Properties of Some Δ_0 Definable Sets on Models of KPU

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Abstract. It is a standard result in recursion theory that sets of natural numbers recognized by Turing Machines can be mapped bijectively to Δ_1 sets in HF (i.e. the hereditarily finite sets). On the other hand, there are other sets in HF, such as the Δ_0 sets, whose relationship to concrete computational models may be of interest for further research. Unlike Δ_1 sets, Δ_0 sets are defined by bounded quantification relative to certain interpretations of a language. In the standard set-theoretic language L that involves only the set membership symbol (ϵ), some sets can be shown to be not Δ_0 using a model-theoretic proof. In this paper however, we investigate how membership in these sets can be defined by a Δ_0 formula under a new language L^+ that incorporates additional function symbols f, g, h and symbols s, t. The paper studies models of KPU (i.e. the Kripke Platek Axioms with Urelements) that have well-defined interpretations of the symbols f, g, h. We present several corollaries that follow from this result along with some generalizations of Δ_0 sets. Among the results indicate that well-foundedness of elements of sets defined by Δ_0 formulas of L^+ is independent of KPU.

Keywords: logic, computability, recursion theory

1 Introduction

In the standard set theoretic framework (i.e. the language L that involves only the set membership symbol ϵ with the usual interpretation), it has been shown that the set of natural numbers recognized by Turing Machines can be mapped bijectively to the Δ_1 hereditarily finite sets. Standard proofs for this result can be found in [2] [4]. However, the hereditarily finite sets (usually denoted as HF in the literature) is a rich area for investigation, and there are several other types of sets (even simpler ones), such as the Δ_0 sets that may be of interest. Some sets in HF have been shown to be not Δ_0 or even definable under standard set-theory using a model-theoretic argument. For instance, the set of even numbers E is not Δ_0 since it would violate automorphism properties over elementary extensions. In this paper, we investigate how incorporating additional function symbols f, g, h and symbols s, t to L (to come up with another language

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L^+), would render previously non-definable sets (in standard set-theory) as now definable by some Δ_0 formula of L^+. These function symbols are meant to be interpreted as functions with certain specific behaviours, and the models that meet the interpretations of such functions are assumed to be models that satsify the Kripke Platek Axioms with urelements (KPU). Briefly, models of KPU are models that satisfy weaker Axioms than ZFC but which are strong enough to form a general recursion theory as shown in [2], [4]. Some of the literature that work on such models are those of [6], [7], and [8]. Specifically, using models of KPU that satisfy these functions f,g, and h, we prove how E is defined by some Δ_0 formula. Moreover, if such models of KPU likewise satisfy the Axiom of Infinity, then E exists *inside* the model. We also arrive at the result that while Δ_0 formulas of L^+ are rich enough as to define E, they are still not strong enough to discriminate between standard members of E, and non-standard members of E - given non-standard models of KPU (see [3] for a discussion of non-standard models).

The results of this paper have been initially presented as an (unpublished) poster in [5], and this paper provides several additional details along with additional results regarding generalizations of E to E^m, where E^m denotes the set of multiples of m. There are also some additional results on the relationship between Δ_0 and Δ_1 sets.

2 Preliminaries

Following the approach of [2], we fix ZFC as the meta-theory of this paper. Let L be some finite language with equality, and let $L^* = L(\epsilon, \dots)$ be a finite expansion of L by adding the symbol ϵ and possibly other symbols. A structure $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \dots)$ for L^* consists of: (1) A structure $\mathfrak{M} = \langle M, \dots \rangle$ for L, where $M = \emptyset$ is a possiblity, and the elements of M are the *urelements* of $\mathfrak{A}_{\mathfrak{M}}$, (2) A nonempty set A disjoint from M, (3) A relation $E \subseteq (M \cup A) \times A$ which interprets the membership symbol ϵ, and (4) other functions, relations, constants on $M \cup A$ to interpret the $\dots$ in L^*. We adopt a multi-sorted language where the variables $p, q, \dots$ range over M (the urelements), variables $a, b, c, \dots$ range over A (the sets), and variables $x, y, z, \dots$ range over $M \cup A$. It could be shown however that this multi-sorted language has an equivalent single-sorted language, so that standard Lowenheim-Skolem-Tarski Theorems over single-sorted languages apply given the ZFC meta-theory (see [2]).

