About the Method for Solving the Navier-Stokes Equations: Mathematical Modeling of the Hydrodynamic Bearings

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Abstract. A methodology is provided for solving the Navier-Stokes equations with certain assumptions applied to the flow of lubricant between the bearing liner of a sliding bearing and the shaft journal. A plane problem of velocity distribution of oil particles filling the space between the shaft journal of a rotating crankshaft and the bearing liner is solved based on the hydrodynamic lubrication theory. Based on this solution, dependencies for determining the friction moment arising in the main bearing of a piston machine are derived. A methodology for determining the friction moment and load-bearing capacity of a hydrodynamic bearing with finite length under constant load is developed. The problem of determining the friction moment and load-bearing capacity of sliding bearings under the action of variable loads based on hydrodynamic lubrication theory arises when designing and calculating main bearings of piston compressor machines. As is well known, in these machines, liquid friction predominantly occurs, governed by the laws of hydrodynamic lubrication theory. However, these bearings operate under variable loads, caused by periodic pressure forces of gases in the cylinder and inertia forces of moving parts in piston compressor machines. The purpose of this work is to develop a methodology for solving the Navier-Stokes equations and determining the friction moment in the sliding bearing under constant load. The object of the study is the sliding bearings of piston compressors used in the oil and gas industries, operating under liquid friction conditions.

Keywords: Hydromechanics, Navier-Stokes equations, hydrodynamic bearing, friction moment, load-bearing capacity.

1 Introduction

The Navier–Stokes equations, sometimes referred to as the Reynolds equations, are among the most important tools in hydrodynamics and are widely used in the mathematical modeling of many engineering problems, including the determination of velocity distributions of oil particles occupying the space between the journal of a rotating crankshaft and a sliding bearing. In the main bearings of piston machines, as is well known, liquid friction predominantly occurs and is governed by the principles of hydrodynamic lubrication theory [1–4].

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The problem of determining the friction moment and load-bearing capacity of sliding bearings under constant or variable loads based on hydrodynamic lubrication theory arises in the design and calculation of main bearings of piston compressor machines. The reliability of piston machines operating at high temperatures and in corrosive environments largely depends on the stable performance of the sliding bearing. These bearings operate under relative sliding between the journal surface and the bearing liner surface and represent one of the most heavily loaded moving pairs in the machine [5–7].
In piston compressor machines, sliding bearings are subjected to variable loads caused by periodic gas pressure forces in the cylinder and inertia forces of moving components [8-11]. Therefore, the determination of the friction moment and load-bearing capacity of sliding bearings under variable loading conditions based on hydrodynamic lubrication theory remains a critical issue in their design and analysis [12-14].
The purpose of this work is to develop a methodology for solving the Navier-Stokes equations and determining the friction moment in a sliding bearing under constant load conditions. The object of the study is the sliding bearings of piston compressors used in the oil and gas industry, operating under conditions of liquid friction.

2 Methodology for solving the Navier-Stokes equations and evaluating the load capacity of a hydrodynamic bearing

To determine the velocity distribution of oil particles occupying the space between the journal and the sliding bearing, the equations of motion and continuity of the oil film (the Navier-Stokes equations) are employed. By neglecting convective inertia forces and body forces acting on the oil particles-since they are negligible compared with the viscous friction forces between oil layers-the equations in polar coordinates reduce to the following form:

$$\frac{\partial V_\theta}{\partial t} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r^2}\right) \tag{1}$$

$$\frac{\partial V_r}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{V_r}{r^2}\right) \tag{2}$$

$$\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r} = 0 \tag{3}$$

Where p, ρ, ν are the pressure, density, and kinematic viscosity coefficient of the oil, V_θ, V_r are the velocity components of oil particles along the θ and r (Fig. 1).

