Applications of Partial Differential Equations

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Abstract. Partial differential equations (PDEs) are descriptions of continuous changes. Using methods/algorithms, PDE-based models are applied in physical calculations, image processing, machine learning, etc. This paper discusses the applications of PDEs, especially in image denoising and inpainting. When denoising, the PDE diffusion model relies on local smoothing to balance noise suppression and edge protection. When inpainting, the PDE inpainting model relies on propagating structural information to make the visual reconstruction of the defective area more consistent. In computational physics (comp scenarios), the partial differential equation (PDE) model makes the features of the deep model more stable and easier to understand. However, there are challenges in the current application of PDE, such as high computational cost, insufficient numerical stability, difficult parameter selection, and limited model generation ability. Future research on partial differential equations (PDEs) will focus more on integrating data models, efficient algorithms, and interdisciplinary applications. This paper aims to demonstrate the diversity and importance of partial differential equations (PDEs) and provide references for subsequent research on partial differential equations (PDEs).

Keywords: Partial Differential Equation, Image Denoising, Image Restoration, Numerical Analysis.

1 Introduction

Partial differential equations (PDEs) play a very crucial role in modern mathematics and are also widely used in fields such as natural science and technology engineering. From computer image processing, machine learning to physical equations, PDEs act as a bridge connecting mathematical theory and real life through appropriate applications. It can describe the changing relationships between multivariable functions, and show accurate and unique expression abilities when people use it on explaining natural phenomena and rules.

After the 21st century, with the development of computing power and numerical algorithms, the application fields of partial differential equations have continued to develop. In many fields such as image analysis, signal processing, biological modeling, financial engineering, and deep learning, PDEs play an irreplaceable role. For example, in the field of image processing, methods based on PDE can remove

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noise effectively, preserve edge features, and achieve image restoration and reconstruction.

However, although PDE has achieved plenty of achievements in applications, current research still faces several challenges: how to design more efficient numerical solution methods, how to maintain stability of computing in high-dimensional data environments... Towards those difficulties, the academic community continues to propose and improve mathematical tools and models to create foundations for the development of PDE in the future.

The goal of this paper is to demonstrate some typical applications of PDEs systematically, especially on areas such as image denoising and restoration. By summarizing existing literature and algorithmic models, to analyzing their basic ideas, research progress, and current limitations. At the same time, by showing the universality and development potential of PDEs, hopes this papaer can provide a reference for understanding the importance of PDEs in fields like engineering and scientific research, and improve the cooperation of mathematical theory and modern technology

2 Basic Information

Ordinary Differential Equation (ODE) describes the relationship between an independent variable and its derivative. It describes how a physical quantity changes over time or space, and its a fundamental mathematical tool for studying systems that change continuously. A general first-order ordinary differential equation can be written as:

$$\frac{dy}{dx}=f(x,y) \tag{1}$$

In this equation, y=y(x) is the unknown function and f(x,y) describes the changing rate of y with the independent variable x. Higher-order differential equations will contain higher-order derivatives.

ODEs can be divided into two kinds: linear equations and nonlinear equations. In linear equations, the unknowns and their derivatives appear in linear form and can usually be solve by analysis. Nonlinear equations are relatively complex, and it is often necessary to apply methods of numerical analysis or approximate analysis.

In the fields of physics and engineering, ordinary differential equations (ODEs) are often used. For example, particle motion equations, radioactive decay models, and population growth patterns are all used to describe time-varying systems. For example:

$$\frac{dy}{dt}=ky \tag{2}$$

Equations are used to describe the situation of exponential growth or decay and are applied in the models of physics, chemistry and biology.

Solving ordinary differential equations (ODEs) often requires finding a function y(x) that satisfies the equation and given initial or boundary conditions. These conditions define physical scenarios, such as the initial position or velocity of an object. With the development of computer technology, numerical algorithms like the Euler method, Runge - Kutta method, and finite difference method are more important when solving complex ordinary differential equations.

Partial differential equations (PDE) is a generalized ordinary differential equation, which often used to study the relationship between functions with multiple independent variables and their partial derivatives. A general form of PDE can be written as:

$$F\left(x_1,x_2,\dots,x_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial^2 u}{\partial x_i \partial x_j},\dots\right)=0 \tag{3}$$

Where $u(x_1,x_2,\dots,x_n)$ is unknown functions.