Throughout this paper, language symbols will be used interchangeably with symbols that represent relations / functions in structures, with the intention that if such symbols are used in structures, they serve as relation / function interpretations of the corresponding language symbols.

Definition 1. Given $L(\epsilon, \dots)$, the collection of Δ_0 formulas is the smallest collection containing the atomic formulas and closed under $\neg, \wedge, \vee$ and bounded quantification $\forall u \in v$ and $\exists u \in v$ for variables u and v. A set S is a Δ_0 set if it is defined by some Δ_0 formula, i.e. for some Δ_0 ϕ, we have $x \in S \leftrightarrow \phi(x)$. In general, given $L(\epsilon, \dots)$, a set A is defined by a formula ϕ of L iff $x \in A \leftrightarrow \phi(x)$.

Definition 2. Given $L(\epsilon, \dots)$, a *Σ-formula* is a formula of L that is built up from the operations involved in Δ_0 formulas plus the operation $(\exists x)$. The negation of a Σ-formula is a *Π-formula*, or equivalently a formula built up from the operations used for building Δ_0-formulas plus the operation $(\forall x)$. A *Σ_1-formula* is a formula of L of the form $(\exists x)\phi(x)$, where ϕ is a Δ_0-formula. The negation of a Σ_1-formula is a *Π_1-formula*, which is of the form $(\forall x)\phi(x)$, where ϕ is Δ_0. A *Δ_1-formula* is a formula of L such that both it and its complement is Σ_1.

Definition 3. The theory of KPU as defined in [2] relative to $L(\epsilon, \dots)$ consists of universal closures of the following:

- **(A1).** *Extensionality*: $\forall x(x \in a \leftrightarrow x \in b) \to a = b$
- **(A2).** *Foundation*: $\exists x\phi(x) \to \exists x[\phi(x) \wedge \forall y \in x \neg\phi(y)]$ for $\phi(x)$ in which y is not free
- **(A3).** *Pair*: $\exists a(x \in a \wedge y \in a)$
- **(A4).** *Union*: $\exists b \forall y \in a \forall x \in y(x \in b)$
- **(A5).** Δ_0 *Separation*: $\exists b \forall x(x \in b \leftrightarrow x \in a \wedge \phi(x))$ for Δ_0 formulas ϕ in which b is not free
- **(A6).** Δ_0 *Collection*: $(\forall x \in a)(\exists y)\phi(x, y) \to (\exists b)(\forall x \in a)(\exists y \in b)[\phi(x, y)]$ for all Δ_0 formulas ϕ in which b does not occur free.

Definition 4. If ϕ is a Σ formula of L, and ψ is a Π formula of L and it can be shown that $KPU \vdash \phi \leftrightarrow \psi$, then the set defined by ϕ (and equivalently ψ) is a Δ set.

Definition 5. [2]: Given an arbitrary collection M of urelements, we define by recursion the proper class:

$V_M(0) = 0$
$V_M(\alpha + 1) =$ Power Set of $(M \cup V_M(\alpha))$
$V_M(\lambda) = \bigcup_{\alpha<\lambda} V_M(\alpha)$ if λ is a limit ordinal
$\mathbb{V}_{\mathfrak{M}} = \bigcup_{\alpha} V_M(\alpha)$

Definition 6. Given a structure $\mathfrak{M}$ for L, an *admissible set over* $\mathfrak{M}$ is a model of KPU of the form $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \epsilon, \dots)$ such that $M \cup A$ is transitive, and where ϵ is the membership relation in $\mathbb{V}_{\mathfrak{M}}$.