This paper considers and solves the problem of velocity distribution of oil particles in the layer filling the space between the journal of the crankshaft and the main bearing liner of the piston machine under constant force [1,2]. The schematic drawing of the journal inside the bearing liner of the main hydrodynamic bearing of the piston machine is shown in Fig. 1.

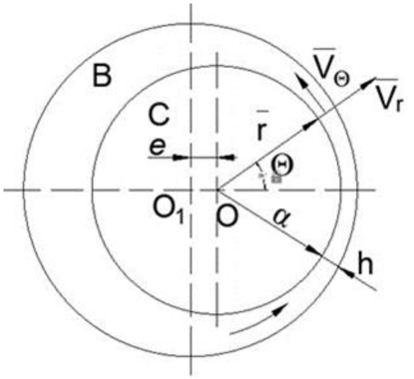

Fig. 1. Schematic diagram of the hydrodynamic bearing with the position of the journal during operation

In other words, a steady two-dimensional problem of hydrodynamic lubrication theory is considered. By neglecting inertia forces and body forces acting on the oil particles, as these are insignificant compared with the viscous friction forces between fluid layers, the Navier-Stokes equations (1), (2), and (3) can be rewritten in the following form:

$$\frac{\partial p}{\partial \theta} = \mu r \left(\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r^2} \right) \tag{4}$$

$$\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r} = 0 \tag{5}$$

To solve the differential equations (4) and (5), we will apply the boundary conditions, which for this problem are as follows:

$$\begin{aligned} r = a, V_\theta = u,\ V_r = 0 \\ r = r_B, V_\theta = 0,\ V_r = 0 \end{aligned} \tag{6}$$

Where u is the tangential velocity of points on the shaft journal.

We will seek the solution to the differential equation (4) in the following form:

$$V_\theta = u\varphi(r)\psi(\theta) + \tau(r) \tag{7}$$

Where $\psi(\theta)$ is a function depending on the angular coordinate θ, $\varphi(r)$ and $\tau(r)$ are functions depending only on the radial coordinate *r*.

As shown in Fig. 1, the shaft journal inside the bearing liner occupies a symmetric position relative to the line $O_1 O$, from which the angle θ is measured. Therefore, the function $\psi(\theta)$ must be an even function, taking the same value when the sign of θ is changed. Consequently, we will seek the function $\psi(\theta)$ in the form of the following trigonometric series containing only cosines of the angle θ

$$\psi(\theta) = c_0 + c_1 cos\theta + c_2 cos2\theta \tag{8}$$

Where $c_{0,}c_{1\,,}\, c_2$ are unknown coefficients.

Regarding the functions $\varphi(r)$and $\tau(r)$, we will represent them as power series containing the radial coordinate *r*:

$$\begin{aligned}&\varphi(r) = a_0 + a_1 r + a_2 r^2\\&\tau(r) = b_0 + b_1 r + b_2\end{aligned} \tag{9}$$

The coefficients $a_0, a_1, a_2, b_0, b_1, b_2$ appearing in these expressions are also unknown and need to be determined.
We now proceed to determine the unknown coefficients appearing in expressions (8) and (9). By substituting expressions (7) into the boundary conditions (6), the following relations are obtained:

$$u\varphi(a)\psi(\Theta) + \tau(a) = u$$
$$u\varphi(r_B)\psi(\Theta) + \tau(r_B) = 0$$

From these relations, we get:

$$\varphi(a) = 0, \tau(a) = u \tag{10}$$
$$u\varphi(r_B)\psi(\Theta) + \tau(r_B) = 0 \tag{11}$$

As can be seen from Fig. 1, the amount of oil flowing through any section of the channel between the journal and the bearing remains constant. Based on this, we can write:

$$\int_a^{r_C} V_\Theta \, dr = C$$

Where *C* is a constant representing the oil flow rate through the channel.
Substituting expression (7) into this, we calculate the integral, considering the relations in (9), and obtain the following equation:

$$\int_a^{r_C} V_\Theta \, dr = u\psi(\Theta) \left| a_0 r + \frac{a_1}{2} r^2 + \frac{a_2}{3} r^3 \right|_a^{r_C} - \left| b_0 r + \frac{b_1}{2} r^2 + \frac{b_2}{3} r^3 \right|_a^{r_C} = C$$