PDEs can be divided into first-order and higher-order equations based on the highest order of their derivatives. In practical applications, second-order partial differential equations are the most common, for example:

Heat Equation, used to describe heat diffusion;

$$\frac{\partial u}{\partial t}=\alpha \nabla^2 u \tag{4}$$

Wave Equation, used to describe the propagation of sound waves or vibrations;

$$\frac{\partial^2 u}{\partial t^2}=c^2 \nabla^2 u \tag{5}$$

Laplace Equation, used in electrostatic fields and fluid mechanics.

$$\nabla^2 u=0 \tag{6}$$

In those equations, u represents an unknown function, typically a physical quantity such as temperature, displacement, or electric potential, t represents the time variable, α represents thermal diffusivity, c represents the wave propagation speed, and ∇^2 represents the Laplacian operator, used to describe the rate of change in a second-order space.

3 Applications of Partial Differential Equations

3.1 Image Denoising

Image Denoising -- One of the fundamental problems in the field of digital image processing. Its function and goal is to remove noise as much as possible while preserving edges, textures and other structural features of the image as much as possible. Traditional mean filtering and Gaussian filtering methods can reduce random noise to a certain extent, but their global or fixed coefficient smoothing mechanism often leads to blurred image details and lost boundaries, thus damaging visual quality.

To solve this problem, a denoising method based on partial differential equations (PDE) was introduced and widely used. Its core function is to use the diffusion equation to simulate the evolution process of the image grayscale field in space, and by adjusting the diffusion coefficient, effectively distinguish between noise smoothing and edge preservation.

One of the most representative denoising models that based on PDE is the “Anisotropic Diffusion” model proposed by Pietro Perona and Jitendra Malik in 1990. and its mathematical form can be expressed as:

$$\frac{\partial I}{\partial t} = \nabla \cdot \left(c(x,y,t)\, \nabla I \right) \tag{7}$$

In this equation, I(x,y,t) represents the evolve of image grayscale over time t. The diffusion coefficient c(x,y,t) is usually designed based on the image gradient $\|\nabla \mathrm{I}\|$. Suppress the diffusion at the edges while enhancing the diffusion in the smooth regions. This model is a typical example of balancing noise removal and edge preservation [1]. Another classic model is the "total variation (TV)" denoising method proposed by Leonard Rudin, Stanley Osher, and Emad Fatemi. It is necessary to suppress noise and preserve discontinuous edges by minimizing the integral of the image gradient magnitude (total variation). This model is recognized due to its mathematical analysis and wide application effects [2].

Recently, there are two directions in the research of image denoising based on partial differential equations (PDE): expanding classic models (such as Perona-Malik model) in terms of numerical solution and stability. For example, Wielgus' research points out that the existence, uniqueness and stability of Perona-Malik equation are still a challenge [3]. Second, is to generalizing the model to more complex forms, such as introducing nonlocal terms, curvature-driven diffusion, and fractional diffusion, further enhances detail preservation and edge protection. Recent research also shows that the ability to preserve structural features is increasing with the effect of non-local and fractional PDE [4].

Not only the theory, but also in applications, image denoising methods based on PDEs are constantly developing. For example, Yuan Ping (2025) introduced an improved diffusion model in the modeling of visual spot image denoising, by adjusting the diffusion control function it achieved a more precise balance between noise suppression and structure preservation [5]. In addition, Wu Shufan and Wu Qingping (2022) systematically summarized the application of PDE in medical image and remote sensing image denoising, and pointed out that the diffusion model based on PDE is more suitable for real-world scenarios that require diffusion with edge protection [6].

In general, the PDE method has become an important part of the image denoising field due to its advantages in mathematical interpretability (such as model derivation, energy functional, gradient flow interpretation) and physical intuitiveness (such as diffusion process, smoothing-preserving mechanism).

3.2 Image Restoration

Image restoration (or inpainting) aims to fill damaged or missing areas in an image so that the restored image is visually consistent with the surrounding structure. This problem is particularly important in applications such as restoring old photos, repairing images of cultural relics, and completing missing video frames. The key challenge is in introducing appropriate structural continuity in the missing areas while avoiding the "blurring" or "distortion" effects caused by simple smoothing.

In this field, the role of the PDE method is to extend the surrounding known information into the unknown area through diffusion or propagation mechanisms by treating the defect area as a continuous field evolution problem with known boundaries and an interior to be filled. Typical methods include: the smooth continuation model based on the Laplace equation $(\nabla^2 I=0)$, the geometric repair model based on curvature-driven diffusion, and the flow field-type PDE repair model proposed by Marcelo Bertalmío et al. based on the Navier–Stokes equations (fluid dynamics). For example, Bertalmío et al. proposed in 2001 to regard the image grayscale field as a "stream function" of a two-dimensional incompressible flow, and to continuously push the contour line from the boundary to the interior of the defect through the Navier–Stokes equation [7]. The core effect of this method is to use the pixel gradient field and boundary conditions to "continue isophotes" in the defect area and match the boundary gradient, so that the repaired area and the surrounding structure are more unified in visual and geometric directions.