Lemma 1. *[2]: In models of KPU, every Σ formula is equivalent to a Σ_1 formula. In particular, if $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \epsilon, \dots)$ is an admissible set for L and if ϕ is some Σ formula of L, then there exists a Σ_1 formula ψ of L such that $\phi(x) \leftrightarrow \psi(x)$ for all $x \in A$.*

The following definition and theorems below would serve as material for the proof of Lemma 4 below.

Definition 7. Let $\mathfrak{M}$ and $\mathfrak{N}$ be structures for a language L with $\mathfrak{M} \subseteq \mathfrak{N}$ (i.e. $\mathfrak{M}$ is a substructure of $\mathfrak{N}$). If ϕ is a formula of L, then $\mathfrak{M} \preccurlyeq_\phi \mathfrak{N}$ means that $\mathfrak{M} \models \phi[\sigma]$ iff $\mathfrak{N} \models \phi[\sigma]$ for all assignments σ for ϕ in A. If this holds for all formulas ϕ of L, then $\mathfrak{M} \preccurlyeq \mathfrak{N}$, or that $\mathfrak{M}$ is an elementary submodel of $\mathfrak{N}$ and $\mathfrak{N}$ is an elementary extension of $\mathfrak{M}$.

Theorem 1. *[4], [1]: Upward Lowenheim-Skolem-Tarski Theorem. Let $\mathfrak{M}$ be any infinite structure for a first order language L. Fix $\kappa \geq \max(|L|, |A|)$. Then there is a structure $\mathfrak{N}$ for L with $\mathfrak{M} \preccurlyeq \mathfrak{N}$ and $|\mathfrak{N}| = \kappa$.*

Lemma 2. *[4]: If $A \subseteq B$ and A is transitive, then $A \preccurlyeq_\phi B$ for all Δ_0 formulas ϕ in the language of set theory.*

3 A Non Δ_0 Definable Set given $L^* = \{\epsilon\}$

Definition 8. [2]: Given a set M of urelements, a pure set in $\mathbb{V}_M$ is a set a with empty support. In particular, let $TC(a)$ denote the transitive closure of a (see [2] or [4] for more details on transitive closure), then $TC(a) \cap M = \emptyset$

Lemma 3. *[2]: The set $\mathbb{HF} := \{a \in \mathbb{V} : a \text{ is a pure hereditarily finite set}\}$ is the smallest admissible set, and is part of any other admissible set.*

Lemma 4. *(ZFC): Let $L^* = \{\epsilon\}$, and let $\mathfrak{A} = (\emptyset; A, \epsilon)$ be admissible. The set $E = \{n : n \text{ is an even natural number}\}$ is not a Δ_0 definable subset of $\mathfrak{A}$.*

Proof. The proof for this result follows an outline provided in [4] and in [9]. By Lemma 2, we need only to show that E is not a Δ_0 subset of ω, given that $\omega \subseteq \mathbb{HF}$ and $\mathbb{HF} \subseteq A$ from Lemma 3. Suppose to the contrary that E is a Δ_0 definable subset of ω. By the Upward Lowenheim-Skolem-Tarski Theorem (Theorem 1), there exists an elementary extension $\mathfrak{M}'$ of the model (ω, ϵ) of cardinality greater than $\aleph_0$. Define the automorphism: $f : A' \to A'$ by $f(a) = a$ if $a \in \omega$ and $f(a) = S(a)$ if $a \notin \omega \wedge a$ is a natural number, where $S(a)$ stands for the successor of a (note that $S(a) = a \cup \{a\}$ is well-defined in (ω, ϵ)). Here, "x is a natural number" is given by the formula $(\text{Ord}(x) \wedge \forall y \in x[\neg\text{Lim}(y)] \wedge \neg\text{Lim}(x))$, such that $(\text{Ord}(x) \leftrightarrow x$ is an ordinal$)$, and $(\text{Lim}(x) \leftrightarrow x$ is a limit ordinal$)$. Suppose that E is defined by some Δ_0 formula ϕ of L^*, then $\mathfrak{A} \models \forall x[\phi(x) \leftrightarrow \neg\phi(S(x))]$. Given that $\mathfrak{A} \preccurlyeq \mathfrak{A}'$, the same sentence should hold in $\mathfrak{A}'$, but this yields a contradiction given the automorphism f defined above and the fact that automorphisms preserve truth of sentences. Hence, E is not definable in the model (ω, ϵ) and is not a Δ_0 definable subset of $\mathfrak{A}$ by Lemma 2, where it has been shown that it is not even definable by any formula in $\mathfrak{A}$