From this expression, we obtain:

$$u\psi(\Theta) \left| a_0 r + \tfrac{a_1}{2} r^2 + \tfrac{a_2}{3} r^3 \right|_a^{r_B} - \langle b_0 r_B + \tfrac{b_1}{2} {r_B}^2 + \tfrac{b_2}{3} {r_B}^3 \rangle = 0 \tag{12}$$
$$b_0 a + \tfrac{b_1}{2} a^2 + \tfrac{b_2}{3} a^3 = C \tag{13}$$

Next, we note that within the confined oil film, the pressure *p* must be a continuous periodic function of the angle Θ, that is:

$$\int_a^{2\pi} \frac{\partial p}{\partial \theta} d\theta = 0 \tag{14}$$

Substituting the dependency (12) into equation (9), for $r = a$ we get:

$$\frac{\partial p}{\partial \theta} = \mu \left[au\varphi''(a) + u\varphi'(a) - \frac{u}{a}\varphi(a) \right] \psi(\theta) + \mu \frac{u}{a}\varphi(a)\psi''(a) + \mu\tau'(a) - \mu\frac{1}{a}\tau'(a) \tag{15}$$

Substituting the value of $\frac{\partial p}{\partial \theta}$ from (15)into the integrand expression, and after calculating the value of the integral (14) taking into account the dependency (8) for $r = a$, we obtain:

$$uc_0 \left[a\varphi''(a) + \varphi'(a) - \frac{1}{a}\varphi(a) \right] + \left[a\tau''(a) + \tau'(a) - \frac{1}{a}\tau(a) \right] = 0 \tag{16}$$

Thus, to determine the unknown coefficients in expressions (8) and (9), we have dependencies (10), (11), (12), (13), and (16), of which identities (12) and (13) contain the variable angle θ. These identities must be satisfied for any value of the angle θ. By substituting the values of $\theta = 0, \frac{\pi}{2}$, or π at characteristic points of the oil layer, we obtain six equations. These equations, together with the four equations (10), (13), and (16), form a closed system of ten equations containing ten unknowns: a_0, a_1, a_2, b_0, $b_1, b_2,$ $c_0, c_1,$ c_2 and c. By solving this system of equations, we can find the numerical values of the ten unknown coefficients in these equations.

It should be noted that the stress tensor components $p_{r\theta}$ and p_{rr} of the surface forces are directed along the θ and r - axes, respectively, of the adopted polar coordinate system. The component $p_{r\theta}$ represents the tangential (shear) stress acting on the bearing surface, while p_{rr} corresponds to the normal stress associated with the oil film pressure. These stress components play a fundamental role in the mechanical interaction between the oil film and the contacting surfaces of the journal and bearing liner.

Having determined the pressure distribution and shear stresses within the lubricating layer, we can now proceed to evaluate the principal performance characteristics as load ability of the crankshaft bearing. In particular, the frictional moment is obtained by integrating the tangential stress over the contact surface, while the load-bearing capacity of the bearing is determined by integrating the pressure distribution over the bearing area. These quantities provide a quantitative assessment of the bearing's ability to support external loads and ensure stable operation under hydrodynamic lubrication conditions.

3 Conclusion

A method is presented for solving the Navier-Stokes equations under specific assumptions describing lubricant flow between the sliding bearing liner and the shaft journal. Mathematical modeling of hydrodynamic friction in a sliding bearing based on these equations is carried out.

A method is developed for determining the load ability of the crankshaft bearing of a finite-length sliding bearing of piston machine operating under constant load conditions.

Declaration. I confirm that the use of any third-party materials in the manuscript does not require permission.

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