i	y_j	$i, j = [1, t]$, dan $i \neq j$	x_i, a, y_j
6.	x_i	y_j	$i, j = [1, t]$, dan $i \neq j$	y_i, a, y_j

2. Theorem of rainbow antimagic connection number of $Amal(W_3, v, t)$: For every integer $n = 3$, rainbow antimagic connection number on graph $amal(W_3, v, t)$ is found to be

$$3t \leq rac(amal(W_3, v, t)) \leq 3t + 2$$

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