4 A Δ_0 Formula of L^+ for E

Given $L^* = \{\epsilon\}$, we expand this to $L^+ = \{\epsilon, f, g, h, s, t\}$, whose intended interpretations in a structure $\mathfrak{A}_\mathfrak{M} = (\mathfrak{M}; A, \epsilon, f, g, h, s, t)$ are as follows, where the constants s and t are among the urelements of $\mathfrak{A}_{\mathfrak{M}}$ (i.e. $\{s, t\} \subseteq M$), and where we use $\sim$ as a notation for the function: $\sim s = t$, $\sim t = s$, identity otherwise. For f, g, h below, assume that all other elements of A that do not meet the condition would result in $\emptyset$.

☞ The function $f : A \to A$ is such that:

$$f(a) = \text{rank}(a)$$

☞ the function $g : A \times M \to A$ is such that for all $b \in A$ and $p \in M$, then:

$$g(b, p) = (b, \sim p)$$

☞ The function $h : A \to M$ is such that $(\emptyset, s) \in h$ and if $a \neq \emptyset$, we have:

$$h(a) = p \leftrightarrow (a, p) = g(a, h(b)) \wedge (\forall x \in a)(\forall y \in a)[x = b \to f(x) \geq f(y)]$$

As a note on the underlying logic, the functions f, g, h above are well-defined under the ZFC meta-theory and can be seen as sets. Hence, under ZFC, it makes sense to state that a structure $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \epsilon, f, g, h, s, t)$ is an admissible set. However, as shown in Corollary 2 below, such functions are still well-defined using only the KPU Axioms, but f, g, h in this case can be proper classes.

Theorem 2. *Let $L^+ = \{\epsilon, f, g, h, s, t\}$ and let $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \epsilon, f, g, h, s, t)$ be an admissible set with a finite number of urelements that satisfies the interpretations intended for f, g, h, s, t above. Furthermore, suppose that A consists of hereditarily finite sets. There exists a Δ_0 formula ϕ of L^+ such that given $E = \{n : n$ is an even natural number$\}$, we have $x \in E \leftrightarrow \phi(x)$ for all $x \in A$.*

Proof. Let $n \in E \leftrightarrow \phi(n)$, where:

$$\phi(n) \leftrightarrow n \text{ is a natural number} \wedge h(n) = s$$

ϕ is a Δ_0 formula given that (n is a natural number) is Δ_0, and h is a function symbol of L^+. We prove that if n is an even natural number, then $n \in E$ by induction on the rank of sets in A. For $0 = \emptyset$, we have $h(\emptyset) = s$ so that $0 \in E$. Suppose that the induction hypothesis holds for all sets of rank i. Let $a \in A$ be any set of rank $i + 1$. If a is not a natural number, then $\neg\phi(a)$ by definition so that $a \notin E$. If a is odd, then $P(a)$ (i.e. the predecessor of a) is even so that $P(a) \in E$ and $h(P(a)) = s$. Using the b in the definiton of h, we have $P(a) = b \wedge (\forall x \in a)(\forall y \in a)[(x = b = P(a)) \to f(x) \geq f(y)]$ since $P(a)$ is the greatest natural number in a. It follows that $g(a, h(P(a))) = g(a, s) = (a, t)$ so that $(a, t) \in h$. Since $h(a) = t$, we have $\neg\phi(a)$ so that $a \notin E$ and a is an odd natural number. A similar reasoning applies if a is even.