Research on image restoration based on PDEs has been developed well. Bertalmío et al.'s early work is considered classic; they established a mathematical link between image inpainting and fluid simulation. After that, researchers have combined PDE models with level set methods, total variation methods, and even deep learning frameworks to enhance its restoration accuracy and adapt it to more complex defect scenarios. A recent review points out that traditional PDE models still have strong structural expression capabilities, but their numerical implementation often needs omplex boundary condition design and stable algorithm support [7]. Even so, PDE-based image restoration technology still has advantages in both theoretical depth and application breadth, especially in scenarios that have strong demands for structural continuity and high-quality restoration.

3.3 In Other Fields

In recent years, the concept of partial differential equations (PDEs) commonly used in machine learning is often used to control the propagation of model features and keep training stable. The deep network layer structure similar to ResNet can be represented in the form of discretized ordinary differential equations/partial differential equations, so that network training can be regarded as solving a time-evolution equation. Then training the deep model becomes solving the time-evolution process, which provides a clearer theoretical framework for explaining network behavior and also enhances stability. This form alleviates problems such as gradient vanishing/exploding, and improves the interpretability and generation ability of the model [7].

4 Limitations and Future Perspectives

In general, there have been relatively great progress in fields such as physical modeling, image processing, and machine learning. However, the practical applications and theoretical developments of partial differential equations are still often limited. The problem lies in situations such as relatively high computational costs, insufficient numerical stability, excessive dependence on model parameters, and difficulties in combining with data-driven methods.

4.1 Current Limitations

High computational complexity can solve most partial differential equations, especially nonlinear and high-dimensional partial differential equations, but the cost is relatively high. The finite difference method and the finite element method are widely applied, but often require very small step sizes and fine grids to ensure stable accuracy, resulting in low efficiency, making it difficult for partial differential equation models to be applied to real-time tasks, such as image and video processing [8, 9].

There are problems of stability and error in the discretization and iterative solution of PDE models. The diffusion equation is relatively sensitive to the time step and boundary conditions; a small imprecision may lead to numerical oscillation or false diffusion. In the current research, balancing stability and accuracy is a key challenge. The generalization ability of traditional PDE models is not good, and to a large extent, they rely on parameters. These models are based on specific assumptions and parameters (such as diffusion coefficients or boundary conditions). It is difficult to accurately obtain these parameters in complex systems. This leads to poor adaptability in different fields and makes cross-field generalization difficult [10].

There is a situation of lack of data-driven integration in partial differential equation (PDE) models. Big data and artificial intelligence are developing relatively rapidly. Traditional partial differential equations are difficult to handle high-dimensional, incomplete or noisy data. PDEs have the characteristic of good physical interpretability, but their adaptability is relatively limited. Fusing PDE physical constraints with neural network learning ability is a key direction for future research.

4.2 Future Perspectives

There are certain limitations, but the research prospects of partial differential equations (PDEs) are relatively broad, and some directions are showing new development trends at present.

Combination of Partial Differential Equations and Deep Learning: Researchers integrate the theory of partial differential equations with the hybrid mode of deep networks. Taking the neural network layer structure as the discrete representation of partial differential equations, a "physics-informed" mode, that is, a mode constrained by physics and driven by data, can be constructed, providing new solutions for scientific computing and image analysis.

The key to the development of efficient numerical algorithms in the future lies in creating more efficient and stable numerical solutions. For example, new technologies such as adaptive multi-resolution (AMR), multi-grid method, and GPU parallel computing can greatly improve the calculation speed and accuracy, enabling partial differential equation (PDE) models to achieve real-time or high-resolution applications.

The interdisciplinary applications of PDEs are expanding. In computational biology, it simulates tumor growth and neural signals; in quantum mechanics, it is the basis of the Schrödinger - like equation model; in data science, the PDE - based methods enhance the generation power of machine learning models.

Challenges exist in computation and modeling for partial differential equations, but the advantages that the theory can be solved and the modeling is universal still make it occupy a core position in the future scientific research and engineering fields.

5 Conclusions

This paper Introduces the basic theory of partial differential equations and their applications in various fields. In field of image analysis, models based on PDE show strong structure preservation and control abilities to smoothness on denoising and restoration. Moreover, researches on combining PDEs with deep learning models have large potential value in improving model interpretability and training stability. However, challenges such as high computational complexity, parameter difficult, and strong sensitivity on numbers still need more researches and Investment. Future researches will more focus on developing efficient numerical solution algorithms, combining data-driven mechanisms with PDE models, and deeper explorations in other fields. In all, as an important bridge that connecting mathematical theory and real applications, further developments will show the advantages of PDEs in more fields

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