Corollary 1. *(ZFC): Let B be the minimal transitive set which has $\{\omega \times \{s,t\}\} \cup \{s,t\}$ as a subset. Let $\mathfrak{B} = (B, \epsilon, f, g, h, s, t)$ be a structure where ϵ is the usual membership symbol and whose interpretations for symbols f, g, h, s, t satisfy the definitions above. Then E is a Δ_0 definable set of $\mathfrak{B}$.*

Corollary 2. *Let $L^* = \{\epsilon\}$. For any even n, $\big(KPU + \exists$ at least 2 urelements $\vdash$ n is even $\big)$ has a proof in L^* that does not use the $+$ operation.*

Proof. The functions f, g, h in L^+ are equivalent to Σ operations in L^*. These functions are well-defined in KPU models by Σ recursion, as shown in [2].

Corollary 3. *Let $(\mathfrak{A}_{\mathfrak{M}}) = (\mathfrak{M}; A, \epsilon)$ be an admissible set for $L^* = \{\epsilon\}$ with at least two urelements and which satisfies the Axiom of Infinity. Then $E \in A$.*

Proof. Any Σ function is also Π so that f, g, h under L^* are Δ functions (this result uses the definition of functions). KPU models satisfy Σ collection, where for any $a \in A$, if $(\forall x \in a)(\exists y)[\phi(x,y)]$, then $(\exists b)[(\forall x \in a)(\exists y \in b)[\phi(x,y)]]$. Also, KPU models satisfy Δ separation (see [2] for more details): for any Δ set defined by some Σ (or Π) formula ϕ, and $a \in A$, there is a set $b = \{x \in a : \phi(x)\}$. The result follows.

Corollary 4. *Let $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \epsilon, f, g, h, s, t)$ be an admissible set for L^+, and let $\mathfrak{A}_{\mathfrak{N}} = (\mathfrak{N}; A', \epsilon)$ be an admissible set for L^* such that $A \subseteq A'$ and both $\mathfrak{M}$ and $\mathfrak{N}$ contain a finite number of at least two urelements. Let ϕ be any Δ_0 formula of L^+. Then there exists a Σ_1 formula ψ of L^* and a Π_1 formula φ of L^* such that for all $x \in A$, $\phi(x) \leftrightarrow \psi(x) \leftrightarrow \varphi(x)$.*

Proof. As stated in the proof of Corollary 2, the functions f, g, h of L^+ are equivalent to Σ operations in L^*. From Lemma 1, given that $\mathfrak{N}$ is admissible, for each Σ operation defined in L^*, there exists a Σ_1 operation that is equivalent to it. It follows that there are well-defined Σ_1 operations in $\mathfrak{N}$ that are equivalent to f, g, h. Since Σ_1 functions are also Π_1 functions (using the definition of functions), the result follows.

Corollary 5. *Let $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \epsilon, f, g, h, s, t)$ be an admissible set for L^+, and let $\mathfrak{A}_{\mathfrak{N}} = (\mathfrak{N}; A', \epsilon)$ be an admissible set for L^* such that $A = A'$ and $\mathfrak{M}$ contains a finite number of at least two urelements. Any Δ_0 set in $\mathfrak{A}_{\mathfrak{N}}$ defined by some Δ_0 formula of L^+ is a Δ_1 set in $\mathfrak{A}_{\mathfrak{M}}$ defined by some Σ_1 (or Π_1) formula of L^*.*

Proof. This follows from Corollary 4.

Corollary 6. *Let $L^{++} = L^+ \cup c$, where c is a constant symbol not found in L^+. The statement "$h(c) = s$ and c is well founded" is independent of KPU models for L^{++}.*

Proof. Let c be an even standard natural number in a standard hereditarily finite model of KPU. Then $h(c) = s$ following the proof in Theorem 2. Then let c be a non-standard natural number in some non-standard model of KPU. Such a non-standard model exists as shown in Theorem 1 under the ZFC meta-theory (i.e. using the Upward Lowenheim-Skolem Theorem). In more detail, let $\mathbb{HF}$ be the standard hereditarily finite model which is admissible by Lemma 3. Using Theorem 1, there exists an admissible set $\mathfrak{A}$ of uncountable cardinality such that $\mathbb{HF} \preccurlyeq \mathfrak{A}$. Let ϕ be the sentence

$$\phi := (\forall x)(\exists y)[(f(y) > f(x)) \wedge (y \text{ is a natural number})]$$

Let ψ be the sentence:

$$\psi := (\forall x)[(x \text{ is an odd natural number}) \rightarrow (S(x) \text{ is an even natural number})]$$

Let γ be the sentence:

$$\gamma := (\forall x)[(x \text{ is an even natural number}) \rightarrow (S(x) \text{ is an odd natural number})]$$

Let φ be the sentence:

$$\varphi := (\forall x)[(x \text{ is an even natural number}) \rightarrow (h(x) = s)]$$

where "x is an odd natural number" and "x is an even natural number" are both definable by Δ_0 formulas in L^+ as shown in Theorem 2. It follows that $\mathbb{HF} \models \phi$, $\mathbb{HF} \models \psi$, $\mathbb{HF} \models \gamma$ and $\mathbb{HF} \models \varphi$ so that $\mathfrak{A} \models \phi$, $\mathfrak{A} \models \psi$, $\mathfrak{A} \models \gamma$ and $\mathfrak{A} \models \varphi$ as well given that $\mathbb{HF} \preccurlyeq \mathfrak{A}$. The sentences ϕ and ψ imply that in both $\mathbb{HF}$ and $\mathfrak{A}$, there exists an infinite number of even natural numbers c of arbitrary cardinality less than $|\mathbb{HF}|$ and $|\mathfrak{A}|$ respectively such that $h(c) = s$. However, we have a non-standard even natural number c (of rank greater than ω) in $\mathfrak{A}$ such that $h(c) = s$ and so c is not well founded.

5 Generalizing E

In this section, we present a simple generalization from E (set of even natural numbers) to E^m (set of multiples of m), and show that the same results as shown in Theorem 2 apply in this case. That is, E^m is not a Δ_0 set in the standard set theoretic language L, but if we augment L by adding function symbols f, g, h and $s_1, \ldots, s_m$ constant symbols corresponding to m urelements, then $x \in E^m \leftrightarrow \phi(x)$ for some Δ_0 formula ϕ under the new language L^m.

Lemma 5. *(ZFC): Let* $L^* = \{\epsilon\}$, *and let* $\mathfrak{A} = (\emptyset; A, \epsilon)$ *be admissible. The set* $E_m = \{n : n \text{ is a multiple of } m\}$ *is not a* Δ_0 *definable set of* $\mathfrak{A}$

Proof. The proof for this Lemma is a simple re-application of the proof for Lemma 4. Once more, applying Lemma 2, we need only consider the model

(ω, ϵ) given that $\omega \subseteq \mathbb{HF} \subseteq A$ (from Lemma 3). By the Upward Lowenheim-Skolem-Tarski Theorem (Theorem 2), we have an elementary extension $\mathfrak{A}' = (\emptyset; A', \epsilon)$ of (ω, ϵ) of cardinality $\kappa > \aleph_0$. Similar to the proof for Lemma 4, let the automorphism $f : A' \to A'$ be $f(a) = a$ if $a \in \omega$, and $f(a) = S(a)$ if $a \notin \omega \wedge a$ is a natural number, where $S(a)$ stands for the successor of a. Here, "a is a natural number" follows the same formula described in the proof for Lemma 4. If there were a Δ_0 formula ψ defining E^m, then let ϕ be the sentence:

$$\forall x \Big[\psi(x) \leftrightarrow \Big(\neg\psi(S(x)) \wedge \neg\psi(S(S(x))) \wedge \cdots \wedge \neg\psi(S^{m-1}(x)) \Big) \Big]$$

where S^{m-1} is an abbreviation for applying the successor operation S to x by $m-1$ times. Then $\mathfrak{A} \models \phi$ but $\mathfrak{A}' \not\models \phi$ giving rise to a contradiction as $\mathfrak{A} \preccurlyeq \mathfrak{A}'$ and we have the automorphism f defined above. Thus, E^m is not at all definable in the model (ω, ϵ), and is therefore not Δ_0 given Lemma 2.

To show our generalization, given $L^* = \{\epsilon\}$, we expand this to:

$$L^m = \{\epsilon, f, g, h, s_1, \ldots, s_m\}$$

whose intended interpretations in a structure $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \epsilon, f, g, h, s_1, \ldots, s_m)$ are as follows, where the constants $s_1, \ldots, s_m$ are the urelements of $\mathfrak{A}_{\mathfrak{M}}$ (i.e. $M = \{s_1, \ldots, s_m\}$), and where we use $\sim$ as a notation for the function: $\sim s_i = s_{i+1}$ for $1 \leq i \leq m-1$, and $\sim s_m = s_1$, identity otherwise. Given this revised definition for $\sim$, interpretations for the functions f, g, h in Theorem 3 below are basically the same as before (in Theorem 2).

Theorem 3. *Let* $L^m = \{\epsilon, f, g, h, s_1, \ldots, s_m\}$ *and let*

$$\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \epsilon, f, g, h, s_1, \ldots, s_m)$$

be an admissible set with a finite number of m *urelements that satisfies the interpretations intended for* $f, g, h, s_1, \ldots, s_m$ *above. Furthermore, suppose that* A *consists of hereditarily finite sets. Let* E^m *be the set:*

$$E_m = \{n : n \textit{ is a multiple of } m\}$$

There exists a Δ_0 *formula* ϕ *of* L^m *such that* $\phi(x) \leftrightarrow x \in E_m$ *for all* $x \in A$.

Proof. Following the proof in Theorem 2, let $n \in E^m \leftrightarrow \phi(n)$, where:

$$\phi(n) \leftrightarrow n \text{ is a natural number} \wedge h(n) = s_1$$

As usual, ϕ is a Δ_0 formula given that "n is a natural number" is Δ_0, and h is a function symbol of L^m. We prove that if n is a natural number that is a multiple of m, then $n \in E^m$ by induction on the rank of sets in A. For $0 = \emptyset$, we have $h(\emptyset) = s_1$ so that $0 \in E^m$. For $1 \leq i \leq m-1$, we have $h(i) = p$ such that:

$$(i, p) = g(i, h(b)) \text{ and } P(i) = b \wedge (\forall x \in i)(\forall y \in i)[(x = b = P(i)) \to f(x) \geq f(y)]$$

so that $g(i, h(P(i))) = (i, \sim s_i) = (i, s_{i+1})$. It follows that $p \in \{s_2, \ldots, s_m\}$ for $1 \le i \le m-1$ so that $i \notin E^m$ as desired. For $i = m$, using the same reasoning, we have:

$$h(m) = p \leftrightarrow (m, p) = g(m, h(P(m))) = (m, \sim s_m) = (m, s_1)$$

so that $p = s_1$ and $\phi(m)$ by definition, i.e. $m \in E^m$. Suppose that the induction hypothesis holds for all sets of rank $i > m$, whereby for natural numbers c that are not multiples of m, we have $c \notin E^m$ and $h(c) \in \{s_2 \ldots s_m\}$. But if c is a multiple of m, we have $c \in E^m$ and $h(c) = s_1$. Let $a \in A$ be any set of rank $i+1$. If a is not a natural number, then $\neg\phi(a)$ by definition so that $a \notin E^m$. If a is a natural number and a is a multiple of $1 \le i \le m-1$, we can apply the same reasoning as the previous paragraph for sets from 1 to $m-1$, applying the induction hypothesis where needed, i.e.

$$h(a) = p \leftrightarrow (a, p) = g(a, h(P(a))) = (a, \sim s_j) = (a, s_{j+1})$$

for some $1 \le j \le m-1$ so that $p \in \{s_2, \ldots, s_m\}$ and $a \notin E^m$. If a is a natural number and is a multiple of m, then by the induction hypothesis, we have $h(a) = p \leftrightarrow (a, s_1) = g(a, h(P(a))) = (a, \sim s_m)$ so that $p = s_1$ and $a \in E^m$.

Corollary 7. *Let $L^* = \{\epsilon\}$. For any even n, $\big(KPU + \exists$ at least m urelements $\vdash$ n is a multiple of $m\big)$ has a proof in L^* that does not use the $+$ operation.*

Proof. Similar to the proof in Corollary 2.

Corollary 8. *Let $L^{m+} = L^m \cup c$, where c is a constant symbol not found in L^m. The statement "$h(c) = s_1$ and c is well founded" is independent of KPU models for L^{m+}.*

Proof. Similar to the proof in Corollary 6, changing "even" to "multiple of m" and s to s_1.

6 Further Results

Theorem 4. *Let L' be a language that has the standard set membership symbol ϵ along with a finite number of n function symbols $f_1, \ldots, f_n$ and a finite number of symbols denoting urelements. Suppose that the interpretation in some admissible set $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \epsilon, f_1, \ldots, f_n, \ldots)$ for L' of each of the symbols $f_1, \ldots, f_n$ is equivalent to a Δ_1 operation defined by some formula involving none of $f_1, \ldots, f_n$. Then any Δ_0 set $S \subseteq A$ defined by a Δ_0 formula of L' is also a Δ_1 set defined by some Σ_1 (or Π_1) formula involving none of $f_1, \ldots, f_n$.*

Proof. Let $L = \{\epsilon, \dots\}$ be $L' \setminus \{f_1, \dots, f_n\}$. From the given, for f_i (with $1 \leq i \leq n$) and $x, y \in A$, we have $f_i(x) = y \leftrightarrow \phi_i(x, y) \leftrightarrow \psi_i(x, y)$ for some Σ_1 formula ϕ_i and Π_1 formula ψ_i of L^*. Suppose that ϕ is a Δ_0 formula of L' that defines some set $S \subseteq A$. To arrive at the result, substitute each subformula of ϕ of the form $f_i(x) = y$ (for $1 \leq i \leq n$ and variables or constants x, y) with $\phi_i(x, y)$. Likewise, for each subformula of ϕ of the form $f_i(x) \in y$ (for $1 \leq i \leq n$ and variables or constants x, y), substitute it with $\phi_i(x, z) \wedge z \in y$ for some z not in ϕ_i. For functional composition, we have the result (see [2]) that the composition of Σ_1 functions in models of KPU is still a Σ_1 function and therefore there exists a Σ_1 formula that is equivalent to whatever composition the functions $f_1, \dots, f_n$ may have - allowing for substitution as well. The result follows using the given that each Σ_1 formula ϕ_i of L^* for $1 \leq i \leq n$ that satisfies the said condition is equivalent to some Π_1 formula ψ_i of L^*.

Corollary 9. *Assume the same assumptions of Theorem 4 but suppose that A is uncountable. Then there are sets of A that are not Δ_0 definable by formulas of L'.*

Proof. From Theorem 4, each set $S \subseteq A$ that is definable by some Δ_0 formula ϕ of L' is also defined by some Σ_1 (or Π_1) formula φ of L (where $L = L' \setminus \{f_1, \dots, f_n\}$). However, given that L is finite, the number of possible Σ_1 formulas φ is countable, whereas A is uncountable.

7 Conclusion and Future Work

In this paper we presented function symbols f, g, h and symbols s, t that can be added to the set-membership symbol ϵ to form a language L^+ such that non-Δ_0 sets in standard set-theory L, would become Δ_0 under L^+. These function symbols involve specific interpretations that the paper studies under models of KPU. Moreover, we showed that if we form a language $L^{++} = L^+ \cup c$, where c is a constant symbol, then the statement "$h(c) = s$ and c is well founded" is independent of KPU models for L^{++} - implying that the additional function symbols in L^+ are not strong enough to discriminate between standard and non-standard elements. We also presented a generalization from E to E^m. For future work, it is of interest if the recursive sets in $\langle \mathcal{N}, +, \cdot \rangle$ can be mapped to Δ_0 sets of some structure for L^+ using only the symbols f, g, h, s, t.

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